Determination of the Number of Roots of a Polynomial Lying in a Given Algebraic Domain

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ABSTRACT

The problem of finding the number of roots of a polynomial $f(z = x + iy)$ satisfying a given system of algebraic inequalities $g_1(x, y) > 0, \dots, g_k(x, y) > 0$ is considered. The method proposed is based on elimination theory and on the Hermite approach to the problem. The algorithm uses a finite number of elementary algebraic operations on the coefficients of the polynomials involved.

INTRODUCTION

Let a polynomial with complex coefficients be given:

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad \text{where} \quad a_0 \neq 0, \quad z = x + iy,$$

and let $g(x, y)$ be a polynomial in x and y with real coefficients.

PROBLEM 1. Find the number of roots of $f(z)$ which satisfy the inequality $g(x, y) > 0$.

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The problem stated has its historical background in the studies of Cauchy, Sturm, Jacobi, and Hermite on the general problem of separation of roots of an algebraic equation.

The importance of the problem for the stability theory of differential equations induced investigations into special types of domains in the complex plane: $g(x, y) \equiv -x$ (Routh, Hurwitz, and others), $g(x, y) \equiv 1 - x^2 - y^2$ (Schur, Cohn, and others). We shall not go into details regarding history; for surveys and recent developments in this area we refer to [2, 5, 8, 10].

In the present paper we shall, however, be interested in the general statement of the problem. Its solutions may be useful for some problems of control theory [1, 2, 8]. We shall discuss also the following generalization:

PROBLEM 2. Find the number of roots of $f(z)$ which satisfy the system of inequalities $g_1(x, y) > 0, \dots, g_k(x, y) > 0$. Here $g_j(x, y)$ $(j = 1, \dots, k)$ are polynomials in x and y with real coefficients.

In 1853–56 Hermite proposed a method for solving Problem 1 for several classes of $g(x, y)$ [6, 7]. Hermite's method is constructive and purely algebraic, i.e., the algorithm consists of a finite number of operations on the coefficients of f and g.

Our approach will be based on the Hermite method of solving Problem 1 for the real roots of $f(z)$ (Section 2). For this case the problem can be reduced to finding the positive index for the quadratic form in the real variables $x_0, \dots, x_{n-1}$:

$$\sum_{j=1}^{n} g(\lambda_j, 0)\left[x_0 + \lambda_j x_1 + \lambda_j^2 x_2 + \cdots + \lambda_j^{n-1} x_{n-1}\right]^2, \tag{1}$$

where $\lambda_1, \dots, \lambda_n$ are the roots of $f(z)$. Coefficients of the form (1) are symmetric polynomials in $\lambda_1, \dots, \lambda_n$, and so they can be expressed rationally in terms of the coefficients of f and g (Theorem 1.1). Problem 2 can be reduced to Problem 1 with the help of the Markov formula (2.4).

To solve Problem 1 for the nonreal roots of $f(z) = 0$ we shall consider, first, the reduction of this equation to the system of real algebraic equations

$$F_1(x, y) = 0, \qquad F_2(x, y) = 0, \tag{2}$$

where $F_1 \equiv \operatorname{Re} f(z)$, $F_2 \equiv \operatorname{Im} f(z)$. Then we shall use the following fact from the theory of algebraic equations (Theorem 1.5): the y-component of a solution of an algebraic equation system is a real rational function of the

x-component. Thus, the system (2) may be replaced by the following one:

$$\mathscr{X}(x) = 0, \qquad y = r(x).$$

Here $r(x)$ is the abovementioned rational function, and $\mathscr{X}(x)$ is the resultant of F_1 and F_2 obtained by the elimination of y. Under some assumptions, a one-to-one correspondence between the set of roots of $f(z)$ and the set of real roots of $\mathscr{X}(x)$ can be established, and so Problem 1 can be reduced to the case considered in the previous paragraph. Thus, Problems 1 and 2 can be solved in a finite number of operations on the coefficients of the polynomials involved. We shall treat in detail the case when $f(z)$ is a polynomial with real coefficients and $g(x, y)$ is an even polynomial in y, since this is of particular interest for control theory.

NOTATION.

1. For the quadratic form $A(X, X)$: $\mathscr{A}$ denotes its matrix, $\sigma(A)$ its signature, and $n_+(A)$ [or $n_-(A)$] its positive [or negative] index.
2. For polynomials $f(x), G(x), G_1(x), \ldots, G_k(x)$: $\mathscr{D}(f)$ is the discriminant of f, $\mathscr{R}(f, G)$ is the resultant of f and G, $\gcd(f, G)$ is the greatest common divisor of f and G, and $\operatorname{nrr}\{f = 0 \mid G_1 > 0, \ldots, G_k > 0\}$ is the number of real roots of $f(x)$ which satisfy the system of inequalities $G_1(x) > 0, \ldots, G_k(x) > 0$.
3. For polynomials $f_1(x, y), f_2(x, y)$ considered as polynomials in y [or x]: $\mathscr{X}(x)$ [or $\mathscr{Y}(y)$] denotes the resultant of f_1 and f_2.

REMARK. Henceforth, any polynomials with real (complex) coefficients will be called a real (complex) polynomial. A rational function will be called real if its numerator and denominator are real polynomials.

1. PRELIMINARY RESULTS

We recall here some results from the theory of algebraic equations (symmetric polynomials of the roots, resultant, subresultants, common roots, and elimination of variables) [2, 3, 13].

Consider two complex polynomials

$$f_1(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad \text{and}$$

$$f_2(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m \tag{1.1}$$

($a_0 \neq 0, b_0 \neq 0$). Denote by $\lambda_1, \ldots, \lambda_n$ the roots of f_1, and consider them as functions of $a_0, \ldots, a_n$.

THEOREM 1.1. *Let $\mathscr{F}(x_1, \ldots, x_n)$ be a symmetric polynomial of its variables, i.e., let its value be unchanged when any two of the variables are interchanged. If we substitute in $\mathscr{F}(\lambda_1, \ldots, \lambda_n)$ instead of $\lambda_1, \ldots, \lambda_n$ their representations in terms of $a_0, \ldots, a_n$, then the function obtained, $F(a_0, \ldots, a_n)$, is a rational one in its arguments (a polynomial if $a_0 = 1$).*

DEFINITION. The expression $a_0^m f_2(\lambda_1) \cdots f_2(\lambda_n)$ is a polynomial with respect to $a_0, \ldots, a_n, b_0, \ldots, b_m$; it is called the *resultant* of f_1 and f_2, and is denoted by $\mathscr{R}(f_1, f_2)$.

THEOREM 1.2. *$\mathscr{R}(f_1, f_2) = 0$ if and only if f_1 and f_2 have a common root.*

DEFINITION. The expression $(-1)^{n(n-1)/2}\mathscr{R}(f_1, f_1')/a_0$ is a polynomial with respect to $a_0, \ldots, a_n$; it is called the *discriminant* of f_1 and is denoted by $\mathscr{D}(f_1)$.

COROLLARY TO THEOREM 1.2. *$\mathscr{D}(f_1) = 0$ if and only if f_1 has a multiple root.*

There exist several methods for the representation of $\mathscr{R}(f_1, f_2)$ in terms of the coefficients of f_1 and f_2. Here is one that is widely used:

THEOREM 1.3. *$\mathscr{R}(f_1, f_2) = \det M$, where*

$$M = \left[\begin{array}{cccccccccccccccc}
a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_n & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & a_n \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_m \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_m & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_0 & b_1 & \cdot & \cdot & \cdot & b_m & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} m \text{ rows} \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} n \text{ rows}
\end{array}
\tag{1.2}$$

Suppose that $\mathscr{R}(f_1, f_2) = 0$, i.e., f_1 and f_2 have a common root, or, in other words, the degree of the greatest common divisor of f_1 and f_2 [$\gcd(f_1, f_2)$] is at least 1. In order to find an explicit formula for $\gcd(f_1, f_2)$ in terms of a_j, b_k we introduce the following

DEFINITION. The matrix M_1 of dimension $m + n - 2$ obtained on deleting the first and the last rows and the first and the last columns in the matrix M is called the *first inner* of M [8]. On deleting further, we obtain the inners $M_2, M_3, \ldots$ of dimensions $m + n - 4, m + n - 6, \ldots$ respectively. The determinants of inners are called *subresultants*.

EXAMPLE 1.1. For $n = 5$, $m = 3$ we have three inners:

$$M = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_0 & b_1 \\ 0 & b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & b_0 \\ b_0 & b_1 \end{bmatrix}.$$

THEOREM 1.4. *If* $\det M = \det M_1 = \cdots = \det M_{k-1} = 0$ *and* $\det M_k \neq 0$, *then* $\deg\{\gcd(f_1, f_2)\} = k$. *In this case* $\gcd(f_1, f_2)$ *equals the determinant of the matrix obtained from* M_k *by replacing the last column in it with the column*

$$\left[z^{m-k-1}f_1(z), z^{m-k-2}f_1(z), \ldots, f_1(z), f_2(z), z f_2(z), \ldots, z^{n-k-1}f_2(z)\right]^T.$$

Supposing in Example 1.1 that $\deg\{\gcd(f_1, f_2)\} = 1$, we get

$$\gcd(f_1, f_2) = \det\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & z f_1(z) \\ 0 & a_0 & a_1 & a_2 & a_3 & f_1(z) \\ 0 & 0 & 0 & b_0 & b_1 & f_2(z) \\ 0 & 0 & b_0 & b_1 & b_2 & z f_2(z) \\ 0 & b_0 & b_1 & b_2 & b_3 & z^2 f_2(z) \\ b_0 & b_1 & b_2 & b_3 & 0 & z^3 f_2(z) \end{bmatrix}.$$

Using the fact that $\gcd(f_1, f_2)$ must be of the form $Az + B$, we may simplify the expression to

$$\gcd(f_1, f_2) = \det\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 z \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 z + a_5 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 z + b_3 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 z \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \end{bmatrix}.$$

Thus, $\gcd(f_1, f_2) = Az + B$, where $A = \det M_1$ and

$$B = \det\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_5 \\ 0 & 0 & 0 & b_0 & b_1 & b_3 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \end{bmatrix}.$$

In the general case we have

COROLLARY TO THEOREM 1.4. *If f_1 and f_2 have a single common root λ ($\det M = 0$, $\det M_1 \neq 0$), then*

$$\lambda = -\frac{\det \tilde{M}_1}{\det M_1}, \tag{1.3}$$

where $\tilde{M}_1$ equals the matrix obtained from M_1 by replacing the last column in it with the column

$$[\underbrace{0, \ldots, 0}_{m-2}, a_n, b_m, \underbrace{0, \ldots, 0}_{n-2}]^T.$$

REMARK 1.1. Since there exist different methods for finding $\gcd(f_1, f_2)$ [2, 3], it is possible to write down other formulae for a common root of f_1 and f_2 instead of (1.3). Though these formulae may look very different while we consider the coefficients a_j, b_k as indeterminate, they will be equivalent, of course, when we substitute numerical values of coefficients in them.

Consider now a system of real polynomial equations:

$$f_1(x, y) = 0, \quad f_2(x, y) = 0 \qquad (\deg f_j = n_j). \tag{1.4}$$

We shall suppose that the coefficient of y^{n_j} in the development of f_j in powers of x, y differs from zero. Arranging f_1 and f_2 in powers of y, we construct the resultant of these polynomials, $\mathscr{X}(x) = \mathscr{R}_y(f_1, f_2)$ (elimination of y). Let $\mathscr{X}(x) \not\equiv 0$, i.e., f_1 and f_2 be coprime. Then the system (1.4) has a finite number of solutions. $\deg \mathscr{X}(x)$ generally equals $N = n_1 n_2$ (*Bézout theorem*).

ASSUMPTION 1. Let $\deg \mathscr{X}(x) = N = n_1 n_2$.

Under this assumption, the system (1.4) has exactly N solutions. Let α be a root of $\mathscr{X}(x)$. Then $f_1(\alpha, y)$ and $f_2(\alpha, y)$ have a nontrivial gcd, which may be found from Theorem 1.4. Thus, to every root α of $\mathscr{X}(x)$ corresponds at least one of value β that is a root of $\gcd(f_1(\alpha, y), f_2(\alpha, y))$, and every such pair will be a solution of (1.4).

ASSUMPTION 2. Henceforth we shall consider the case when $\mathscr{X}(x)$ has no multiple roots [i.e., by Corollary to Theorem 1.3, $\mathscr{D}(\mathscr{X}) \neq 0$].

In this case $\deg\{\gcd(f_1(\alpha, y), f_2(\alpha, y))\} = 1$, and to find the single value of β we may use the Corollary to Theorem 1.4.

THEOREM 1.5. *Under Assumption* 2, *the component* β *of the solution* (α, β) *of the system* (1.4) *is a real rational function of* α:

$$\beta = \mathscr{r}(\alpha). \tag{1.5}$$

The function $\mathscr{r}(x)$ *is determined by the formula* (1.3).

EXAMPLE 1.2. Solve the system of equations

$$\begin{aligned} f_1(x, y) &\equiv y^2 + x^2 - 6x = 0, \\ f_2(x, y) &\equiv 2y^3 - 6xy^2 + 8y^2 + 9x - 9y = 0. \end{aligned} \tag{1.6}$$

Solution

Let us construct the resultant $\mathscr{X}(x)$:

$$\mathscr{X}(x) = \det\begin{bmatrix} 1 & 0 & x^2 - 6x & 0 & 0 \\ 0 & 1 & 0 & x^2 - 6x & 0 \\ 0 & 0 & 1 & 0 & x^2 - 6x \\ 0 & 2 & 8 - 6x & -9 & 9x \\ 2 & 8 - 6x & -9 & 9x & 0 \end{bmatrix}$$

$$= 40x^6 - 600x^5 + 3088x^4 - 6312x^3 + 4626x^2 - 486x.$$

The components α_j of solutions (α_j, β_j) of the system (1.6) are the roots of $\mathscr{X}(x)$. Their approximate values are

$$0, \qquad 0.12515, 1.40357, 2.16417, 5.60133, 5.70575.$$

Once α_j are found, β_j can be immediately obtained from (1.3):

$$\beta_j = -\frac{\det \tilde{M}_1(\alpha_j)}{\det M_1(\alpha_j)},$$

where

$$\det M_1(x) = \det\begin{bmatrix} 1 & 0 & x^2 - 6x \\ 0 & 1 & 0 \\ 2 & 8 - 6x & -9 \end{bmatrix} = -2x^2 + 12x - 9,$$

$$\det \tilde{M}_1(x) = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x^2 - 6x \\ 2 & 8 - 6x & 9x \end{bmatrix} = 6x^3 - 44x^2 + 57x.$$

So the corresponding values of β_j are

$$0, \qquad 0.85744, -2.54005, 2.88119, -1.49381, 1.29492.$$

REMARK 1.2. Referring to Remark 1.1, we may repeat that it is possible to write down other formulae instead of (1.5) connecting the two components of solution of the system (1.4). One of these formulae is due to Liouville; it may be found in [12].

2. SEPARATION OF REAL ROOTS

Let us consider now two real polynomials ($a_0 \neq 0, b_0 \neq 0$)

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \quad \text{and}$$

$$G(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m.$$

Suppose that $f(x)$ and $G(x)$ have no common roots, i.e., by Theorem 1.2, $\mathscr{R}(f, G) \neq 0$. Let $\lambda_1, \ldots, \lambda_n$ be the roots of f, suppose that they are distinct, i.e., by Corollary to Theorem 1.2, $\mathscr{D}(f) \neq 0$.

PROBLEM 1(R). Find the number of real roots of f that satisfy the condition $G > 0$ (we shall denote this number by $\operatorname{nrr}\{f = 0 \mid G > 0\}$).

In order to solve the above problem, let us construct two quadratic forms in real variables $x_0, \ldots, x_{n-1}$:

$$S(x_0, \ldots, x_{n-1}) = \sum_{j=1}^{n} \left[x_0 + \lambda_j x_1 + \lambda_j^2 x_2 + \cdots + \lambda_j^{n-1} x_{n-1} \right]^2 = X^T \mathscr{S} X$$

and

$$H(x_0, \ldots, x_{n-1}) = \sum_{j=1}^{n} G(\lambda_j) \left[x_0 + \lambda_j x_1 + \lambda_j^2 x_2 + \cdots + \lambda_j^{n-1} x_{n-1} \right]^2$$

$$= X^T \mathscr{H} X \tag{2.1}$$

with *Hankel matrices*

$$\mathscr{S} = \left[s_{j+k} \right]_{j,k=0}^{n-1} \quad \text{and} \quad \mathscr{H} = \left[h_{j+k} \right]_{j,k=0}^{n-1}. \tag{2.2}$$

Since $s_k = \sum_{j=1}^{n} \lambda_j^k$ and $h_k = \sum_{j=1}^{n} G(\lambda_j) \lambda_j^k$ are symmetric polynomials in $\lambda_1, \ldots, \lambda_n$, they permit, by Theorem 1.1, rational representations in terms of the coefficients $a_0, \dots, a_n$:

$$s_0 = n, \qquad s_1 = -a_1/a_0,$$

$$s_k = -\frac{a_1 s_{k-1} + a_2 s_{k-2} + \cdots + a_{k-1} s_1 + a_k k}{a_0} \quad \text{if} \quad k \leqslant n,$$

$$s_k = -\frac{a_1 s_{k-1} + a_2 s_{k-2} + \cdots + a_n s_{k-n}}{a_0} \quad \text{if} \quad k > n$$

[the Newton sums of $f(x)$], and

$$h_k = b_0 s_{k+m} + b_1 s_{k+m-1} + \cdots + b_m s_k.$$

THEOREM 2.1 (Jacobi) [4, 9].

$$\operatorname{nrr}\{f = 0\} = \sigma(S) = n_+(S) - n_-(S).$$

THEOREM 2.2 (Hermite, Sylvester) [9, 14].

$$\operatorname{nrr}\{f = 0 \mid G > 0\} = n_+(H) - q,$$

where $q = [n - \sigma(S)]/2$ is the number of pairs of complex-conjugate roots of $f(x)$.

REMARK 2.1. The numbers $\sigma(S)$ and $n_+(H)$ may be determined by means of the number P of permanences and the number V of variations of sign in the sequence of leading principal minors of corresponding matrices:

$$\begin{aligned} \sigma(S) &= P(1, \mathcal{S}_1, \dots, \mathcal{S}_n) - V(1, \mathcal{S}_1, \dots, \mathcal{S}_n), \\ n_+(H) &= P(1, \mathcal{H}_1, \dots, \mathcal{H}_n), \end{aligned} \tag{2.3}$$

if none of them is zero. If there exists a zero in one of these sequences, then $\sigma(S)$ and $n_+(H)$ should be calculated with the help of the *rule of Frobenius* [4].

REMARK 2.2. Theorems 2.1 and 2.2 may be extended to the cases when $f(x)$ has multiple roots [9, 14] and/or when $f(x)$ and $G(x)$ have common roots. To use the Theorem 2.2 in the latter case, we have to divide $f(x)$ and $G(x)$ beforehand by $\gcd(f, G)$ (which can be determined by Theorem 1.4).

Consider now

PROBLEM 2(R). For real polynomials $f(x), G_1(x), \dots, G_k(x)$ find

$$\operatorname{nrr}\{f = 0 \mid G_1 > 0, \dots, G_k > 0\}.$$

We shall suppose that polynomials f and G_j are coprime, i.e., $\mathscr{R}(f, G_j) \neq 0$ for $j = 1, \dots, k$. Problem 2(r) can be immediately solved by applying the following

THEOREM 2.3 (Markov [11]).

$$\begin{aligned}
&\operatorname{nrr}\{f = 0 | G_1 > 0, \dots, G_k > 0\} \\
&\quad = \frac{1}{2^{k-1}} \Bigg[\sum_{1 \leqslant j_1 \leqslant k} \operatorname{nrr}\{f = 0 | G_{j_1} > 0\} \\
&\qquad - \sum_{1 \leqslant j_1 < j_2 \leqslant k} \operatorname{nrr}\{f = 0 | G_{j_1} G_{j_2} < 0\} \\
&\qquad + \sum_{1 \leqslant j_1 < j_2 < j_3 \leqslant k} \operatorname{nrr}\{f = 0 | G_{j_1} G_{j_2} G_{j_3} > 0\} - \cdots \\
&\qquad + (-1)^{k-1} \operatorname{nrr}\{f = 0 | (-1)^{k-1} G_1 G_2 \cdots G_k > 0\} \Bigg]. \qquad (2.4)
\end{aligned}$$

Idea of the proof. Let $k = 2$. Then

$$\begin{aligned}
\operatorname{nrr}\{f = 0 | G_1 > 0\} &= \operatorname{nrr}\{f = 0 | G_1 > 0, G_2 > 0\} \\
&\quad + \operatorname{nrr}\{f = 0 | G_1 > 0, G_2 < 0\}, \\
\operatorname{nrr}\{f = 0 | G_2 > 0\} &= \operatorname{nrr}\{f = 0 | G_2 > 0, G_1 > 0\} \\
&\quad + \operatorname{nrr}\{f = 0 | G_2 > 0, G_1 < 0\},
\end{aligned}$$

$$\begin{aligned}
&\operatorname{nrr}\{f = 0 | G_1 > 0, G_2 < 0\} + \operatorname{nrr}\{f = 0 | G_1 < 0, G_2 > 0\} \\
&\quad = \operatorname{nrr}\{f = 0 | G_1 G_2 < 0\}.
\end{aligned}$$

Adding the first and the second equalities and taking into account the third one, we obtain the required formula. Induction on k completes the proof. ■

Equation (2.4) reduces Problem 2(r) to Problem 1(r). Thus, Theorem 2.2 solves Problems 1 and 2 for the real roots of a polynomial $f(z)$ $[G_j(x) \equiv g_j(x, 0)$ for $j = 1, \dots, k]$.

REMARK 2.3. The results of this section could be obviously extended to the case where $G(x)$ and $G_1(x), \dots, G_k(x)$ are real rational functions of x. If, for example, $G(x) \equiv A(x)/B(x)$, where $A(x)$ and $B(x)$ are real polynomials such that $\mathscr{R}(A, B) \neq 0$, $\mathscr{R}(f, A) \neq 0$, and $\mathscr{R}(f, B) \neq 0$, then

$$\operatorname{nrr}\{f(x) = 0 \mid G(x) > 0\} = \operatorname{nrr}\{f(x) = 0 \mid A(x)B(x) > 0\}.$$

The formula (2.4) is also valid if the system of inequalities $G_1 > 0, \dots, G_k > 0$ is incompatible.

3. SEPARATION OF NONREAL ROOTS

Consider a real polynomial $f(z)$ of degree n without multiple roots. Let us represent it as

$$f(z) = f(x + iy) = F_1(x, y) + iF_2(x, y),$$

where

$$F_1(x, y) = \operatorname{Re} f(z) = f(x) - \frac{1}{2!}f''(x)y^2 + \frac{1}{4!}f^{(4)}(x)y^4 - \cdots$$

and

$$F_2(x, y) = \operatorname{Im} f(z) = y\left[f'(x) - \frac{1}{3!}f^{(3)}(x)y^2 + \frac{1}{5!}f^{(5)}(x)y^4 - \cdots\right].$$

Consider the system

$$F_1(x, y) = 0, \qquad F_2(x, y) = 0. \tag{3.1}$$

The following result is apparently due to Lagrange [12]:

THEOREM 3.1. *The set of solutions of the system* (3.1) *coincides with the set*

$$\{(\alpha_{jk}, \beta_{jk}) \mid j, k = 1, \dots, n\}, \tag{3.2}$$

where

$$\alpha_{jk} = \tfrac{1}{2}(\lambda_j + \lambda_k), \qquad \beta_{jk} = \frac{1}{2i}(\lambda_j - \lambda_k),$$

and $\lambda_1, \dots, \lambda_n$ *are the roots of* $f(z)$.

Proof. Let us prove initially that (3.2) contains every solution (α, β) of (3.1). We have $f(\alpha + i\beta) = F_1(\alpha, \beta) + iF_2(\alpha, \beta) = 0$; thus, $\alpha + i\beta$ coincides with a root of $f(x)$, say λ_j. If $\beta = 0$, then taking $k = j$ in (3.2), we get the claimed representation. Let $\beta \neq 0$. Since $f(z)$ is a real polynomial, there exists an index k such that $\lambda_k = \overline{\lambda}_j = \alpha - i\beta$, wherefrom we obtain (3.2).

Let us show now that every pair from (3.2) is a solution of (3.1). For any $j, k, j \neq k$, we have

$$0 = f(\lambda_j) = f\left(\frac{\lambda_j + \lambda_k}{2} + i\frac{\lambda_j - \lambda_k}{2i}\right) = F_1(\alpha_{jk}, \beta_{jk}) + iF_2(\alpha_{jk}, \beta_{jk}),$$

$$0 = f(\lambda_k) = f\left(\frac{\lambda_j + \lambda_k}{2} + i\frac{\lambda_k - \lambda_j}{2i}\right) = F_1(\alpha_{jk}, -\beta_{jk}) + iF_2(\alpha_{jk}, -\beta_{jk}).$$

Since $F_1(x, y)[F_2(x, y)]$ is an even [odd] function in y, comparing these equations gives the claimed result.

To prove the same in the case $j = k$ we only note that $F_1(x, 0) \equiv f(x)$ and $F_2(x, 0) \equiv 0$. ∎

This theorem establishes the one-to-one correspondence between the set of the roots of $f(z)$ and the set of real solutions of (3.1):

$$\lambda_k \leftrightarrow (\lambda_k, 0) \qquad \text{for real } \lambda_k,$$

$$\lambda_k \leftrightarrow (\operatorname{Re}\lambda_k, \operatorname{Im}\lambda_k) \qquad \text{for nonreal } \lambda_k.$$

Thus, Problem 1 is reduced to finding the number of real solutions of (3.1) which satisfy the inequality $g(x, y) > 0$. Since in the previous section we have solved that problem for real roots of $f(z)$, we may divide $F_2(x, y)$ by y and investigate the following system:

$$F_1(x, y) = 0, \qquad F_2^{(1)}(x, y) = 0, \tag{3.3}$$

where F_1 and $F_2^{(1)}$ are even polynomials in y.

3.1

Let $g(x, y)$ be an even polynomial in y [the domain in the complex plane defined by the inequality $g(x, y) > 0$ is symmetric with respect to the real axis]. Denote $Y = y^2$, $B_{jk} = \beta_{jk}^2$, $\Phi_1(x, Y) \equiv F_1(x, y)$, $\Phi_2(x, Y) \equiv F_2^{(1)}(x, y)$, and $G(x, Y) \equiv g(x, y)$. Consider the system

$$\Phi_1(x, Y) = 0, \qquad \Phi_2(x, Y) = 0. \tag{3.4}$$

We shall investigate its solutions with the help of results from Section 1.

Let us eliminate Y from (3.4). Consider

$$\mathscr{X}(x) = \mathscr{R}_Y(\Phi_1, \phi_2). \tag{3.5}$$

It could be easily shown that

$$\mathscr{R}_y\left(F_1, F_2^{(1)}\right) \equiv \mathscr{R}_Y^2(\Phi_1, \Phi_2) \equiv \mathscr{X}^2(x);$$

thus, $\deg \mathscr{X} = n(n-1)/2$.

THEOREM 3.2. *System* (3.4) *possesses* $n(n-1)/2$ *solutions* $(\alpha_{jk}, \beta_{jk})$, $1 \leqslant j < k \leqslant n$. *Here* α_{jk} *is a root of* $\mathscr{X}(x)$. *The set of real solutions of* (3.4) *consists of two subsets*:

$$\left\{\left(\operatorname{Re} \lambda_k, (\operatorname{Im} \lambda_k)^2\right) \middle| \textit{nonreal roots } \lambda_k \textit{ of } f(z), \operatorname{Im} \lambda_k > 0\right\}, \tag{3.6}$$

and provided that $f(z)$ *has at least two real roots, then*

$$\left\{\left(\frac{\lambda_j + \lambda_k}{2}, -\left(\frac{\lambda_j - \lambda_k}{2}\right)^2\right) \middle| \textit{pairs } (\lambda_j, \lambda_k), j < k, \textit{ of real roots of } f(z)\right\}. \tag{3.7}$$

Proof. The first and the second assertions of the theorem follow from Theorem 3.1 and the results of Section 1. The third one can be proved by using the representation (3.2) for α_{jk} and $B_{jk} = \beta_{jk}^2$. ■

Thus, the above theorem reduce the investigation of real solutions of the system (3.3) to that of the real solutions of the system (3.4). The latter, however, possesses the undesirable solutions (3.7).

ASSUMPTION 1. We shall suppose that $\mathscr{X}(x)$ does not have multiple roots, i.e., $\mathscr{D}(\mathscr{X}(x)) \neq 0$.

From this condition it follows, in particular, that the nonreal roots of $f(z)$ have distinct imaginary parts.

If this condition is fulfilled, then we may use the result of Theorem 1.5 and express the second component of solution of (3.4) as a real rational function of the first one:

$$B_j = \mathscr{r}(\alpha_j), \qquad j = 1, \dots, N = \frac{n(n-1)}{2} \tag{3.8}$$

[for the sake of simplicity we shall henceforth use a single index for the solutions of (3.4)].

For the set (3.6) we have $\mathscr{r}(x) > 0$, while for the set (3.7) we have $\mathscr{r}(x) < 0$.

Now we can solve Problem 1 for the complex roots of $f(z)$:

THEOREM 3.3. *The number of nonreal roots of $f(z)$ that satisfy the condition $g(x, y) > 0$ equals*

$$2\,\mathrm{nrr}\{\mathscr{X}(x) = 0 \mid \mathscr{r}(x) > 0,\, G(x, \mathscr{r}(x)) > 0\}. \tag{3.9}$$

The number (3.9) can be found by using Theorem 2.3 and Remark 2.3. Thus, Theorem 2.2 and 3.3 solve Problem 1.

Applying the formula (2.4) ($k = 2$) to (3.9) and taking into account the equality

$$2\,\mathrm{nrr}\{\mathscr{X}(x) = 0 \mid \mathscr{r}(x) > 0\} \overset{(3.6)}{=} n - \mathrm{nrr}\{f(z) = 0\},$$

we obtain the following

COROLLARY. *The number* (3.9) *equals*

$$\tfrac{1}{2}(n - \mathrm{nrr}\{f(z) = 0\}) + \mathrm{nrr}\{\mathscr{X}(x) = 0 | G(x, \mathscr{y}(x)) > 0\}$$
$$- \mathrm{nrr}\{\mathscr{X}(x) = 0 | \mathscr{y}(x)\, G(x, \mathscr{y}(x)) < 0\}. \quad (3.10)$$

EXAMPLE. For the polynomial

$$f(z) = z^4 - z^3 - 2z^2 + 6z - 4,$$

find the number of its roots that satisfy the condition

$$g(x, y) \equiv 2y^2 - x > 0.$$

Solution. By using Theorem 2.2 let us find first

$$\mathrm{nrr}\{f(z) = 0 | g(x, 0) > 0\}.$$

According to Remark 2.1, we need to calculate the leading principal minors of the matrices (2.2) with the elements

$$\mathscr{s}_0 = 4, \quad \mathscr{s}_1 = 1, \quad \mathscr{s}_2 = 5, \quad \mathscr{s}_3 = -11, \quad \mathscr{s}_4 = 9,$$
$$\mathscr{s}_5 = -39, \quad \mathscr{s}_6 = 65, \quad (\mathscr{s}_7 = -111);$$
$$\mathscr{h}_0 = 1, \quad \mathscr{h}_1 = 5, \quad \mathscr{h}_1 = -11, \quad \mathscr{h}_3 = 9,$$
$$\mathscr{h}_4 = -39, \quad \mathscr{h}_5 = 65, \quad \mathscr{h}_6 = -111.$$

The leading principal minors of those matrices are

$$\mathscr{S}_1 = 4, \qquad \mathscr{S}_1 = 19, \qquad \mathscr{S}_3 = -548, \qquad \mathscr{S}_4 = -3600;$$
$$\mathscr{H}_1 = 1, \qquad \mathscr{H}_2 = -36, \qquad \mathscr{H}_3 = 1664, \qquad \mathscr{H}_4 = 14400.$$

By Theorem 2.2 and (2.3), $\mathrm{nrr}\{f(z) = 0\} = \sigma(S) = 2$, and according to Theorem 2.3 $\mathrm{nrr}\{f(z) = 0 \,|\, g(x, 0) > 0\} = 1$. Thus, one of the two real roots of $f(z)$ satisfies the inequality $g(x, 0) > 0$.

To find whether two nonreal roots satisfy the given inequality, let us consider the system (3.4):

$$\begin{aligned} \Phi_1(x,Y) &\equiv Y^2 - \frac{1}{2!}f''(x)Y + f(x) = 0, \\ \Phi_2(x,Y) &\equiv -\frac{1}{3!}f^{(3)}(x)Y + f'(x) = 0. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathscr{X}(x) &\equiv \mathscr{R}_Y(\Phi_1, \Phi_2) \\ &= -4(16x^6 - 24x^5 - 4x^4 + 14x^3 + 10x^2 - 7x - 5). \end{aligned}$$

For simplicity, subsequently we shall consider $\mathscr{X}(x)$ without the factor -4. Now $\mathscr{D}(\mathscr{X}) \approx 2.01852 \times 10^{13} \neq 0$; therefore $\mathscr{X}(x)$ has no multiple roots, and we may apply Theorem 1.5. According to that theorem, solutions of (3.11) are connected by the formula (3.8) where

$$\mathscr{r}(x) = \frac{6f'(x)}{f^{(3)}(x)}. \tag{3.12}$$

To find the number (3.9) we shall use first its representation (3.10), and secondly Remarks 2.3 and 2.1:

$$\begin{aligned} \operatorname{nrr}\{\mathscr{X}(x) = 0 | G(x, \mathscr{r}(x)) > 0\} &= \operatorname{nrr}\{\mathscr{X} = 0 | 2\mathscr{r}(x) - x > 0\} \\ &= \operatorname{nrr}\{\mathscr{X} = 0 | (12f'(x) - xf^{(3)}(x))f^{(3)}(x) > 0\} \\ &= \operatorname{nrr}\{\mathscr{X} = 0 | G_1(x) \equiv 32x^4 - 48x^3 - 18x^2 + 55x - 12 > 0\}. \end{aligned}$$

The leading principal minors of the form (2.2) constructed for $\mathscr{X}(x)$ and $G_1(x)$ are

$$\mathscr{H}_1 = -139, \qquad \mathscr{H}_2 = -12504.5, \qquad \mathscr{H}_3 \approx -862032,$$

$$\mathscr{H}_4 \approx 3.0092 \times 10^7, \qquad \mathscr{H}_5 \approx -2.85266 \times 10^8, \qquad \mathscr{H}_6 \approx -2.08108 \times 10^9.$$

By Theorem 3.2, $\mathscr{X}(x)$ has two real roots. Thus, by Theorem 2.2,

$$\operatorname{nrr}\{\mathscr{X} = 0 | G_1(x) > 0\} = 1,$$

i.e., on one of these roots the polynomial $G_1(x)$ is positive. Furthermore,

$$\begin{aligned}
\operatorname{nrr}\{\mathscr{X}(x) &= 0 \,|\, \mathscr{I}(x) G(x, \mathscr{I}(x)) < 0\} \\
&= \operatorname{nrr}\left\{\mathscr{X} = 0 \,\middle|\, f'\left[f^{(3)}\right]^2 (12 f' - x f^{(3)}) < 0\right\} \\
&= \operatorname{nrr}\{\mathscr{X} = 0 \,|\, f'(12 f' - x f^{(3)}) < 0\} \\
&= \operatorname{nrr}\{\mathscr{X} = 0 \,|\, 2\mathscr{X}(x) - (16x^5 + 22x^4 - 129x^3 + 88x^2 \\
&\qquad\qquad + 76x - 82) < 0\} \\
&= \operatorname{nrr}\{\mathscr{X} = 0 \,|\, G_2(x) \equiv 16x^5 + 22x^4 - 129x^3 + 88x^2 \\
&\qquad\qquad + 76x - 82 > 0\}.
\end{aligned}$$

In exactly the same manner as we found $\operatorname{nrr}\{\mathscr{X} = 0 \mid G_1(x) > 0\}$ we can find that

$$\operatorname{nrr}\{\mathscr{X} = 0 \,|\, G_2(x) > 0\} = 0.$$

Thus, by (3.10), the number of nonreal roots of $f(z)$ satisfying the given inequality equals 2.

Finally, three roots of $f(z)$ satisfy the condition $2y^2 - x > 0$.

Check. The roots of $f(z)$ are -2 $(g > 0)$, 1 $(g < 0)$, $1 \pm i$ $(g > 0)$.

REMARK. Hermite proposed another method for solving Problem 1 for the domain $g(x, y) > 0$ with the *rationally parametrizable boundary curve* $g(x, y) = 0$ [7]. Briefly, the method consists of calculating the Cauchy index of the curve $g(x, y) = 0$ and reduces Problem 1 to the similar one for $g(x, y) \equiv x$.

To conclude this subsection we mention an interesting application of the resultant $\mathscr{X}(x)$ defined by (3.5):

THEOREM 3.4 (Routh) [8, p. 293]. *Let the coefficients of z^n in $f(z)$ and of x^N in $\mathscr{X}(x)$ be positive. For the roots of $f(z = x + iy)$ to satisfy the condition $x < 0$ it is necessary and sufficient that the coefficients of $f(z)$ and $\mathscr{X}(x)$ should all be positive.*

Let us consider now the general case.

3.2

Let $g(x, y)$ be an arbitrary polynomial. Consider the system (3.3). Let us eliminate x from it: consider

$$\mathscr{Y}(y) = \mathscr{R}_x(F_1, F_2^{(1)}),$$

$\deg \mathscr{Y} = n(n-1) = 2N$. Instead of Assumption 1 consider

ASSUMPTION 2. Let $\mathscr{D}(\mathscr{Y}) \neq 0$.

Under this assumption all the real solutions of the system (3.3) correspond to the nonreal roots of $f(z)$. As in the previous case, we may use Theorem 1.4. According to that theorem, the first component of the solution of (3.3) is a real rational function of the second one:

$$\alpha_j = r_1(\beta_j), \qquad j = 1, \dots, 2N.$$

THEOREM 3.5. *The number of nonreal roots of a real polynomial* $f(z)$ *which satisfy an algebraic condition* $g(x, y) > 0$ *equals*

$$\operatorname{nrr}\{\mathscr{Y}(y) = 0 \mid g(r_1(y), y) > 0\}. \tag{3.13}$$

Though the formula (3.13) seems to be simpler than (3.9), the degree of $\mathscr{Y}(y)$, is twice as high as that of $\mathscr{X}(x)$.

Extensions of the Theorems 3.3 and 3.5 to Problem 2 can be easily obtained by applying the formula (2.4). Thus, Problem 2 can be solved in finite number of elementary algebraic operations on the coefficients of the polynomials involved.

3. OTHER APPROACHES TO PROBLEM 1

The general algorithm for Problem 1 proposed above can be simplified for some particular families of algebraic domains in the complex plane. Hermite himself proposed other procedures for the following cases [7].

(a) $g(x, y) \equiv \operatorname{Re} \Phi(z)$ where $\Phi(z)$ is a complex rational function. All the roots $\lambda_1, \dots, \lambda_n$ of $f(z) = a_0 z^n + \cdots + a_n$ satisfy $g(x, y) > 0$ if and only if $F(z) = a_0[z + \Phi(\lambda_1)] \cdots [z + \Phi(\lambda_n)]$ has its roots in the left half plane. Using the Theorem 1.1 and the Routh-Hurwitz criterion, we obtain a solution for Problem 1.

(b) The equation $g(x, y) = 0$ provides a *rationally parametrizable curve*, i.e., it is equivalent to $x = \varphi(t)$, $y = \psi(t)$, $t \in (-\infty, +\infty)$, where φ and ψ are real rational functions. For this case Hermite made use of the *Cauchy index* [12, pp. 123, 290].

See also [5] for the development of these and some other approaches.

4. RELATION TO THE PROBLEM OF SEPARATION OF REAL SOLUTIONS OF AN ALGEBRAIC SYSTEM

For the complex polynomial $f(z)$ Problem 1 can be reduced to the problem of finding the number of real solutions of the system of real polynomial equations (3.1) satisfying the inequality $g(x, y) > 0$ (it could be solved by the procedure from Section 3.2 as well).

Under some additional conditions, the latter problem can be solved by investigation of the following quadratic form [analogue of (2.2)]:

$$\sum_{j=1}^{M} g(\alpha_j, \beta_j)\left[x_0 + \beta_j x_1 + \beta_j^2 x_2 + \cdots + \beta_j^{M-1} x_{M-1}\right]^2,$$

where $M = n^2$ and (α_j, β_j) $(j = 1, \dots, M)$ are solutions of (3.1). The method, which is also due to Hermite, was discussed in [14]. Markov's formula (2.4) is also valid for the polynomials in two variables.

5. CONCLUSION

The problem under consideration for a fixed polynomial is not hard from a computational point of view—it suffices to calculate all the roots of $f(z)$ and to check if they belong to the prescribed domain. However, the algorithm proposed may be useful for the case of a parameter dependent polynomial [1]:

$$f(z, \Omega) = a_0(\Omega) z^n + \cdots + a_n(\Omega).$$

Here a vector of uncertain parameters, $\Omega = [\omega_1, \dots, \omega_p]$, belongs to some algebraic set Q (i.e., the set defined by a system of algebraic inequalities), and $a_0(\Omega), \dots, a_n(\Omega)$ are real polynomials in Ω. By applying the algorithm for $f(z, \Omega)$, we will obtain two systems of algebraic inequalities in Ω, one of which describes Q, and the other the necessary and sufficient conditions for Problem 1. We hope to discuss the problem of comparison of these sets in subsequent papers.

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