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Maximum order-index of matrices over commutative inclines: An answer to an open problem ☆

Song-Chol Han [a,b], Hong-Xing Li [b,*]

[a] *Department of Mathematics and Mechanics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea*

[b] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China*

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Abstract

This paper proves that the maximum order-index of $n \times n$ matrices over an arbitrary commutative incline equals $(n-1)^2+1$. This is an answer to an open problem "Compute the maximum order-index of a member of $M_n(L)$", proposed by Cao, Kim and Roush in a monograph *Incline Algebra and Applications*, *1984*, where $M_n(L)$ is the set of all $n \times n$ matrices over an incline L.

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1. Introduction

Inclines are additively idempotent semirings in which products are less than or equal to either factor. Boolean algebra, fuzzy algebra and distributive lattice are examples of inclines. Inclines

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* Corresponding author. Fax: +86 10 58807482.
E-mail address: lhxqx@bnu.edu.cn (H.-X. Li).

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and incline matrices have good vistas of applications in diverse areas such as automata theory, graph theory, medical diagnosis, informational systems, complex systems modelling, decision-making theory, dynamical programming, control theory, nervous system, clustering and so on. Incline algebra and incline matrix theory have been extensively studied by many authors [1–17] (inclines are also called simple semirings, refer to [7] for example).

Cao et al. [3] introduced the notion of the order-index of an element in a partially ordered semigroup, and proposed an open problem "Compute the maximum order-index of a member of $M_n(L)$", where $M_n(L)$ is the set of all $n \times n$ matrices over an incline L (see the first problem of paragraph 5.5 in [3]).

In this paper, we prove that the maximum order-index of $n \times n$ matrices over an arbitrary commutative incline equals $(n-1)^2 + 1$. This is an answer to the above open problem.

2. Preliminaries

Definition 2.1 [3]. A nonempty set L with two binary operations + and $\cdot$ is called an incline if it satisfies the following conditions:

(1) $(L, +)$ is a semilattice,
(2) $(L, \cdot)$ is a semigroup,
(3) $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$ for all $x, y, z \in L$,
(4) $x + xy = x + yx = x$ for all $x, y \in L$.

In an incline L, define a relation $\leqslant$ by $x \leqslant y \Leftrightarrow x + y = y$. It is easy to see that $\leqslant$ is a partial order on L and that for any $x, y \in L$, the element $x + y$ is the least upper bound of $\{x, y\} \subseteq L$. It follows that $xy \leqslant x$ and $yx \leqslant x$ for all $x, y \in L$ and that for any $x, y, z \in L$, $y \leqslant z$ implies $xy \leqslant xz$ and $yx \leqslant zx$. If an incline L has an additive identity 0, then 0 is called the zero of L. Then $x + 0 = 0 + x = x$, $0 \leqslant x$ and $0x = x0 = 0$ for all $x \in L$. If an incline L has a multiplicative identity 1, then 1 is called the identity of L. Then $x1 = 1x = x$, $x \leqslant 1$ and $1 + x = x + 1 = 1$ for all $x \in L$. By an incline with zero and identity we mean an incline L that has both zero and identity satisfying $0 \neq 1$. An incline L is said to be commutative if $xy = yx$ for all $x, y \in L$.

The Boolean algebra $(\{0, 1\}, \vee, \wedge)$ is an incline. In general, every distributive lattice is an incline. The fuzzy algebra $([0, 1], \vee, T)$ is an incline, where T is a t-norm. The tropical algebra $(\mathbb{R}_0^+ \cup \{\infty\}, \wedge, +)$ is an incline, where $\mathbb{R}_0^+$ is the set of all nonnegative real numbers.

From now on, L always denotes any given commutative incline with zero and identity, n denotes any given positive integer greater than or equal to 2, $\underline{n}$ stands for the set $\{1, 2, \ldots, n\}$, and $[n]$ denotes the least common multiple of integers $1, 2, \ldots, n$. For a nonnegative integer l, $\underline{l}^0$ denotes the set of integers 0 through l.

We denote by $M_n(L)$ the set of all $n \times n$ matrices over L. Given $A = (a_{ij}) \in M_n(L)$ and $B = (b_{ij}) \in M_n(L)$, we define the product $A \cdot B \in M_n(L)$ by $A \cdot B := \left(\sum_{v \in \underline{n}} a_{iv} b_{vj}\right)$. And we denote $A \leqslant B$ when $a_{ij} \leqslant b_{ij}$ for all $i, j \in \underline{n}$.

Then $(M_n(L), \leqslant, \cdot)$ forms a partially ordered semigroup, i.e. for all $A, B, C, D \in M_n(L)$,

(1) $(AB)C = A(BC)$,
(2) $A \leqslant B$ and $C \leqslant D \Rightarrow AC \leqslant BD$.

Definition 2.2 [3]. Let S be a partially ordered semigroup and $a \in S$. If there are some positive integers k and d satisfying $a^{k+d} \leqslant a^k$, then the least such positive integers k and d are called the order-index of a and the order-period of a, respectively.

In this paper, the order-index of a matrix $A \in M_n(L)$ is denoted by $oi(A)$.

3. Reduction of walks

Let $V : v_0, v_1, \ldots, v_l$ be a sequence of positive integers such that $v_i \in \underline{n}$ for all $i \in \underline{l}^0$. We call V a *walk on* $\underline{n}$, $l(V) := l$ the *length* of V, and v_i $(i \in \underline{l}^0)$ the *terms* of V. Below, the walk on $\underline{n}$ shall be called the walk briefly. When $l \geqslant 2$, the walk $v_1, \ldots, v_{l-1}$ is called the *interior* of V. We call V a *closed* walk if $l \geqslant 1$ and $v_0 = v_l$. A closed walk V is called a *cycle* when $v_i = v_j$ $(i < j)$ implies $i = 0$ and $j = l$.

If V includes two closed walks $T_1 : v_i, \ldots, v_j$ and $T_2 : v_{i'}, \ldots, v_{j'}$, and if $j \leqslant i'$ or $j' \leqslant i$, then we say that T_1 and T_2 are *independent* in V. If V includes closed walks $T_1, T_2, \ldots, T_k$ $(k \geqslant 3)$, and if T_i and T_j are independent in V for any $i \neq j$, then we say that $T_1, T_2, \ldots, T_k$ are *independent* in V.

For any $p \in \underline{n}$, we put $m(V; p) := |\{i \in \underline{l}^0 | v_i = p\}|$. When $l \geqslant 1$, for any $p, q \in \underline{n}$, we put $m(V; p, q) := |\{i \in \underline{(l-1)}^0 \mid v_i = p, v_{i+1} = q\}|$.

Let $U : u_0, u_1, \ldots, u_h$ be another walk. U is called a *reduction* of V if $u_0 = v_0$, $u_h = v_l$ and $m(U; p, q) \leqslant m(V; p, q)$ for all $p, q \in \underline{n}$. U is said to be *equivalent* to V if U is a reduction of V and V is a reduction of U simultaneously. All the equivalent walks shall be considered as the same one. If $v_l = u_0$, then we denote by $V + U$ the walk $v_0, \ldots, v_{l-1}, u_0, \ldots, u_h$.

Let $T : t_0, t_1, \ldots, t_r$ be a closed walk. If $v_i = t_j$ for some i and j, then we denote by $V + T$ the walk

$$v_0, \ldots, v_i, t_{j+1 \bmod r}, t_{j+2 \bmod r}, \ldots, t_{j+r \bmod r}, v_{i+1}, \ldots, v_l.$$

If V includes a closed walk $T : v_i, \ldots, v_j$, then $V - T$ denotes the walk $v_0, \ldots, v_i, v_{j+1}, \ldots, v_l$.

Lemma 3.1. *For two walks V, U and a closed walk T, the following hold when the corresponding operations are defined*:

(1) $l(V + U) = l(V) + l(U)$,
(2) $l(V \pm T) = l(V) \pm l(T)$,
(3) $m(V + U; p, q) = m(V; p, q) + m(U; p, q)$ *for all* $p, q \in \underline{n}$,
(4) $m(V \pm T; p, q) = m(V; p, q) \pm m(T; p, q)$ *for all* $p, q \in \underline{n}$.

Proof. It follows immediately from the definition of the operations. □

For a walk V and a closed walk T, the results $V + T$ are not necessarily unique, but they are equivalent to each other. The similar statement holds for $V - T$ as well.

Lemma 3.2. *Let $S \subseteq \underline{n}$ with $|S| = s \geqslant 1$. If a walk V contains an element of S and $l(V) \geqslant (n-1)^2 + 1 + s$, then there exists a reduction V' of V such that $l(V') < l(V)$, $l(V') \equiv l(V) \pmod s$, and V' contains an element of S.*

Proof. Let $V : v_0, v_1, \ldots, v_{m+s}$ and $m \geqslant (n-1)^2 + 1$. Choose a number $p \in \underline{n}$ satisfying $m(V; p) = \max\{m(V; p') \mid p' \in \underline{n}\}$, and put $h := m(V; p)$. Obviously, $h \geqslant 2$. Denote by U the walk consisting of v_0 through the first p, by T_i the closed walk consisting of the ith p through the $(i+1)$th p $(1 \leqslant i \leqslant h-1)$, and by W the walk consisting of the last p through v_{m+s}, i.e.

$$V : v_0 \underbrace{, \ldots ,}_{U} p \underbrace{, \ldots ,}_{T_1} p, \ldots \ldots, p \underbrace{, \ldots ,}_{T_{h-1}} p \underbrace{, \ldots ,}_{W} v_{m+s}.$$

Then $V = U + T_1 + \cdots + T_{h-1} + W$. We divide the proof into three cases.

Case 1: $s \leqslant n-2$. Then $h \geqslant \frac{m+s+1}{n} \geqslant \frac{(n-1)^2+s+2}{n} \geqslant \frac{(s+1)(n-1)+s+2}{n} = s + 1 + \frac{1}{n}$, so $h \geqslant s+2$. Since V contains an element of S, there exists a closed walk T_i such that $U + T_i + W$ contains an element of S. Denote by $T'_1, \ldots, T'_{h-2}$ the closed walks except T_i. Consider $h-2$ numbers $l(T'_1), l(T'_1 + T'_2), \ldots, l(T'_1 + T'_2 + \cdots + T'_{h-2})$. Since $h - 2 \geqslant s$, there is a j such that $l(T'_1 + \cdots + T'_j)$ is a multiple of s, or there exist a and b $(a < b)$ such that $l(T'_1 + \cdots + T'_a) \equiv l(T'_1 + \cdots + T'_b) \pmod s$. In the first case, we put $V' := V - T'_1 - \cdots - T'_j$; in the second case, we put $V' := V - T'_{a+1} - \cdots - T'_b$. In any case, V' is a reduction of V to be found.

Case 2: $s = n-1$. Then $h \geqslant \frac{(n-1)^2+n+1}{n} = n - 1 + \frac{2}{n}$, so $h \geqslant n$. If $h \geqslant n+1 = s+2$ or $U + W$ contains an element of S, then we obtain the conclusion similarly to Case 1. Suppose that $h = n$ and $U + W$ does not contain any elements of S. Then $S = \underline{n} \setminus \{p\}$, so $V = T_1 + T_2 + \cdots + T_{n-1}$. Since $l(T_1) + l(T_2) + \cdots + l(T_{n-1}) = l(V) \geqslant (n-1)^2 + n$, there exists a closed walk T_i with $l(T_i) \geqslant \frac{(n-1)^2+n}{n-1} = n + \frac{1}{n-1}$, i.e. $l(T_i) \geqslant n+1$. Since the interior of T_i has at least n terms and does not contain p, there exists a $q \in S$ such that the interior of T_i includes a closed walk $T'_i : q, \ldots, q$. Put $T'_j := T_j (j \neq i)$. Consider $n-1$ numbers $l(T'_1), l(T'_1) + l(T'_2), \ldots, l(T'_1) + l(T'_2) + \cdots + l(T'_{n-1})$. Since $n - 1 = s$, there is a j such that $l(T'_1) + \cdots + l(T'_j)$ is a multiple of s, or there exist a and b $(a < b)$ such that $l(T'_1) + \cdots + l(T'_a) \equiv l(T'_1) + \cdots + l(T'_b) \pmod s$. In the first case, we put $V' := V - T'_1 - \cdots - T'_j$; in the second case, we put $V' := V - T'_{a+1} - \cdots - T'_b$. In any case, V' contains $q \in S$, so it is a reduction of V to be found.

Case 3: $s = n$. Then $S = \underline{n}$. Assume that the conclusion does not hold. Then neither the length of any closed walk in V nor the sum of lengths of any independent closed walks in V is a multiple of n. Besides, it is easy to see that V does not include n independent closed walks. Since $h \geqslant \frac{(n-1)^2+n+2}{n} = n - 1 + \frac{3}{n}$, we have $h \geqslant n$, so $h = n$.

We now show that if $T'_1, T'_2, \ldots, T'_{n-2}, T$ and $T'_1, T'_2, \ldots, T'_{n-2}, T'$ are two groups of $n-1$ independent closed walks in V, then $l(T) \equiv l(T') \pmod n$. In fact, when we divide $n-1$ numbers $l(T'_1), l(T'_1) + l(T'_2), \ldots, l(T'_1) + l(T'_2) + \cdots + l(T'_{n-2}) + l(T)$ by n, the set of their remainders equals $\underline{n-1}$. The same statement is also true for $n-1$ numbers $l(T'_1), l(T'_1) + l(T'_2), \ldots, l(T'_1) + l(T'_2) + \cdots + l(T'_{n-2}) + l(T')$. Hence, $l(T'_1) + l(T'_2) + \cdots + l(T'_{n-2}) + l(T) \equiv l(T'_1) + l(T'_2) + \cdots + l(T'_{n-2}) + l(T') \pmod n$, so $l(T) \equiv l(T') \pmod n$.

Put $M := T_1 + T_2 + \cdots + T_{n-1}$. Then M is a closed walk and $V = U + M + W$.

We first show that p is the unique common term of U, M and W. In fact, assume that U, M and W contain another common term r except p. Noticing that the rearrangement of $T_1, T_2, \ldots, T_{n-1}$ in M yields a closed walk equivalent to M, we may as well suppose that T_1 contains r. Then

$$V : v_0 \underbrace{, \ldots, r, \ldots ,}_{U} p \underbrace{, \ldots, r, \ldots ,}_{T_1} p \underbrace{, \ldots ,}_{T_2} p, \ldots \ldots, p \underbrace{, \ldots ,}_{T_{n-1}} p \underbrace{, \ldots, r, \ldots ,}_{W} v_{m+n}.$$

Put $T'_1 := (r, \ldots, p \text{ in } U) + (p, \ldots, r \text{ in } T_1)$, i.e.

$$V : v_0, \ldots, r \underbrace{, \ldots, p, \ldots,}_{T_1'} r, \ldots, p \underbrace{, \ldots,}_{T_2} p, \ldots\ldots, p \underbrace{, \ldots,}_{T_{n-1}} p \underbrace{, \ldots, r, \ldots,}_{W} v_{m+n}.$$

Then $T_1', T_2, \ldots, T_{n-1}$ are independent closed walks in V. Since $T_1, T_2, \ldots, T_{n-1}$ are also independent closed walks in V, we have $l(T_1') \equiv l(T_1) \pmod n$, so $l(r, \ldots, p \text{ in } U) \equiv l(r, \ldots, p \text{ in } T_1) \pmod n$. By interchanging T_{n-1} and T_1, we obtain $l(p, \ldots, r \text{ in } T_1) \equiv l(p, \ldots, r \text{ in } W) \pmod n$. We now put $M' := (r, \ldots, p \text{ in } T_1) + T_2 + \cdots + T_{n-1} + (p, \ldots, r \text{ in } W)$, i.e.

$$V : v_0, \ldots, r \underbrace{, \ldots, p, \ldots,}_{T_1'} r \overbrace{, \ldots, p \underbrace{, \ldots,}_{T_2} p, \ldots\ldots, p \underbrace{, \ldots,}_{T_{n-1}} p, \ldots,}^{M'} r, \ldots, v_{m+n}.$$

Then $l(M') \equiv l(M) \pmod n$. Consider $n-1$ numbers

$$l(T_1') \bmod n, l(T_1') + l(T_2) \bmod n, \ldots, l(T_1') + l(T_2 + \cdots + T_{n-1}) \bmod n.$$

Since $T_1', T_2, \ldots, T_{n-1}$ are independent closed walks in V, the set of the above $n-1$ numbers equals $\underline{n-1}$. Meanwhile, $l(T_1') + l(M) \bmod n = l(T_1') + l(M') \bmod n = l(T_1' + M') \bmod n \neq 0$ since $T_1' + M'$ is a closed walk in V. Hence, $l(T_1') + l(M) \bmod n \in \underline{n-1}$, so $l(T_1') + l(M) \bmod n$ coincides with a number among the above $n-1$ numbers. However, this is impossible because $T_1, T_2, \ldots, T_{n-1}$ are independent closed walks in V. Consequently, U, M and W have no any common terms except p.

We next show that all T_i $(1 \leqslant i \leqslant n-1)$ are cycles. Without the loss of generality, we assume that T_1 is not a cycle. Then T_1 includes a closed walk T with maximal length in the interior of T_1, and $l(T_1) \equiv l(T) \pmod n$, so $l(T_1 - T)$ is a nonzero multiple of n. Moreover, $T_1 - T$ is a cycle since T_1 cannot include two independent closed walks and T is the closed walk with maximal length in the interior of T_1. Hence, $l(T_1 - T) = n$, and all of $1, 2, \ldots, n$ are terms of $T_1 - T$. If T includes a closed walk T', then $l(T) \equiv l(T') \pmod n$, so $l(T - T')$ is a multiple of n. Since $T_1 - T$ contains all $1, 2, \ldots, n$ as terms, a walk $U + (T_1 - T + T') + T_2 + \cdots + T_{n-1} + W$ is defined and a reduction of V. Its length $l(V) - l(T - T')$ is congruent to $l(V)$ modulo n. Hence, $l(T - T') = 0$, so $T = T'$. Thus T is a cycle and $l(T_1) = n + l(T)$. If $V - T_1$ and T have the same term, then a walk $V - T_1 + T$ is defined and a reduction of V. Its length is $l(V) - l(T_1) + l(T) = l(V) - n$. This is a contradiction. Hence, $V - T_1$ and T have no common terms. If there is an i $(2 \leqslant i \leqslant n-1)$ such that T_i is a noncyclic closed walk, then T_i also contains all $1, 2, \ldots, n$ as terms, so T and T_i have a common term. This is a contradiction. Therefore, for every $i (2 \leqslant i \leqslant n-1)$, T_i is a cycle, so $l(T_i) \leqslant n - l(T)$. If $U + W$ includes a closed walk $t, \ldots, t$, then $t \neq p$, and both U and W contain t. However, T_1 contains t, so U, W and M have the same term t. This is a contradiction. Hence, $U + W$ consists of different terms, so $l(U + W) \leqslant n - 1 - l(T)$. Thus $l(V) = l(U + W) + l(M) = l(U + W) + l(T_1) + l(T_2 + \cdots + T_{n-1}) \leqslant (n - 1 - l(T)) + (n + l(T)) + (n - l(T))(n-2) \leqslant 2n - 1 + (n-1)(n-2) = (n-1)^2 + n$. This is a contradiction. Hence, T_1 is a cycle. Consequently, all T_i $(1 \leqslant i \leqslant n-1)$ are cycles.

Finally, since V has no n independent closed walks, every element of $\underline{n}$ is contained in $U + W$ at most two times. Let b be the number of elements of $\underline{n}$ which are duplicated in $U + W$. Then $l(U + W) \leqslant n + b - 1$. If $c \in \underline{n}$ is duplicated in $U + W$, then $c \neq p$, and both U and W contain c. Since U, M and W have no common terms except p, c is not contained in every cycle T_i $(1 \leqslant i \leqslant n-1)$. If $b \geqslant 1$, then $l(T_i) \leqslant n - b$ for $1 \leqslant i \leqslant n-1$. Thus $l(V) \leqslant n + b - 1 + (n - b)(n-1) \leqslant (n-1)^2 + n$. This is a contradiction. If $b = 0$, then $l(U + W) \leqslant n - 1$. And for every $i (1 \leqslant i \leqslant n-1), l(T_i) \leqslant n-1$ because $l(T_i) \neq n$. Hence, $l(V) \leqslant (n-1)^2 + n - 1$. This is a contradiction. In all, the proof is completed. □

Put $k := (n-1)^2 + 1$. Denote by t_c the total number of possible cycles (i.e. $t_c = \sum_{i=1}^{n} A_n^i$). Let d be any given multiple of $[n]$ satisfying $d \geqslant nkt_c$.

Lemma 3.3. *Every walk with length $k+d$ has a reduction with length k.*

Proof. Let $V : v_0, v_1, \ldots, v_{k+d}$ be a walk with length $k+d$, and put $f := \frac{d}{n}$. For every i $(1 \leqslant i \leqslant f)$, we denote by V_i the walk $v_{in-n}, v_{in-n+1}, \ldots, v_{in}$. Then

$$V : v_0 \underbrace{, v_1, \ldots ,}_{V_1} v_n \underbrace{, v_{n+1}, \ldots ,}_{V_2} v_{2n}, v_{2n+1}, \ldots\ldots, v_{d-n} \underbrace{, v_{d-n+1}, \ldots ,}_{V_f} v_d, v_{d+1}, \ldots, v_{d+k},$$

and every V_i necessarily includes a cycle T_i. Consider f cycles $T_1, T_2, \ldots, T_f$. Obviously, they are independent in V. If the multiplicity of every cycle T_i among them is less than k, then we have $f < kt_c$. This is a contradiction. Hence, there is a cycle T_{i_0} such that the multiplicity of T_{i_0} among them is at least k. Put $T := T_{i_0}$ and $s := l(T)$. Let S be the set of all terms of T. Then $S \subseteq \underline{n}$ and $|S| = s \geqslant 1$. Since d is a multiple of s, using Lemma 3.2 repeatedly, we can see that there is a reduction U of V such that $l(U) \leqslant k$, $l(U) \equiv k \pmod s$, and U contains an element of S. Now we put $s' := \frac{k-l(U)}{s}$ and $W := U + T + \cdots + T$, where the multiplicity of T is s' $(s' \leqslant k)$. Then $l(W) = k$. For any $p, q \in \underline{n}$, we have $m(W; p, q) = m(U; p, q) + s'm(T; p, q)$. Since T is a cycle, $m(T; p, q) \leqslant 1$. If $m(T; p, q) = 0$, then $m(W; p, q) = m(U; p, q) \leqslant m(V; p, q)$. If $m(T; p, q) = 1$, then $m(W; p, q) = m(U; p, q) + s' \leqslant l(U) + s' = \frac{k}{s} + l(U)\left(1 - \frac{1}{s}\right) \leqslant k = m(T; p, q)k \leqslant m(V; p, q)$. Thus W is a reduction of V with length k. This completes the proof. □

4. Order-index of incline matrices

Theorem 4.1. *If $A \in M_n(L)$, then $A^{k+d} \leqslant A^k$.*

Proof. Let $A = (a_{ij})$. We denote $A^k = \left(a_{ij}^{(k)}\right)$ and $A^{k+d} = (a_{ij}^{(k+d)})$. For every $i, j \in \underline{n}$, we have $a_{ij}^{(k+d)} = \sum_{v_1, v_2, \ldots, v_{k+d-1} \in \underline{n}} a_{iv_1} a_{v_1 v_2} \cdots a_{v_{k+d-1} j}$. Consider any summand $a_{iv_1} a_{v_1 v_2} \cdots a_{v_{k+d-1} j}$ of $a_{ij}^{(k+d)}$. By Lemma 3.3, the walk $i, v_1, v_2, \ldots, v_{k+d-1}, j$ with length $k+d$ has a reduction $i, u_1, u_2, \ldots, u_{k-1}, j$ with length k. Noticing that L is a commutative incline, we obtain $a_{iv_1} a_{v_1 v_2} \cdots a_{v_{k+d-1} j} \leqslant a_{iu_1} a_{u_1 u_2} \cdots a_{u_{k-1} j} \leqslant a_{ij}^{(k)}$. Since this inequality holds for every summand of $a_{ij}^{(k+d)}$, we have $a_{ij}^{(k+d)} \leqslant a_{ij}^{(k)}$. This completes the proof. □

There exists a matrix $A \in M_n(L)$ with $oi(A) = k$.

Example 4.1. Consider a matrix $A = (a_{ij}) \in M_n(L)$, where

$$a_{ii+1} = 1, \quad 1 \leqslant i \leqslant n-1,$$
$$a_{n1} = a_{n2} = 1,$$
$$a_{ij} = 0, \quad \text{otherwise}.$$

Then A has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

For the power sequence $A, A^2, \ldots, A^k$, the multiplicity of 1 among entries of A^x is strictly increasing with respect to x, and $A^k = J_n$, where J_n is the $n \times n$ entire incline matrix, i.e. its every entry is 1. Hence, $A^k = A^{k+1} = J_n$. This implies that $oi(A) = k$.

Theorem 4.2. $\max\{oi(A) \mid A \in M_n(L)\} = k$.

Proof. It follows from Theorem 4.1 and Example 4.1. □

5. Conclusions

This paper gave the maximum order-index of square matrices over a commutative incline.

The following problem is still open: "Compute the maximum order-index of $n \times n$ matrices over a noncommutative incline".

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