The cordiality of one-point union of \( n \) copies of a graph

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Abstract


In this paper we give an equivalent definition of a cordial graph. The definition implies a previous result of Cahit (1986); it also enables us to find infinite families of noncordial graphs, derive some bound on the number of edges in a cordial graph and establish a necessary and sufficient condition for a one-point union of two \( n \)-cliques.

Let \( G \) be a rooted graph. We denote by \( G^{\otimes n} \) the graph obtained from \( n \) copies of \( G \) by identifying their roots. A sufficient condition for \( G^{\otimes n} \) to be cordial is related to the solution of a system involving one equation and two inequalities with their coefficients depending on some binary labellings of \( G \). According to the solvability of the system, we are able to establish a number of necessary and sufficient conditions for the cordiality of \( G^{\otimes n} \) for certain classes of \( G \), such as cycles, complete graphs, wheels, fans and flags.

1. Introduction

In this paper all graphs are finite, simple and undirected. Let \( V(G) \) (or \( V \)) and \( E(G) \) (or \( E \)) be the vertex set and edge set of a graph \( G \). A mapping \( f: V(G) \to \{0, 1\} \) is called a binary labelling of the graph \( G \). For each \( v \in V(G) \), \( f(v) \) is called the (vertex) label of the vertex \( v \) under \( f \), and, for each edge \( x = uv \), the load (or label) on \( x \) under \( f \) is given by \( |f(u) - f(v)| \). The number of vertices (edges) of \( G \) labelled with 0 and 1 under \( f \) will be denoted by \( v_f(0) \) (\( e_f(0) \)) and \( v_f(1) \) (\( e_f(1) \)), respectively. We also define \( \alpha(f) = v_f(0) - v_f(1) \) and \( \beta(f) = e_f(0) - e_f(1) \). A binary labelling \( f \) of a graph \( G \) is said to be cordial if

\[
|\alpha(f)| \leq 1 \quad \text{and} \quad |\beta(f)| \leq 1.
\]
Definition 1.1. A graph $G$ is cordial if it admits a cordial labelling.

Cordial graphs were first introduced by Cahit [2] as a weaker version of both graceful graphs and harmonious graphs.

A fairly obvious reformulation of the definition of a cordial graph is as follows.

Definition 1.2. A graph $G = (V, E)$ is cordial if and only if there exists a partition $\{V_1, V_2\}$ of $V$ such that the two induced subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ satisfy the following condition:

$$|V_1| - |V_2| \leq 1 \quad \text{and} \quad |E| - 2(|E_1| + |E_2|) \leq 1.$$  

Here we shall label the vertices of $V_1$ with 0 and those of $V_2$ with 1, and call the partition $\{V_1, V_2\}$ a cordial partition of $G$.

The following result by Cahit [2] gives an important condition for a graph to be not cordial.

Theorem 1.3. If $G$ is an eulerian graph with $m$ edges, where $m \equiv 2 \pmod{4}$, then $G$ has no cordial labelling.

For example, the cycle $C_n$ is not cordial for $n \equiv 2 \pmod{4}$, the generalized Petersen graph $P(n, k)$ is not cordial for $n \equiv 2 \pmod{4}$ and any $k$ [4], etc.

For example, let $G(n)$ (where $n \geq 3$) be a graph with vertex set

$$V = \{v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$$

and edge set

$$E = \{v_iv_j | 1 \leq i < j \leq 2n\} \backslash \{(v_i, v_{i+1}) | i = 1, 2, \ldots, n\} \cup \{v_1v_{n+2}, v_2v_{n+1}\}.$$  

Obviously, $G(n)$ has order $2n$ and size $4\left(\frac{n}{2}\right) - 2$.

As

$$\deg(v_i) = \begin{cases} 2n - 3, & i = 1, 2, n+1, n+2, \\ 2n - 2, & \text{otherwise}. \end{cases}$$

$G(n)$ is never eulerian but its size $m \equiv 2 \pmod{4}$. Fig. 1 shows a $G(4)$ graph. The graph $G(n)$ is not cordial for all $n \geq 3$. For if $\{V_1, V_2\}$ is any partition of $V$ such that $|V_1| - |V_2| \leq 1$, then the induced subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are either both complete, or both not complete, and $|E| - 2(|E_1| + |E_2|) \geq 2$.

Definition 1.2 also implies the following necessary condition for the cordiality of a graph, which involves only the order and size of $G$. 

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**Theorem 1.4.** If a graph $G$ of order $n \geq 4$ and size $m$ is cordial, then

$$m < \left( \binom{n}{2} \right)^\frac{1}{2},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding the number $x$.

**Proof.** Let $(V_1, V_2)$ be a bipartition of $V$ as in Definition 1.2. If $n = 2k$, then $|V_1| = |V_2| = k$,

$$|E_1| + |E_2| \leq 2 \left( \binom{k}{2} \right) \text{ and } |E| - (|E_1| + |E_2|) \leq k^2.$$

If $n = 2k + 1$, we may assume $|V_1| = k$ and $|V_2| = k + 1$; then

$$|E_1| + |E_2| \leq \left( \binom{k}{2} \right) + \left( \binom{k+1}{2} \right) \text{ and } |E| - (|E_1| + |E_2|) \leq k(k + 1).$$

In both cases the desired result follows immediately. $\Box$

As an immediate consequence of the above result, we have the following.

The complete graph $K_n$ is not cordial for all $n \geq 4$.

**2. One-point union**

A graph $G$ in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of $G$. Let $G$ be a rooted graph. The graph $G^{(0)}$
obtained by identifying the roots of \( n \) copies of \( G \) is called a one-point union of the \( n \) copies of \( G \).

The friendship graph \( \mathcal{F}^{(n)} \) is a one-point union of \( n \) copies of a triangle \( \mathcal{F} \), which is cordial if and only if \( n \neq 2 \ (\text{mod} \ 4) \) [2].

We shall give some sufficient conditions for a one-point union of \( n \) copies of a rooted graph to be cordial, and deduce various conditions under which one-point unions of \( n \) copies of such graphs as a cycle, a complete graph, etc., will be cordial.

We shall relate the cordiality of a one-point union of \( n \) copies of a rooted graph to the solution of a system involving an equation and two inequalities.

**Theorem 2.1.** Let \( f_i, i = 1, 2, \ldots, q \) be \( q \) binary labellings of a graph \( G \) rooted at a vertex \( u \). Let \( \alpha(f_i) = k_i, \beta(f_i) = h_i \) and \( f_i(u) = 0, i = 1, 2, \ldots, q \). Then the one-point union \( G^{(n)} \) of \( n \) copies of \( G \) is cordial if the following system (*) has a nonnegative integral solution for the \( x_i \)'s:

\[
\begin{align*}
\sum_{i=1}^{q} (k_i - 1)x_i + 1 & \leq 1, \\
\sum_{i=1}^{q} h_ix_i & \leq 1, \\
\sum_{i=1}^{q} x_i & = n.
\end{align*}
\]

**Proof.** Suppose \( x_i = d_i, i = 1, 2, \ldots, q \) is a nonnegative integral solution of the system (*). Then we shall label the \( d_i \) rooted graphs \( G_{i(1)}, G_{i(2)}, \ldots, G_{i(d_i)} \) in \( G^{(n)} \) with \( f_i \) \((i = 1, 2, \ldots, q)\). As each of \( G_{ij} \) has the property \( \alpha(f_i) = k_i, \beta(f_i) = h_i \) and \( f_i(u_i) = 0 \) (where \( u_i \) is the root of \( G_{ij}, i = 1, 2, \ldots, q, j = (1), (2), \ldots, (d_i) \)), the resulting one-point union \( G^{(n)} \) of \( n \) copies of \( G \) is evidently cordial.

**Corollary 2.2.** Let \( G \) be a graph rooted at \( u \). Let \( f, g \) and \( h \) be binary labellings of \( G \) such that

\( f(u) = g(u) = h(u) = 0. \)

(i) If \( \alpha(f) = 1 \) and \( \beta(f) = 0 \), then \( G^{(n)} \) is cordial for all \( n \geq 1 \).

(ii) If either (a) \( \alpha(f) = 0, \beta(f) = -1, \alpha(g) = 2 \) and \( \beta(g) = 1 \), or

(b) \( \alpha(f) = 0, \beta(f) = 0, \alpha(g) = -2 \) and \( \beta(g) = 0 \), or

(c) \( \alpha(f) = 1, \beta(f) = 1, \alpha(g) = 1 \) and \( \beta(g) = -1 \),

then \( G^{(n)} \) is cordial for all \( n \geq 1 \).

(iii) If either (a) \( \alpha(f) = 1, \beta(f) = -1, \alpha(g) = 1 \) and \( \beta(g) = 3 \), or

(b) \( \alpha(f) = 1, \beta(f) = 1, \alpha(g) = 1 \) and \( \beta(g) = -3 \), or

(c) \( \alpha(f) = -1, \beta(f) = 1, \alpha(g) = 3 \) and \( \beta(g) = -1 \),

then \( G^{(n)} \) is cordial for all \( n \neq 2 \ (\text{mod} \ 4) \).
(iv) If either (a) \( \alpha(f) = -1, \beta(f) = -2, \alpha(g) = 3 \) and \( \beta(g) = 2 \), or 
(b) \( \alpha(f) = 0, \beta(f) = -2, \alpha(g) = 2 \) and \( \beta(g) = 2 \),
then \( G^{(n)} \) is cordial for all even \( n \).

(v) If \( \alpha(f) = 2, \beta(f) = 0, \alpha(g) = -2 \) and \( \beta(g) = 0 \), then \( G^{(n)} \) is cordial for all \( n \not\equiv 1 \pmod{4} \).

(vi) If \( \alpha(f) = 0, \beta(f) = 0, \alpha(g) = 2, \beta(g) = 2, \alpha(h) = 2, \beta(h) = -2 \), then \( G^{(n)} \) is cordial for all \( n \not\equiv 3 \pmod{4} \).

**Proof.** We shall prove only part (vi). The other parts can similarly be proved.

(vi) The system (*) is as follows:

\[
\begin{align*}
-x_1 + x_2 + x_3 + 1 & \leq 1, \\
0x_1 + 2x_2 - 2x_3 & \leq 1, \\
x_1 + x_2 + x_3 & = n.
\end{align*}
\]

If \( n = 4m \), then \( x_1 = 2m, x_2 = m, x_3 = m \) is a solution.
If \( n = 4m + 1 \), then \( x_1 = 2m + 1, x_2 = m, x_3 = m \) is a solution.
If \( n = 4m + 2 \), then \( x_1 = 2m + 2, x_2 = m, x_3 = m \) is a solution. \( \Box \)

Hence, by Theorem 2.1, \( G^{(n)} \) is cordial for all \( n \not\equiv 3 \pmod{4} \).

**Remark 2.3.** In Corollary 2.2(iii) if, in addition to the specified condition, the graph \( G \) is also eulerian, then the phrase 'for all \( n \not\equiv 2 \pmod{4} \)' can be replaced by the phrase 'if and only if \( n \not\equiv 2 \pmod{4} \)', because in this case when \( n = 2 \pmod{4} \), \( G^{(n)} \) is eulerian and \( |E(G^{(n)})| = 2 \pmod{4} \) and, hence, by Theorem 1.3, \( G^{(n)} \) is not cordial.

### 3. One-point union of cycles

Let \( C_m \) be a cycle of order \( m \). We write \( C_m = C(a_1, a_2, \ldots, a_m) \) to indicate \( V(C_m) = \{a_1, a_2, \ldots, a_m\} \) in which \( a_ia_{i+1} \in E(C_m), i = 1, 2, \ldots, m-1 \). We shall regard \( C_m \) as a rooted graph with vertex \( a_1 \) as its root.

**Theorem 3.1.** Let \( C_m^{(n)} \) be a one-point union of \( n \) copies of a cycle \( C_m \).

(i) If \( m = 0 \pmod{4} \), then \( C_m^{(n)} \) is cordial for all \( n \geq 1 \).

(ii) If \( m = 1 \) or \( 3 \pmod{4} \), then \( C_m^{(n)} \) is cordial if and only if \( n \not\equiv 2 \pmod{4} \).

(iii) If \( m = 2 \pmod{4} \), then \( C_m^{(n)} \) is cordial if and only if \( n \) is even.

**Proof.** Let \( C_m = C(a_1, a_2, \ldots, a_m) \).

(i) Define two binary labellings \( f \) and \( g \) of \( C_m \) as follows:

\[
f(a_i) = \begin{cases} 
0, & i = 1, 2 \pmod{4}, \\
1, & i = 0, 3 \pmod{4}, 
\end{cases}
\]

(ii) Define two binary labellings \( f \) and \( g \) of \( C_m \) as follows:
and
\[ g(a_i) = \begin{cases} f(a_i), & i \neq m, \\ 0, & i = m. \end{cases} \]

We find
\[ v_f(0) = v_f(1) = 2k, \quad e_f(0) = e_f(1) = 2k, \]
\[ v_g(0) = 2k + 1, \quad v_g(1) = 2k - 1 \quad \text{and} \quad e_g(0) = e_g(1) = 2k, \]
which gives
\[ \alpha(f) = 0, \quad \beta(f) = 0, \quad \alpha(g) = 2, \quad \beta(g) = 0 \]
and \( f(a_1) = g(a_1) = 0 \). By Corollary 2.2(ii)(b), \( C_m^{(0)} \) is cordial for all \( n \geq 1 \).

(ii) Case 1: \( m = 1 \pmod{4} \).
Let \( m = 4k + 1, k \geq 1 \). Define two binary labellings \( f \) and \( g \) of \( C_m \) as follows:

\[ f(a_i) = \begin{cases} 0, & i = 0, 1 \pmod{4}, \\ 1, & i = 2, 3 \pmod{4}, \end{cases} \]
and
\[ g(a_i) = \begin{cases} f(a_i), & i \neq 2, m, \\ 0, & i = 2, \\ 1, & i = m. \end{cases} \]

Then we have
\[ \alpha(f) = 1, \quad \beta(f) = 1, \quad \alpha(g) = 1, \quad \beta(g) = -3 \]
and \( f(a_1) = g(a_1) = 0 \).

By Corollary 2.2(iii)(b) and Remark 2.3, \( C_m^{(0)} \) is cordial if and only if \( n \neq 2 \pmod{4} \).

Case 2: \( m = 3 \pmod{4} \), say \( m = 4k + 3 \).
The following binary labellings \( f \) and \( g \) of \( C_m \) can be shown to satisfy the condition in Corollary 2.2(iii)(b):

\[ f(a_i) = \begin{cases} 0, & i = 1, 2 \pmod{4}, i \neq m - 2, \\ 1, & i = 0, 3 \pmod{4}, i \neq m, \\ 1, & i = m - 2, \\ 0, & i = m, \end{cases} \]
and
\[ g(a_i) = \begin{cases} f(a_i), & i \neq m - 2, m - 1, \\ 0, & i = m - 2, \\ 1, & i = m - 1. \end{cases} \]

In view of Remark 2.3, \( C_m^{(0)} \) is cordial if and only if \( n \neq 2 \pmod{4} \).
(iii) $m = 2 \pmod{4}$

The following binary labellings $f$ and $g$ of $C_m$ satisfy the condition of Corollary 2.2(iv)(b):

$$f(a_i) = \begin{cases} 
0, & i = 1, 2 \pmod{4}, \\
1, & i = 0, 3 \pmod{4}, 
\end{cases}$$

and

$$g(a_i) = \begin{cases} 
f(a_i), & i \neq m, \\
1, & i = m.
\end{cases}$$

It follows that $C_m^{(n)}$ is cordial for all even $n$. If $n$ is odd, then $|E(C_m^{(n)})| = 2 \pmod{4}$, and, as $C_m^{(n)}$ is eulerian, $C_m^{(n)}$ cannot be cordial. Hence, $C_m^{(n)}$ is cordial if and only if $n$ is even.

**Example 3.2.** Fig. 2(a) and (b) show a cordial $C_b^{(3)}$ and $C_b^{(4)}$, respectively.

### 4. One-point union of complete graphs

We shall next investigate the cordiality of the one-point union of $n$ copies of a complete graph $K_m$ of order $m$.

Let $G$ be a graph. The join $G + v$ is a graph obtained by joining a new vertex $v$ to every vertex of $G$. We shall say that a vertex is odd (even) if it has odd (even) degree.

**Lemma 4.1.** Let $G$ be a graph such that every vertex is odd. If either (a) $|V(G)| = 0 \pmod{4}$ and $|E(G)| = 2 \pmod{4}$ or (b) $|V(G)| = 2 \pmod{4}$ and $|E(G)| = 0 \pmod{4}$, then $G$ is not cordial.

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Proof. Suppose $G$ is cordial. Then obviously the join $G^* = G + v$ is also cordial. But, since every vertex in $G^*$ is even, $G^*$ is eulerian; furthermore, as $|E(G^*)| = 2 \text{ (mod 4)}$, $G^*$ is not cordial, a contradiction. □

**Theorem 4.2.** Let $K_m^{(n)}$ be the one-point union of $n$ copies of a complete graph $K_m$.

(i) If $m = 0 \text{ (mod 8)}$, then $K_m^{(n)}$ is not cordial for $n = 3 \text{ (mod 4)}$.

(ii) If $m = 4 \text{ (mod 8)}$, then $K_m^{(n)}$ is not cordial for $n = 1 \text{ (mod 4)}$.

(iii) If $m = 5 \text{ (mod 8)}$, then $K_m^{(n)}$ is not cordial for all odd $n$.

Proof. We shall prove only part (ii). The rest can be proved similarly.

(ii) Suppose $m = 8k + 4$ and $n = 4l + 1$. Then, since every vertex of $K_m$ is odd, $|V(K_m^{(n)})| = 0 \text{ (mod 4)}$ and $|E(K_m^{(n)})| = 2 \text{ (mod 4)}$, it follows from Lemma 4.1 that $K_m^{(n)}$ is not cordial.

**Theorem 4.3.** (i) $K_4^{(n)}$ is cordial if and only if $n \neq 1 \text{ (mod 4)}$.

(ii) $K_5^{(n)}$ is cordial if and only if $n$ is even.

Proof. (i) Consider the following two binary labellings $f$ and $g$ of $K_4$ in which the vertex $u$ is the root.

$$f : 0 \quad 0 \quad 0 \quad 0 \quad u$$

and

$$g : 0 \quad 0 \quad 0 \quad 0 \quad u$$

We find

$\alpha(f) = 2, \quad \beta(f) = 0, \quad \alpha(g) = -2, \quad \beta(g) = 0$

and

$f(u) = g(u) = 0$.

By Corollary 2.2(v), $K_4^{(n)}$ is cordial for all $n \neq 1 \text{ (mod 4)}$. When $n = 1 \text{ (mod 4)}, K_4^{(n)}$ is not cordial by Theorem 4.2(ii).

(ii) The following two binary labellings $f$ and $g$ of $K_5$

We find

$\alpha(f) = 2, \quad \beta(f) = 0, \quad \alpha(g) = -2, \quad \beta(g) = 0$

and

$f(u) = g(u) = 0$.

By Corollary 2.2(v), $K_5^{(n)}$ is cordial for all $n \neq 1 \text{ (mod 4)}$. When $n = 1 \text{ (mod 4)}, K_5^{(n)}$ is not cordial by Theorem 4.2(ii).
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give

$$\alpha(f) = -1, \quad \beta(f) = -2, \quad \alpha(g) = 3, \quad \beta(g) = 2$$

and

$$f(u) = g(u) = 0.$$  

By Corollary 2.2(iv)(a), $K_5^{(n)}$ is cordial for all even $n$. But if $n$ is odd, then, since $K_5^{(n)}$ is eulerian and $|E(K_5^{(n)})| = 2 \mod 4$, $K_5^{(n)}$ is not cordial.

Example 4.4. Fig. 3(a) and (b) show a cordial $K_4^{(7)}$ and a cordial $K_5^{(4)}$.

Theorem 4.5. (i) $K_6^{(n)}$ is cordial if and only if $n \geq 3$.
(ii) $K_7^{(n)}$ is cordial if and only if $n \not\equiv 2 \mod 4$.

Proof. Let $V(K_m) = \{u = v_1, v_2, \ldots, v_m\}$ and consider $u$ as the root of $K_m$.

(i) For the following binary labellings $f_i$ ($i = 1, \ldots, 6$) of $K_6$,

$$f_i(v_j) = \begin{cases} 0, & j = 1, 2, \ldots, 6 - i, \\ 1, & j = 6 - i + 1, \ldots, 6, & i = 1, 2, \ldots, 5, \\ f_6(v_j) = 0, & j = 1, 2, \ldots, 6, \end{cases}$$

we find

$$\alpha(f_1) = 6, \quad \beta(f_1) = 15, \quad \alpha(f_2) = 4, \quad \beta(f_2) = 5, \quad \alpha(f_3) = 2, \quad \beta(f_3) = -1, \quad \alpha(f_4) = 0, \quad \beta(f_4) = -3, \quad \alpha(f_5) = -2, \quad \beta(f_5) = -1, \quad \alpha(f_6) = -4, \quad \beta(f_6) = 5.$$
and 
\[ f_i(u) = 0, \quad i = 1, 2, \ldots, 6. \]

The system (\( \ast \)) in this case is as follows:
\[
\begin{align*}
|5x_1 + 3x_2 + x_3 - x_4 - 3x_5 - 5x_6 + 1| &\leq 1, \\
|10x_1 + 5x_2 - x_3 - 3x_4 - x_5 + 5x_6| &\leq 1, \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= n.
\end{align*}
\]

It is known that \( K_6^{(1)} = K_6 \) is not cordial. Since \( f_i, \ i = 0, 1, \ldots, 5 \) are all the possible binary labellings of \( K_6 \) such that \( f_i(u) = 0 \) and the above system has no solution for \( n = 2 \), it follows that \( K_6^{(2)} \) is not cordial.

Now, let \( n \geq 3 \). When \( n = 4m, \ x_2 = x_3 = x_4 = x_5 = m, \ x_1 = x_6 = 0 \) is a solution. When \( n = 4m + 1, \ x_2 = m - 1, \ x_3 = m + 3, \ x_4 = m - 1, \ x_5 = m - 1, \ x_6 = 1, \ x_1 = 0 \) is a solution. When \( n = 4m + 2, \ x_2 = x_3 = m + 4, \ x_4 = m - 1, \ x_5 = m - 1, \ x_6 = 1, \ x_1 = 0 \) is a solution. When \( n = 4m + 3, \ x_2 = m + 1, \ x_3 = m, \ x_4 = m + 1, \ x_5 = m + 1, \ x_1 = x_6 = 0 \) is a solution. Hence, \( K_6^{(2)} \) is cordial for all \( n \geq 3 \).

(ii) Consider the following binary labellings \( f_i (i = 1, 2, \ldots, 5) \) of \( K_7 \):
\[
f_i(v_j) = \begin{cases} 0, & j = 1, 2, \ldots, 7-i, \\ 1, & j = 7 - i + 1, \ldots, 7. \end{cases}
\]

Then we have
\[
\begin{align*}
\alpha(f_1) &= 5, \quad \beta(f_1) = 9, \\
\alpha(f_2) &= 3, \quad \beta(f_2) = 1, \\
\alpha(f_3) &= 1, \quad \beta(f_3) = -3, \\
\alpha(f_4) &= -1, \quad \beta(f_4) = -3, \\
\alpha(f_5) &= -3, \quad \beta(f_5) = 1,
\end{align*}
\]

and
\[ f_i(u) = 0, \quad i = 1, 2, \ldots, 5. \]

The system (\( \ast \)) then becomes
\[
\begin{align*}
|4x_1 + 2x_2 + 0x_3 - 2x_4 - 4x_5 + 1| &\leq 1, \\
|9x_1 + x_2 - 3x_3 - 3x_4 + x_5| &\leq 1, \\
x_1 + x_2 + x_3 + x_4 + x_5 &= n.
\end{align*}
\]

When \( n = 3, \ x_1 = 0, \ x_2 = 1, \ x_3 = 1, \ x_4 = 0, \ x_5 = 1 \) is a solution. When \( n = 4m, \ x_2 = m, \ x_3 = m, \ x_4 = 2m, \ x_5 = 0 \) is a solution. When \( n = 4m + 1, \ x_1 = m - 1, \ x_2 = 3, \ x_3 = m - 1, \ x_4 = 2m - 1, \ x_5 = 1 \) is a solution. When \( n = 4m + 3, \ x_1 = m, \ x_2 = 1, \ x_3 = m + 2, \ x_4 = 2m - 1, \ x_5 = 1 \) is a solution. In case \( n = 4m + 2 \), the reason for which \( K_7^{(m)} \) is not cordial as in Theorem 1.3. Hence, \( K_7^{(m)} \) is cordial if and only if \( n \neq 2 \) (mod 4). \( \square \)
Example 4.6. Fig. 4(a) and (b) show a cordial $K_n^{(7)}$ and a cordial $K_4^{(4)}$.

The following lemma follows easily from Definition 1.2.

**Lemma 4.7.** If $\{V_1, V_2\}$ is a cordial partition of a graph $G$ of even size $m$, then $m = 2(|E_1| + |E_2|)$.

**Theorem 4.8.** The graph $K_n^{(2)}$ is cordial if and only if $n = p^2$ or $p^2 + 1$, where $p$ is a positive integer.

**Proof.** Suppose $K_n^{(2)}$ is cordial. Let $\{V_1, V_2\}$ be a cordial partition of $K_n^{(2)}$ with $|V_1| = n$. Assume first that $w \not\in V_1$ and

$$V_1 = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{m-r-1}, w\},$$

where the $v_i$'s are vertices of one $K_2$ and the $u_j$'s are the vertices of another $K_n$, $w$ is the common vertex, and all these vertices will be labelled with 0.

Without loss of generality, we may assume that $r \leqslant \lfloor (n-1)/2 \rfloor$. Then

$$|E_1| = \binom{r+1}{2} + \binom{n-r}{2} \quad \text{and} \quad |E_2| = \binom{n-(r+1)}{2} + \binom{r}{2},$$

where $E_1$ and $E_2$ denote the edge sets of the induced subgraphs $G_1$ and $G_2$ of $G$ with vertex sets $V_1$ and $V_2$, respectively; therefore,

$$|E_1| + |E_2| = \binom{r+1}{2} + \binom{n-r}{2} + \binom{n-(r+1)}{2} + \binom{r}{2}.$$

Fig. 4.
As \(|E|=2(\frac{n}{2})|\) is even, it follows from Lemma 4.7 that
\[
\binom{r+1}{2} + \binom{n-r}{2} + \binom{n-(r+1)}{2} + \binom{r}{2} = \binom{n}{2},
\]
which reduces to
\[
(n-1)(n-2) = 4r(n-1-r).
\]
Let
\[
h = \left\lfloor \frac{n-1}{2} \right\rfloor - r \quad \text{or} \quad r = \left\lfloor \frac{n-1}{2} \right\rfloor - h.
\]
Substitution of this value of \(r\) into expression (I) gives
\[
n = \begin{cases} 
(2h + 1)^2 + 1, & \text{if } n \text{ is even}, \\
(2h)^2 + 1, & \text{if } n \text{ is odd}.
\end{cases}
\]
Suppose next that \(w \notin V_1\) and
\[
V_1 = \{v_{i_1}, v_{i_2}, \ldots, v_r, u_{j_1}, u_{j_2}, \ldots, u_{j_{n-r}}\}.
\]
Then \(w \in V_2 = V \setminus V_1\). In this case,
\[
2\binom{r}{2} + 2\binom{n-r}{2} = \binom{n}{2},
\]
which simplifies to
\[
n^2 - n = 4r(n-r).
\]
Again, let \(h = \left\lfloor \frac{(n-1)/2} \right\rfloor - r\). As in the case of \(w \in V_1\), we shall find that if \(n\) is even, \(n = (2h+2)^2\), and that if \(n\) is odd, \(n = (2h+1)^2\). Hence, we conclude that if \(K_n^{(2)}\) is cordial, then \(n = p^2\) or \(p^2 + 1\) for some positive integer \(P\).

Conversely, if \(n = p^2\), we shall choose the subset
\[
V_1 = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{n-r}\},
\]
and if \(n = p^2 + 1\),
\[
V_1 = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{n-r-1}, w\},
\]
where \(r = \left\lfloor \frac{(n-1)/2} \right\rfloor - h\), in which \(h\) is determined as follows.

Case 1. \(n - p^2\): If \(n\) is even, \(p = 2h + 2\); if \(n\) is odd, \(p = 2h + 1\).

Case 2. \(n = p^2 + 1\): If \(n\) is even, \(p = 2h + 1\); if \(n\) is odd, \(p = 2h\).

By arguing backwards in the proof of the necessity, we can easily verify that the partition \(\{V_1, V_2\}\), where \(V_2 = V \setminus V_1\) is a cordial partition of \(K_n^{(2)}\).

The following example illustrates the choice of the cordial partition \(\{V_1, V_2\}\) of \(K_n^{(2)}\) for \(n = 9\).
Example 4.9. For $K^{(2)}_9$, $h=1$, $r=3$ and a cordial partition $\{V_1, V_2\}$ of $K^{(2)}_9$ is given by

$V_1 = \{v_1, v_2, v_3, u_1, u_2, u_3, u_4, u_5, u_6\}$

and

$V_2 = \{v_4, v_5, v_6, v_7, v_8, u_7, u_8, w\}$.

Vertices in $V_1$ will be labelled with 0 and those in $V_2$ with 1 to secure a cordial labelling.

Fig. 5 shows a cordial labelling of $K^{(2)}_9$.

5. One-point union of wheels, fans and flags

A wheel $W_m$ is obtained by joining all vertices of a cycle $C_m$ to an extra vertex called the centre. The wheel $W_m$ is cordial if and only if $m \not\equiv 3 \pmod{4}$ [2].

A fan $F_m$ ($m \geq 3$) is obtained by joining all vertices of a path $P_m$ to an extra vertex called the centre, and contains $m + 1$ vertices and $2m - 1$ edges. All fans are cordial [2].

A flag $F_m$ is obtained by joining exactly one vertex (we will call the first vertex) of a cycle $C_m$ to an extra vertex called the root.

The following theorems can be easily verified by using the information in Table 1.

Theorem 5.1. Let $W^{(n)}_m$ be the one-point union of $n$ copies of a wheel $W_m$.

(i) If $m = 0$ or $2 \pmod{4}$, then $W^{(n)}_m$ is cordial for all $n \geq 1$ provided the root is the centre of $W_m$.

(ii) If $m = 3 \pmod{4}$, then $W^{(n)}_m$ is cordial if $n \not\equiv 1 \pmod{4}$ provided the root is the centre.

(iii) If $m = 1 \pmod{4}$, then $W^{(n)}_m$ is cordial if $n \not\equiv 3 \pmod{4}$ provided the root is the centre.

Example 5.2. Fig. 6 shows a cordial $W^{(n)}_4$ rooted at the centre.

Theorem 5.3. The one-point union $F^{(n)}_m$ of $n$ copies of a fan $F_m$ is cordial for all $n \geq 1$ and $m > 2$. 

![Fig. 5.](image-url)
<table>
<thead>
<tr>
<th>Graph</th>
<th>Binary labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_n$ Root at centre $m \equiv 0 \pmod{4}$</td>
<td>$f(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 0, &amp; i \equiv 1, 2 \pmod{4} \ 1, &amp; i \equiv 0, 3 \pmod{4}, i \neq 0 \end{cases}$</td>
</tr>
<tr>
<td>$W_n$ Root at centre $m \equiv 1 \pmod{4}$</td>
<td>$f_1(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 0, &amp; i = 1, 2 \ 1, &amp; i \equiv 3, 4, 5 \ 1, &amp; i \equiv 2 \pmod{4}, i \geq 6 \ 0, &amp; i \equiv 0, 3 \pmod{4}, i \geq 7 \ 1, &amp; i \equiv 1 \pmod{4}, i \geq 9 \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$f_2(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 0, &amp; i = 1, 5 \ 1, &amp; i \equiv 2, 3, 4 \ 0, &amp; i \equiv 2 \pmod{4}, i \geq 6 \ 1, &amp; i \equiv 0, 3 \pmod{4}, i \geq 7 \ 0, &amp; i \equiv 1 \pmod{4}, i \geq 9 \end{cases}$</td>
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<td></td>
<td>$f_3(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 0, &amp; i = 1, 2, 4 \ 1, &amp; i = 3, 5 \ 1, &amp; i \equiv 2 \pmod{4}, i \geq 6 \ 0, &amp; i \equiv 0, 3 \pmod{4}, i \geq 7 \ 1, &amp; i \equiv 1 \pmod{4}, i \geq 9 \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$f_4(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 0, &amp; i = 1, 2, 5 \ 1, &amp; i \equiv 3, 4, 5 \ 1, &amp; i \equiv 2 \pmod{4}, i \geq 6 \ 0, &amp; i \equiv 0, 3 \pmod{4}, i \geq 7 \ 1, &amp; i \equiv 1 \pmod{4}, i \geq 9 \end{cases}$</td>
</tr>
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<td></td>
<td>$f_5(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 1, &amp; i = 1, 2, 5 \ 0, &amp; i = 3, 4 \ 1, &amp; i \equiv 2 \pmod{4}, i \geq 6 \ 0, &amp; i \equiv 0, 3 \pmod{4}, i \geq 7 \ 1, &amp; i \equiv 1 \pmod{4}, i \geq 9 \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$f_6(v_i) = \begin{cases} 0, &amp; i = 0 \text{ centre} \ 1, &amp; i = 1, 5 \ 0, &amp; i = 2, 3, 4 \ 1, &amp; i \equiv 2 \pmod{4}, i \geq 6 \ 0, &amp; i \equiv 0, 3 \pmod{4}, i \geq 7 \ 1, &amp; i \equiv 1 \pmod{4}, i \geq 9 \end{cases}$</td>
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</table>

Use $f_1$ to $f_6$ to form unit one.
The cordiality of one-point union of n copies of a graph

Table 1 (Continued)

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<thead>
<tr>
<th>Graph</th>
<th>Binary labelling</th>
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<tbody>
<tr>
<td></td>
<td>$g_2(v_i) =$</td>
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<tr>
<td></td>
<td>$\begin{cases} 0, &amp; i=0 \text{ centre} \ 1, &amp; i=1,5 \ 0, &amp; i=2,3,4 \ 1, &amp; i=2(\mod 4), i \geq 6 \ 0, &amp; i=0,3(\mod 4), i \geq 7 \ 1, &amp; i=1(\mod 4), i \geq 9 \end{cases}$</td>
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<tr>
<td></td>
<td>$g_3(v_i) =$</td>
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<tr>
<td></td>
<td>$\begin{cases} 0, &amp; i=0 \text{ centre} \ 1, &amp; i=1,4,5 \ 0, &amp; i=2,3 \ 1, &amp; i=2(\mod 4), i \geq 6 \ 0, &amp; i=0,3(\mod 4), i \geq 7 \ 1, &amp; i=1(\mod 4), i \geq 9 \end{cases}$</td>
</tr>
</tbody>
</table>

Use $1g_1$, $1g_2$, and $2g_3$ to form unit two to superimpose on unit one.

$W_m$
Root at centre $m \equiv 2(\mod 4)$

$f(v_i) =$

|       | $\begin{cases} 0, & i=0 \text{ centre} \\ 1, & i=1,6 \\ 0, & i=2,3,4,5 \\ 1, & i=3(\mod 4), i \geq 7 \\ 0, & i=0,1(\mod 4), i \geq 8 \\ 0, & i=2(\mod 4), i \geq 10 \end{cases}$ |

$g(v_i) =$

|       | $\begin{cases} 0, & i=0 \text{ centre} \\ 1, & i=1,4 \\ 0, & i=2,3,5,6 \\ 1, & i=3(\mod 4), i \geq 7 \\ 0, & i=0,1(\mod 4), i \geq 8 \\ 0, & i=2(\mod 4), i \geq 10 \end{cases}$ |

Use $f$ and $g$ alternately.

$W_m$
Root at centre $m = 4k + 3$

$f(v_i) =$

|       | $\begin{cases} 0, & i=0 \text{ centre} \\ 1, & i=1,2,3 \\ 1, & i=0(\mod 4), i \geq 4 \\ 0, & i=1,2(\mod 4), i \geq 5 \\ 1, & i=3(\mod 4), i \geq 7 \end{cases}$ |

$g(v_i) =$

|       | $\begin{cases} 0, & i=0 \text{ centre} \\ 1, & i=1,2 \\ 1, & i=3 \\ 0, & i=0(\mod 4), i \geq 4 \\ 0, & i=1,2(\mod 4), i \geq 5 \\ 1, & i=3(\mod 4), i \geq 7 \end{cases}$ |

When $n = 4k + 2$, use $(k+1)f$ and $(3k+1)g$.
When $n = 4k + 3$, use $(k+1)f$ and $(3k+2)g$.
When $n = 4k$, use $kf$ and $3kg$. 
Table 1 (Continued)

<table>
<thead>
<tr>
<th>Graph</th>
<th>Binary labelling</th>
</tr>
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</table>
| Fan: $F_m$ | $f(v_i) = \begin{cases} 
0, & i = 0 \\
0, & i \equiv 1, 2 \pmod{4} \\
1, & i \equiv 0, 3 \pmod{4}, i \neq 0 \\
0, & i = 0 
\end{cases}$  
$m \equiv 0 \pmod{4}$ |
| Root at centre | $g(v_i) = \begin{cases} 
1, & i \equiv 0, 1 \pmod{4}, i \neq 0 \\
0, & i \equiv 2, 3 \pmod{4} 
\end{cases}$  
Use $f$ and $g$ alternately. |

| $F_m$ | $f(v_i) = \begin{cases} 
0, & i = 0 \text{ centre} \\
0, & i = 1, 2, 4 \\
1, & i = 3, 5, 6 \\
0, & i = 7 \\
1, & i = 1, 2, 4, 5, 6, 7 \\
0, & i \equiv 0, 1, 2, 3, 4, 5 \pmod{4}, \ i \geq 8 \\
1, & i = 1, 2, 4, 5, 6, 7, 8, 9 \\
0, & i \equiv 0, 3, 4 \pmod{4}, \ i > 9 
\end{cases}$  
$m \equiv 1 \pmod{4}$ |
| Root at centre | $g(v_i) = \begin{cases} 
0, & i = 0 \text{ centre} \\
0, & i = 1, 2 \\
1, & i = 3, 5, 6 \\
0, & i = 7 \\
1, & i = 1, 2, 4, 5, 6, 7 \\
0, & i \equiv 0, 1, 2, 3, 4, 5 \pmod{4}, \ i \geq 8 \\
1, & i = 1, 2, 4, 5, 6, 7, 8, 9 \\
0, & i \equiv 0, 3, 4 \pmod{4}, \ i > 9 
\end{cases}$  
Use $f$ and $g$ alternately. |

| $F_m$ | $f(v_i) = \begin{cases} 
0, & i = 0 \text{ centre} \\
0, & i = 1, 2, 4, 5, 6 \\
1, & i = 3, 4, 7 \\
0, & i = 8 \\
1, & i = 1, 2, 4, 5, 6, 7, 8 \\
0, & i \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8 \pmod{4}, \ i \geq 10 
\end{cases}$  
$m \equiv 2 \pmod{4}$ |
| Root at centre | $g(v_i) = \begin{cases} 
0, & i = 0 \text{ centre} \\
0, & i = 1, 2, 4 \\
1, & i = 3, 5, 6, 7 \\
0, & i = 8 \\
1, & i = 1, 2, 4, 5, 6, 7, 8, 9 \\
0, & i \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \pmod{4}, \ i \geq 10 
\end{cases}$  
Use $f$ and $g$ alternately. |

| $F_m$ | $f(v_i) = \begin{cases} 
0, & i = 0 \text{ centre} \\
1, & i = 1, 2 \\
0, & i = 3 \\
0, & i \equiv 0, 1, 2, 3, 4 \pmod{4}, \ i \geq 4 \\
1, & i \equiv 1, 2, 3, 4 \pmod{4}, \ i \geq 5 
\end{cases}$  
$m \equiv 3 \pmod{4}$ |
| Root at centre | $g(v_i) = \begin{cases} 
0, & i = 0 \text{ centre} \\
0, & i = 1, 2 \\
1, & i = 3 \\
1, & i \equiv 0, 1, 2, 3, 4 \pmod{4}, \ i \geq 4 \\
0, & i \equiv 1, 2, 3, 4 \pmod{4}, \ i \geq 5 
\end{cases}$  
Use $f$ and $g$ alternately. |
### Table 1 (Continued)

<table>
<thead>
<tr>
<th>Graph</th>
<th>Binary labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flag: $\mathcal{F}_m$</td>
<td>Use $f$ and $g$ alternately.</td>
</tr>
<tr>
<td>Root at centre $m \equiv 0 \pmod{4}$</td>
<td>$f(v_i) = \begin{cases} f^<em>(v_i), &amp; i \neq 1 \ 0, &amp; i=0 \text{ where } f^</em> \text{ is any cordial labelling of } C_{4k} \end{cases}$ $g(v_i) = \begin{cases} 0, &amp; i \neq 0 \text{ and } f(v_i) = 1 \ 1, &amp; i \neq 0 \text{ and } f(v_i) = 0 \ 0, &amp; i = 0 \end{cases}$ Use $f$ and $g$ alternately.</td>
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</tbody>
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<table>
<thead>
<tr>
<th>$\mathcal{F}_m$</th>
<th>Root at centre $m \equiv 1 \pmod{4}$</th>
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</thead>
<tbody>
<tr>
<td>$f(v_i)$</td>
<td>$0, \quad i=0$ root</td>
<td>Use $f$ and $g$ alternately.</td>
</tr>
<tr>
<td></td>
<td>$1, \quad i=1, 2, 5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0, \quad i=3, 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1, \quad i \equiv 2 \pmod{4}, \ i \geq 6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0, \quad i \equiv 0, 3 \pmod{4}, \ i \geq 7$</td>
<td></td>
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<tr>
<td></td>
<td>$1, \quad i \equiv 1 \pmod{4}, \ i \geq 9$</td>
<td></td>
</tr>
<tr>
<td>$g(v_i)$</td>
<td>$0, \quad i=0$ root</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1, \quad i=1, 2$</td>
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</tr>
<tr>
<td></td>
<td>$0, \quad i=3, 4, 5$</td>
<td></td>
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<tr>
<td></td>
<td>$0, \quad i \equiv 2 \pmod{4}, \ i \geq 6$</td>
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<td>$1, \quad i \equiv 0, 3 \pmod{4}, \ i \geq 7$</td>
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<tr>
<td></td>
<td>$0, \quad i \equiv 1 \pmod{4}, \ i \geq 9$</td>
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<tr>
<th>$\mathcal{F}_m$</th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(v_i)$</td>
<td>$0, \quad i=0$ root</td>
<td>Use $f$ and $g$ alternately.</td>
</tr>
<tr>
<td></td>
<td>$1, \quad i=1, 2, 5$</td>
<td></td>
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<tr>
<td></td>
<td>$0, \quad i=3, 4, 6$</td>
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<tr>
<td></td>
<td>$1, \quad i \equiv 2, 3 \pmod{4}, \ i \geq 7$</td>
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<tr>
<td></td>
<td>$0, \quad i \equiv 0, 1 \pmod{4}, \ i \geq 8$</td>
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<tr>
<td>$g(v_i)$</td>
<td>$0, \quad i=0$ root</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1, \quad i=1, 5, 6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0, \quad i=2, 3, 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1, \quad i \equiv 2, 3 \pmod{4}, \ i \geq 7$</td>
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<td></td>
<td>$0, \quad i \equiv 0, 1 \pmod{4}, \ i \geq 8$</td>
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<thead>
<tr>
<th>$\mathcal{F}_m$</th>
<th>Root at centre $m \equiv 3 \pmod{4}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(v_i)$</td>
<td>$0, \quad i=0$ root</td>
<td>Use $f$ and $g$ alternately.</td>
</tr>
<tr>
<td></td>
<td>$0, \quad i=1$</td>
<td></td>
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<tr>
<td></td>
<td>$1, \quad i=2, 3$</td>
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</tr>
<tr>
<td></td>
<td>$1, \quad i \equiv 0, 3 \pmod{4}, \ i \geq 4$</td>
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<tr>
<td></td>
<td>$0, \quad i \equiv 1, 2 \pmod{4}, \ i \geq 5$</td>
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<tr>
<td>$g(v_i)$</td>
<td>$0, \quad i=0$ root</td>
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<td></td>
<td>$0, \quad i=1, 2$</td>
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<tr>
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<td>$1, \quad i=3$</td>
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<td></td>
<td>$1, \quad i \equiv 0 \pmod{4}, \ i \geq 4$</td>
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<td>$0, \quad i \equiv 1, 2 \pmod{4}, \ i \geq 5$</td>
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<td>$1, \quad i \equiv 3 \pmod{4}, \ i \geq 7$</td>
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</table>
Example 5.4. A cordial $F_8^{(4)}$ and a cordial $F_6^{(5)}$ are shown in Fig. 7(a) and (b), respectively.

Theorem 5.5. The one-point union $S_m^{(n)}$ of $n$ copies of a flag $S_m$ is cordial for all $n \geq 1$.

Example 5.6. A cordial $S_6^{(4)}$ and a cordial $S_5^{(6)}$ are shown in Fig. 8(a) and (b), respectively.

We recall that a wheel $W_m$ is the join $C_m + v_0$, where $C_m = C(v_1, v_2, \ldots, v_m)$. A fan $F_m$ is the join $P_m + v_0$, where the path $P_m = P(v_1, v_2, \ldots, v_m)$, and a flag $S_m$ is obtained by joining $v_0$ to $v_1$ in $C_m = C(v_1, v_2, \ldots, v_m)$.
The cordiality of one-point union of copies of a graph

6. Concluding remark

We relate the cordiality of a one-point union $G^{(n)}$ of $n$ copies of a rooted graph $G$ to the existence of some binary labellings $f_i, i = 1, 2, \ldots, q$, of $G$ having certain property, say,

$$\alpha(f_i) = r_i, \quad \beta(f_i) = t_i \quad \text{and} \quad f_i(u) = 0, \quad i = 1, 2, \ldots, q. \quad (**$$

If there is a rooted graph $G$ having property (**), then there are infinitely many rooted graphs $G^*$ having property (**). Indeed, let $H$ be a cordial graph of even order and even size (there are infinitely many such graphs). Let the graph $G^*$ be obtained by joining an equal number of vertices labelled with 0 and with 1 of $H$ to any one vertex of $G$. Then $G^*$ will be a rooted graph possessing property (**). Repeated applications of this argument will give us infinitely many rooted graphs with property (**). This shows that the results obtained in this paper actually cover a very wide range of graphs.

Acknowledgment

We are very grateful to the referee for detecting some errors in our two papers and for his helpful suggestions how to merge these two papers into just one paper.

References