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Existence of cyclic $(3, \lambda)$-GDD of type g^v having prescribed number of short orbits

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ABSTRACT

In this paper, the necessary and sufficient conditions for the existence of a cyclic $(3, \lambda)$-GDD of type g^v with exactly α short block orbits are determined for all possible parameters λ, g, v and α.

1. Introduction

A (k, λ)-*GDD* of type g^v is an ordered triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of size gv, $\mathcal{G}$ a partition of X into groups of size g, and $\mathcal{B}$ a set of k-subsets of X (called blocks), such that each pair of elements from different groups appears in λ blocks and no block contains two elements from a common group. A GDD is *cyclic* if it admits a cyclic automorphism group G acting sharply transitively on X.

For a cyclic (k, λ)-*GDD* of type g^v, we may assume that $X = Z_{gv}$. Let $B = \{b_1, b_2, \ldots, b_k\}$ be a block of a cyclic (k, λ)-GDD of type g^v. The block orbit generated by B is defined as the set of distinct blocks $B + i = \{b_1 + i, b_2 + i, \ldots, b_k + i\} \pmod{gv}$ for $i \in Z_{gv}$. If a block orbit has gv blocks, then the block orbit is said to be *full*, otherwise *short*. In [13], the necessary and sufficient conditions have been determined for the existence of a cyclic $(3, \lambda)$-GDD of type g^v. In the present paper, we further investigate the existence spectrum of a cyclic $(3, \lambda)$-GDD of type g^v with exactly α short orbits, where α can be any possible value.

A cyclic $(3, \lambda)$-GDD is equivalent to a special difference family which we define below. Throughout this paper, $[a, b]$ denotes the set of integers n such that $a \le n \le b$, and $[a, b]_o$ denotes the set of odd integers in $[a, b]$. For a set S, λS denotes the multiset containing each element of S exactly λ times. A difference family of an abelian group G is a collection $\{B_1, B_2, \ldots, B_t\}$ of k-subsets (called *base blocks*) of G satisfying certain properties. For any base block B of a difference family over an abelian group G, the subgroup

$$\{z \in G : B + z = B\}$$

is called the *stabilizer* of B in G. A base block B is called *full* if its stabilizer is trivial, otherwise it is called *short*. The stabilizer of B is denoted as S_B.

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Let H be a subgroup of order h of an abelian group G of order u. A collection $\{B_1, B_2, \ldots, B_t\}$ of k-subsets (called *base blocks*) of G forms a (u, h, k, λ) *difference family over G and relative to H with α short base blocks* if $\bigcup_{i=1}^{t} \partial B_i$ covers each element of $G-H$ exactly λ times but no element in H, and there are exactly α short base blocks, where $\partial B = \frac{1}{|S_B|}\{a-b : a, b \in B, a \neq b\}$. We denote such a design as $(u, h, k, \lambda)_\alpha$-DF. When the value of short base blocks is not specified, the design is denoted as (u, h, k, λ)-DF. Observe that if k is a prime and G is cyclic, then we could have short base blocks only when k is a divisor of u but not of h. For simplicity, our definition is just a special case of difference families. For general information of difference families, the readers refer to [1]. Note that the base blocks of a $(u, \{h, k_\alpha\}, k, \lambda)$-DF defined in [13] together with exactly α short base blocks $\{0, gv/3, 2gv/3\}$ form a $(u, h, k, \lambda)_\alpha$-DF, and $(u, h, k, 1)_1$-DF is denoted as $(u, \{h, k\}, k, 1)$-DF in [2].

It is not difficult to see that the existence of a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} is equivalent to the existence of a cyclic $(3, \lambda)$-GDD of type g^v with α short block orbits. For a cyclic $(3, \lambda)$-GDD of type g^v, the possible short orbit must be generated by $\{0, gv/3, 2gv/3\}$. Therefore in what follows, we only display the full base blocks for a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv}.

In [7], it is proved by Jiang that there exists a $(gv, g, 3, 1)_0$-DF over Z_{gv} when $g \equiv 0 \pmod{12}$ and $v > 4$, or $g \equiv 6 \pmod{12}$, $v \equiv 0, 1 \pmod 4$ and $v > 4$.

In this paper, we should pay special attention to check those DFs constructed in [12,13] to see whether they are suitable for our purpose. The technique will be implemented all through this paper. Now we need to obtain the necessary conditions for the existence of a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv}.

Lemma 1.1. *If there exists a* $(gv, g, 3, \lambda)_\alpha$-DF *over* Z_{gv}, *then* $v \neq 2, \lambda = \alpha$ *when* $(g, v) = (1, 3), \lambda = 2\alpha$ *when* $(g, v) = (2, 3), \lambda = 4\alpha$ *when* $(g, v) = (1, 6), \lambda \geq 2\alpha$ *when* $(g, v) = (2, 6), \lambda \equiv 0 \pmod 3$ *when* $(g, v) = (1, 9)$ *and* $\lambda = \alpha$.

Proof. Suppose that there exists a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} for $(g, v) = (1, 6)$, then all of the differences in the multiset $\lambda\{1\} \cup (\lambda - \alpha)\{2\} \cup \lambda/2\{3\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{gv}$ except $\{gv/3, gv/3, gv/3\} = \{2, 2, 2\}$. Clearly, the possible triples are the forms of $\{1, 2, 3\}$ and $\{1, 1, 2\}$. From $\lambda - \lambda/2 = 2(\lambda - \alpha - \lambda/2)$, we have $\lambda = 4\alpha$.

Similar to the case $\lambda = 4\alpha$ when $(g, v) = (1, 6)$, we can get the assertion for the other cases. □

By a similar argument as Theorem 3.1 in [10], we can show the following result.

Lemma 1.2. *If a* $(3g, g, 3, \lambda)_\alpha$-DF *over* Z_{3g} *exists, then* $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod 6$.

Proof. When $\alpha = 0$ or $g \equiv 1 \pmod 3$, suppose that there exists a $(3g, g, 3, \lambda)_\alpha$-DF over Z_{3g}. The full base blocks are $\{0, 3a_i + 1, 3b_i + 2\}$, where $a_i, b_i \in [0, g - 1]$ for $1 \leq i \leq (\lambda g - \alpha)/3$. Each base block covers the differences $\{\pm(3x + 1) : x = a_i, b_i - a_i, g - b_i - 1\}$. All of the $(\lambda g - \alpha)/3$ base blocks together cover the difference $\pm(3x + 1)$ for each $x \in \lambda\{0, 1, \ldots, g - 1\} \setminus \alpha\{(g - 1)/3\}$. Note that $a_i + (b_i - a_i) + (g - b_i - 1) \equiv -1 \pmod g$. So we get $-(\lambda g - \alpha)/3 \equiv \lambda\left(\sum_{x=0}^{g-1} x\right) - \alpha(g - 1)/3 \pmod g$. Then we have $\lambda(3g - 1) - 2\alpha \equiv 0 \pmod 6$, i.e., $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod 6$.

When $g \equiv 2 \pmod 3$, similarly we get $-(\lambda g - \alpha)/3 \equiv \lambda\left(\sum_{x=0}^{g-1} x\right) - \alpha(2g - 1)/3 \pmod g$. So we conclude that $\lambda(3g - 1) - 4\alpha \equiv 0 \pmod 6$, that is $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod 6$. □

The proof of Lemma 1.3 is similar to that of Lemma 2 in [5].

Lemma 1.3. *If there exists a* $(gv, g, 3, \lambda)_\alpha$-DF *over* Z_{gv}, *then* $v \not\equiv 2, 3 \pmod 4$ *when* $g \equiv 2 \pmod 4$ *and* $\lambda \equiv 1 \pmod 2$; $v \not\equiv 2 \pmod 4$ *when* $g \equiv 1 \pmod 2$ *and* $\lambda \equiv 2 \pmod 4$.

For $\alpha \in [0, \lambda]$, an obvious necessary condition for the existence of a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} is $\lambda g(v - 1) - 2\alpha \equiv 0 \pmod 6$, and $3 \mid v$ but $3 \nmid g$ when $\alpha \neq 0$. Combining Lemmas 1.1–1.3, we get the following necessary conditions for the existence of a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} for $\alpha \leq \lambda$.

Lemma 1.4. *If there exists a* $(gv, g, 3, \lambda)_\alpha$-DF *over* Z_{gv}, *then*

(1) $\lambda g(v - 1) - 2\alpha \equiv 0 \pmod 6$, $v \neq 2$;
(2) $v \not\equiv 2, 3 \pmod 4$ *when* $g \equiv 2 \pmod 4$ *and* $\lambda \equiv 1 \pmod 2$;
(3) $v \not\equiv 2 \pmod 4$ *when* $g \equiv 1 \pmod 2$ *and* $\lambda \equiv 2 \pmod 4$;
(4) $g \not\equiv 0 \pmod 3$ *and* $v \equiv 0 \pmod 3$ *when* $\alpha \neq 0$;
(5) $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod 6$ *when* $v = 3$;
(6) $\lambda = \alpha$ *when* $(g, v) = (1, 3), \lambda = 2\alpha$ *when* $(g, v) = (2, 3), \lambda = 4\alpha$ *when* $(g, v) = (1, 6), \lambda \geq 2\alpha$ *when* $(g, v) = (2, 6), \lambda \equiv 0 \pmod 3$ *when* $(g, v) = (1, 9)$ *and* $\lambda = \alpha$.

The rest of this paper are organized as follows. In Section 2, we introduce some useful recursive constructions. In Section 3, we investigate the existence of a $(gv, g, 3, \lambda)_0$-DF over Z_{gv}. In Section 4, we establish the necessary and sufficient conditions for the existence of a $(gv, g, 3, 3)_3$-DF over Z_{gv}. In Section 5, we construct a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} for some α and λ which will be used in the next section. Finally in Section 6, we complete the existence spectrum of a cyclic $(3, \lambda)$-GDD of type g^v having α short orbits.

2. Recursive constructions

In this section, we describe some useful recursive constructions that will be required in Sections 3–5. We first introduce the following definition of perfect difference family from [3].

Let g be a divisor of v such that $v = gv_0$. Suppose that $\mathscr{F} = \{B_i : i = 1, 2, \ldots, t\}$ is the family of base blocks of a $(hv, hg, k, \lambda)_0$-DF over Z_{hv} where $B_i = \{0, b_{1i}, b_{2i}, \ldots, b_{k-1,i}\}$ for $i = 1, 2, \ldots, t$. Define $\mathrm{ele}(\mathscr{F}) = \cup_{i=1}^{t}\{b_{1i}, b_{2i}, \ldots, b_{k-1,i}\}$. The $(hv, hg, k, \lambda)_0$-DF over Z_{hv} is said to be *h-perfect*, denoted by (hv, hg, k, λ)-h-PDF over Z_{hv}, if $\mathrm{ele}(\mathscr{F}) \subseteq \left\{a + bv : 0 \le a \le \left\lfloor \frac{v}{2} \right\rfloor, a \ne 0, v_0, 2v_0, \ldots, (g-1)v_0, b = 0, 1, \ldots, h-1\right\}$. When $h = 1$, write (hv, hg, k, λ)-1-PDF over Z_{hv} briefly as (v, g, k, λ)-PDF over Z_v.

Let $(G, \cdot)$ be a finite group of order v and H a subgroup of order h in G. An *H-regular* $(v, k; \lambda)$*-incomplete difference matrix* over G is a $k \times (v-h)\lambda$ matrix $D = (d_{ij})$, $0 \le i \le k-1$, $1 \le j \le \lambda(v-h)$, with entries from G, such that for any $0 \le i < j \le k-1$, the multiset $\{d_{il} \cdot d_{jl}^{-1} : 1 \le l \le \lambda(v-h)\}$ contains every element of $G \setminus H$ exactly λ times. When G is an abelian group, typically additive notation is used, so that the differences $d_{il} - d_{jl}$ are employed. In what follows, we assume that $G = Z_v$, and H is a subgroup of order h in Z_v. Then $H = \{iv/h : 0 \le i \le h-1\}$. We usually denote an H-regular $(v, k; \lambda)$-incomplete difference matrix over Z_v by h-regular ICDM$(k, \lambda; v)$ if $|H| = h$. When $H = \emptyset$ or $h = 0$, an H-regular $(v, k; \lambda)$-incomplete difference matrix over Z_v is termed as CDM$(k, \lambda; v)$. When $\lambda = 1$, write h-regular ICDM$(k, 1; v)$ (or CDM$(k, 1; v)$) briefly as h-regular ICDM$(k; v)$ (or CDM$(k; v)$, respectively). The following simple result can be found in [6] (also see [8]). For more general results on difference matrices the readers refer to [4].

Lemma 2.1 (*[6]*). *Let v and k be positive integers such that* $\gcd(v, (k-1)!) = 1$. *Let $d_{ij} \equiv ij \pmod v$ for $i = 0, 1, \ldots, k-1$ and $j = 0, 1, \ldots, v-1$. Then $D = (d_{ij})$ is a* CDM$(k; v)$. *In particular, if v is an odd prime number, then there exists a* CDM$(k; v)$ *for integer k, $2 \le k \le v$.*

Since there exists a 2-regular ICDM$(4; 2^n)$ for $n \ge 3$ from Lemma 3.6 in [3], the following fact is evidently true.

Lemma 2.2. *There exists a* 2*-regular* ICDM$(3; 2^n)$ *for any integer* $n \ge 3$.

Theorem 2.3 (*[3, Theorem 2.5]*). *If there are a* (v, g, k, λ)-PDF *over* Z_v, *a* (hv, hg, k, λ)-h-PDF *over* Z_{hv} *and an* h*-regular* ICDM$(k; m)$, *then there is an* (mv, mg, k, λ)-m-PDF *over* Z_{mv}.

Theorems 2.4 and 2.5 can be derived with similar technique as Construction 4.1 in [15]. Here we only exhibit the results and omit their proofs.

Theorem 2.4. *If there are a* $(ghv, gh, 3, \lambda)_0$-DF *over* Z_{ghv} *and a* $(gh, g, 3, \lambda)_\alpha$-DF *over* Z_{gh}, *then there is a* $(ghv, g, 3, \lambda)_\alpha$-DF *over* Z_{ghv}.

Theorem 2.5. *If there are a* $(ghv, gh, 3, \lambda)_\alpha$-DF *over* Z_{ghv} *and a* $(gh, g, 3, \lambda)_0$-DF *over* Z_{gh}, *then there is a* $(ghv, g, 3, \lambda)_\alpha$-DF *over* Z_{ghv}.

The following construction serves to combine known DFs into a new one. The proof is similar to that of Construction 4.2 in [15].

Theorem 2.6. *If there are a* $(gv, g, 3, \lambda)_\alpha$-DF *over* Z_{gv}, *a* $(3m, m, 3, \alpha)_\alpha$-DF *over* Z_{3m} *and a* CDM$(3; m)$, *then there is an* $(mgv, mg, 3, \lambda)_\alpha$-DF *over* Z_{mgv}.

Proof. Suppose that $\mathscr{F}, \mathscr{E}$ be the families of full base blocks of the given $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} and $(3m, m, 3, \alpha)_\alpha$-DF over Z_{3m}, respectively. Let $D = (d_{ij})$ be a CDM$(3; m)$ where $d_{ij} \in Z_m$ for $0 \le i \le 2$ and $0 \le j \le m-1$. For each base block $A = \{0, a_1, a_2\} \in \mathscr{F}$ we take m base blocks $A_j = \{0, a_1 + gvd_{1j}, a_2 + gvd_{2j}\}$ for $j = 0, 1, \ldots, m-1$, where the additive operation is performed in Z_{mgv}. For each $B = \{0, b_1, b_2\} \in \mathscr{E}$ we take one base block $uB = \{0, ub_1, ub_2\} \pmod{mgv}$ where $u = gv/3$. It can be checked that the family $\{A_j : A \in \mathscr{F}, j = 0, 1, \ldots, m-1\} \cup \{uB : B \in \mathscr{E}\}$ forms the full base blocks of the desired $(mgv, mg, 3, \lambda)_\alpha$-DF over Z_{mgv}. □

The following result is a corollary of Theorem 2.6 with $\alpha = 0$.

Theorem 2.7. *Suppose that both a* $(v, g, 3, \lambda)_0$-DF *over* Z_v *and a* CDM$(3; m)$ *exist. Then there exists an* $(mv, mg, 3, \lambda)_0$-DF *over* Z_{mv}.

3. $(gv, g, 3, \lambda)_0$-DFs

In [12], it is shown that the necessary and sufficient conditions for the existence of a $(gv, g, 3, \lambda)$-DF over Z_{gv} are (1) $\lambda g(v-1) \equiv 0 \pmod 6$, $v \ne 2$; (2) $\lambda \equiv 0 \pmod 6$, or $\lambda \equiv 3 \pmod 6$ and $g \equiv 1 \pmod 2$ when $v = 3$; (3) $v \not\equiv 2, 3 \pmod 4$ when $g \equiv 2 \pmod 4$ and $\lambda \equiv 1 \pmod 2$; (4) $v \not\equiv 2 \pmod 4$ when $g \equiv 1 \pmod 2$ and $\lambda \equiv 2 \pmod 4$. Note that $\{0, gv/3, 2gv/3\}$ may be contained in the base blocks of a $(gv, g, 3, \lambda)$-DF over Z_{gv} only if $g \not\equiv 0 \pmod 3$, $v \equiv 0 \pmod 3$ and $\lambda \equiv 0 \pmod 3$. Therefore, in order to present the sufficiency for a $(gv, g, 3, \lambda)_0$-DF over Z_{gv}, we only need to consider the conditions of $g \not\equiv 0 \pmod 3$, $v \equiv 0 \pmod 3$ and $\lambda \equiv 0 \pmod 3$ within Lemma 1.4.

The $(gv, g, 3, 3)$-DFs in Z_{gv} from Lemmas 4.2, 4.4 and 4.5 of [12] contain no base block $\{0, gv/3, 2gv/3\}$ and hence we have the following result.

Lemma 3.1 *([12])*.

(1) *There exists a* $(2v, 2, 3, 3)_0$-DF *over* Z_{2v} *for* $v \equiv 0, 9 \pmod{12}$*;*
(2) *There exists a* $(8v, 8, 3, 3)$-*2*-PDF *over* Z_{8v} *for* $v \equiv 0 \pmod 3$ *and* $v > 3$*;*
(3) *There exists a* $(16v, 16, 3, 3)_0$-DF *over* Z_{16v} *for* $v \equiv 0 \pmod 3$ *and* $v > 3$.

The following Lemmas 3.2 and 3.3 are proved in [14].

Lemma 3.2. (1) *For* $g \equiv 1, 5 \pmod 6$ *and* $g > 1$, *there exists a* $(3g, g, 3, 3)_0$-DF *over* Z_{3g}.
(2) *For* $v \equiv 3 \pmod 6$ *and* $v > 3$, *there exists a* $(v, 1, 3, 3)_0$-DF *over* Z_v.
(3) *For* $v \equiv 0 \pmod{12}$, *there exists a* $(v, 1, 3, 6)_0$-DF *over* Z_v.
(4) *For* $v \equiv 6 \pmod{12}$ *and* $v > 6$, *there exists a* $(v, 1, 3, 12)_0$-DF *over* Z_v.
(5) *For* $g \equiv 2, 4 \pmod 6$ *and* $g > 2$, *there exists a* $(3g, g, 3, 6)_0$-DF *over* Z_{3g}.
(6) *For* $g \equiv 1, 5 \pmod 6$ *and* $g > 1$, *there exists a* $(6g, g, 3, 12)_0$-DF *over* Z_{6g}.

Lemma 3.3. *There exists a* $(4v, 4, 3, 3)$-PDF *over* Z_{4v} *which is also a* $(2v, 2, 3, 6)_0$-DF *over* Z_{2v} *for* $v \equiv 0 \pmod 3$ *and* $v > 3$.

Lemma 3.4. *A* $(gv, g, 3, 3)_0$-DF *over* Z_{gv} *exists for*

(1) $v \equiv 3 \pmod 6$ *when* $g \equiv 1, 5 \pmod 6$, $(g, v) \neq (1, 3)$*;*
(2) $v \equiv 0, 9 \pmod{12}$ *when* $g \equiv 2, 10 \pmod{12}$*;*
(3) $v \equiv 0 \pmod 3$ *and* $v > 3$ *when* $g \equiv 4, 8 \pmod{12}$.

Proof. (1) When $v = 3$, a $(3g, g, 3, 3)_0$-DF over Z_{3g} exists by Lemma 3.2 for $g \equiv 1, 5 \pmod 6$ and $g > 1$. When $v \equiv 3 \pmod 6$ and $v > 3$, since a $(v, 1, 3, 3)_0$-DF over Z_v exists from Lemma 3.2, applying Theorem 2.7 with a CDM$(3; g)$ from Lemma 2.1, we can get a $(gv, g, 3, 3)_0$-DF over Z_{gv}.

(2) By Lemma 3.1(1), we know that a $(2v, 2, 3, 3)_0$-DF over Z_{2v} exists for $v \equiv 0, 9 \pmod{12}$, hence we apply Theorem 2.7 with a CDM$(3; g/2)$ from Lemma 2.1 to obtain a $(gv, g, 3, 3)_0$-DF over Z_{gv}.

(3) First we prove that a $(2^n v, 2^n, 3, 3)_0$-DF over $Z_{2^n v}$ exists for $v \equiv 0 \pmod 3$, $v > 3$ and $n \geq 2$. For $n = 2, 3, 4$, the conclusion follows by Lemmas 3.3 and 3.1(2) and (3). For $n \geq 5$, since a $(4v, 4, 3, 3)$-PDF over Z_{4v} and a $(8v, 8, 3, 3)$-2-PDF over Z_{8v} exist by Lemmas 3.3 and 3.1(2), applying Theorem 2.3 with a 2-regular ICDM$(3; 2^{n-2})$ from Lemma 2.2 gives a $(2^n v, 2^n, 3, 3)$-2^{n-2}-PDF over $Z_{2^n v}$. That is a $(2^n v, 2^n, 3, 3)_0$-DF over $Z_{2^n v}$.

When $g \equiv 4, 8 \pmod{12}$, g can be written as $g = 2^n g'$ where $n \geq 2$ and g' is odd. Start with a $(2^n v, 2^n, 3, 3)_0$-DF over $Z_{2^n v}$ and apply Theorem 2.7 with a CDM$(3; g')$ from Lemma 2.1 to get a $(2^n g' v, 2^n g', 3, 3)_0$-DF over $Z_{2^n g' v}$ for $n \geq 2$, odd integer g' and $v > 3$, which conclude to a $(gv, g, 3, 3)_0$-DF over Z_{gv}. □

Now we give the necessary and sufficient conditions for the existence of a $(gv, g, 3, \lambda)_0$-DF over Z_{gv}.

Lemma 3.5. *A* $(gv, g, 3, \lambda)_0$-DF *over* Z_{gv} *exists if and only if*

(1) $\lambda g(v - 1) \equiv 0 \pmod 6$, $v \neq 2$*;*
(2) $v \not\equiv 2, 3 \pmod 4$ *when* $g \equiv 2 \pmod 4$ *and* $\lambda \equiv 1 \pmod 2$*;*
(3) $v \not\equiv 2 \pmod 4$ *when* $g \equiv 1 \pmod 2$ *and* $\lambda \equiv 2 \pmod 4$*;*
(4) $\lambda(3g - 1) \equiv 0 \pmod 6$ *when* $v = 3$*;*
(5) $(g, v) \neq (1, 3), (2, 3), (1, 6)$.

Proof. The necessity follows from Lemma 1.4. For the sufficiency, we only need to prove the existence of a $(gv, g, 3, \lambda)_0$-DF over Z_{gv} when $g \not\equiv 0 \pmod 3$, $v \equiv 0 \pmod 3$ and $\lambda \equiv 0 \pmod 3$ in the following three cases.

Case 1: $g \equiv 1, 5 \pmod 6$. When $\lambda \equiv 0 \pmod 3$, $v \equiv 3 \pmod 6$ and $(g, v) \neq (1, 3)$, repeat the base blocks of a $(gv, g, 3, 3)_0$-DF over Z_{gv} $\lambda/3$ times from Lemma 3.4(1). When $\lambda \equiv 0 \pmod 6$ and $v \equiv 0 \pmod{12}$, since a $(v, 1, 3, 6)_0$-DF over Z_v exists from Lemma 3.2, applying Theorem 2.7 with a CDM$(3; g)$ from Lemma 2.1 we obtain a $(gv, g, 3, 6)_0$-DF over Z_{gv}. Then repeat the base blocks of a $(gv, g, 3, 6)_0$-DF over Z_{gv} $\lambda/6$ times. When $\lambda \equiv 0 \pmod{12}$, $v \equiv 6 \pmod{12}$ and $v > 6$, we apply Theorem 2.7 with a $(v, 1, 3, 12)_0$-DF over Z_v from Lemma 3.2 and a CDM$(3; g)$ from Lemma 2.1 to get a $(gv, g, 3, 12)_0$-DF over Z_{gv}, and then repeat the base blocks of a $(gv, g, 3, 12)_0$-DF over Z_{gv} $\lambda/12$ times. When $\lambda \equiv 0 \pmod{12}$, $g > 1$ and $v = 6$, repeat the base blocks of a $(6g, g, 3, 12)_0$-DF over Z_{6g} $\lambda/12$ times from Lemma 3.2.

Case 2: $g \equiv 2, 10 \pmod{12}$. When $\lambda \equiv 3 \pmod 6$ and $v \equiv 0, 9 \pmod{12}$, repeat the base blocks of a $(gv, g, 3, 3)_0$-DF over Z_{gv} $\lambda/3$ times from Lemma 3.4(2). When $\lambda \equiv 0 \pmod 6$, $v \equiv 0 \pmod 3$ and $v > 3$, since a $(2v, 2, 3, 6)_0$-DF over Z_{2v} exists from Lemma 3.3, we apply Theorem 2.7 with a CDM$(3; g/2)$ from Lemma 2.1 to obtain a $(gv, g, 3, 6)_0$-DF over Z_{gv}, and then repeat the base blocks of a $(gv, g, 3, 6)_0$-DF over Z_{gv} $\lambda/6$ times. When $\lambda \equiv 0 \pmod 6$, $g > 2$ and $v = 3$, repeat the base blocks of a $(3g, g, 3, 6)_0$-DF over Z_{3g} $\lambda/6$ times from Lemma 3.2.

Case 3: $g \equiv 4, 8 \pmod{12}$. When $\lambda \equiv 0 \pmod 3$, $v \equiv 0 \pmod 3$ and $v > 3$, repeat the base blocks of a $(gv, g, 3, 3)_0$-DF over Z_{gv} $\lambda/3$ times from Lemma 3.4(3). When $\lambda \equiv 0 \pmod 6$ and $v = 3$, repeat the base blocks of a $(3g, g, 3, 6)_0$-DF over Z_{3g} $\lambda/6$ times from Lemma 3.2. This completes the proof. □

4. $(gv, g, 3, 3)_3$-DFs

By Lemma 1.4, the necessary conditions for the existence of a $(gv, g, 3, 3)_3$-DF over Z_{gv} are: (1) $v \equiv 3 \pmod 6$ when $g \equiv 1, 5 \pmod 6$; (2) $v \equiv 0 \pmod 3$ and $v > 3$ when $g \equiv 4, 8 \pmod{12}$; (3) $v \equiv 0, 9 \pmod{12}$ when $g \equiv 2, 10 \pmod{12}$. In this section, we are mainly to prove that the necessary conditions for the existence of a $(gv, g, 3, 3)_3$-DF over Z_{gv} are also sufficient.

The following Lemma 4.1 is proved in [14].

Lemma 4.1. (1) *For* $v \equiv 9 \pmod{12}$, *there exists a* $(2v, 2, 3, 3)_3$-DF *over* Z_{2v}.
(2) *For* $v \equiv 3 \pmod 6$ *and* $v > 3$, *there exists a* $(8v, 8, 3, 3)_3$-DF *over* Z_{8v}.
(3) *For* $g \equiv 5 \pmod 6$, *there exists a* $(3g, g, 3, 3)_3$-DF *over* Z_{3g}.
(4) *For* $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 3$ *and* $6 \le v \le 21$, *there exists a* $(gv, g, 3, 3)_3$-DF *over* Z_{gv}.

Lemma 4.2. *For* $v \equiv 3 \pmod 6$ *and* $v \ge 27$, *there exist ordered pairs* (x_l, y_l), $1 \le l \le v-1$, *such that* $y_l - x_l \in [4v/3+1, 10v/3-1]_o \setminus \{3v\}$, $x_l \in ([2v/3+1, 5v/3] \setminus \{v, v+1\}) \cup \{1\}$, $y_l \in [3v+1, 4v-2] \cup \{3v-1\}$.

Proof. Let $v = 6s+3$ where $s \ge 4$. Then for $1 \le l \le 6s+2$, $y_l - x_l \in [8s+5, 20s+9]_o \setminus \{18s+9\}$, $x_l \in ([4s+3, 10s+5] \setminus \{6s+3, 6s+4\}) \cup \{1\}$, $y_l \in [18s+10, 24s+10] \cup \{18s+8\}$. The desired ordered pairs (x_l, y_l) are listed below:

- $s \equiv 0 \pmod 4$ and $s \ge 4$:
  $(1, 20s+10)$, $(5s+2, 20s+9)$, $(5s+3, 22s+14)$, $(11s/2+2, 43s/2+13)$, $(11s/2+3, 43s/2+12)$, $(6s+5, 18s+8)$,
  $(10s+5-r, 18s+10+r)$, $r \in [0, 2s-2]$,
  $(5s+1-r, 23s+12+r)$, $r \in [0, s-2]$,
  $(8s+6-r, 20s+11+r)$, $r \in [0, 3s/2]$,
  $(13s/2+5-r, 43s/2+14+r)$, $r \in [0, s/2-1]$,
  $(6s+2-2r, 22s+15+2r)$, $r \in [0, s/4-1]$,
  $(6s+1-2r, 22s+16+2r)$, $r \in [0, s/4-2]$ ($r \in \emptyset$ when $s = 4$),
  $(11s/2+1-2r, 45s/2+14+2r)$, $r \in [0, s/4-2]$ ($r \in \emptyset$ when $s = 4$),
  $(11s/2-2r, 45s/2+15+2r)$, $r \in [0, s/4-2]$ ($r \in \emptyset$ when $s = 4$).
- $s \equiv 2 \pmod 4$ and $s \ge 6$:
  $(1, 20s+10)$, $(5s+2, 20s+9)$, $(5s+3, 22s+14)$, $(11s/2+2, 43s/2+13)$, $(11s/2+3, 43s/2+12)$, $(6s+5, 18s+8)$,
  $(10s+5-r, 18s+10+r)$, $r \in [0, 2s-2]$,
  $(5s+1-r, 23s+12+r)$, $r \in [0, s-2]$,
  $(8s+6-r, 20s+11+r)$, $r \in [0, 3s/2]$,
  $(13s/2+5-r, 43s/2+14+r)$, $r \in [0, s/2-1]$,
  $(6s+2-2r, 22s+15+2r)$, $r \in [0, (s-6)/4]$,
  $(6s+1-2r, 22s+16+2r)$, $r \in [0, (s-6)/4]$,
  $(11s/2+1-2r, 45s/2+14+2r)$, $r \in [0, (s-6)/4]$,
  $(11s/2-2r, 45s/2+15+2r)$, $r \in [0, (s-10)/4]$ ($r \in \emptyset$ when $s = 6$).
- $s \equiv 1 \pmod 4$ and $s \ge 5$:
  $(1, 20s+10)$, $(5s+2, 22s+14)$, $(5s+3, 20s+9)$, $((11s+3)/2, (43s+25)/2)$, $((11s+5)/2, (43s+23)/2)$, $(6s+5, 18s+8)$,
  $(10s+5-r, 18s+10+r)$, $r \in [0, 2s-2]$,
  $(5s+1-r, 23s+12+r)$, $r \in [0, s-2]$,
  $(8s+6-r, 20s+11+r)$, $r \in [0, (3s-1)/2]$,
  $((13s+11)/2-r, (43s+27)/2+r)$, $r \in [0, (s-1)/2]$,
  $(6s+2-2r, 22s+15+2r)$, $r \in [0, (s-5)/4]$,
  $(6s+1-2r, 22s+16+2r)$, $r \in [0, (s-5)/4]$,
  $((11s+1)/2-2r, (45s+29)/2+2r)$, $r \in [0, (s-9)/4]$ ($r \in \emptyset$ when $s = 5$),
  $((11s-1)/2-2r, (45s+31)/2+2r)$, $r \in [0, (s-9)/4]$ ($r \in \emptyset$ when $s = 5$).
- $s \equiv 3 \pmod 4$ and $s \ge 7$:
  $(1, 20s+10)$, $(5s+2, 22s+14)$, $(5s+3, 20s+9)$, $((11s+3)/2, (43s+25)/2)$, $((11s+5)/2, (43s+23)/2)$, $(6s+5, 18s+8)$,
  $(10s+5-r, 18s+10+r)$, $r \in [0, 2s-2]$,
  $(5s+1-r, 23s+12+r)$, $r \in [0, s-2]$,
  $(8s+6-r, 20s+11+r)$, $r \in [0, (3s-1)/2]$,
  $((13s+11)/2-r, (43s+27)/2+r)$, $r \in [0, (s-1)/2]$,
  $(6s+2-2r, 22s+15+2r)$, $r \in [0, (s-3)/4]$,
  $(6s+1-2r, 22s+16+2r)$, $r \in [0, (s-7)/4]$,
  $((11s+1)/2-2r, (45s+29)/2+2r)$, $r \in [0, (s-7)/4]$,
  $((11s-1)/2-2r, (45s+31)/2+2r)$, $r \in [0, (s-11)/4]$ ($r \in \emptyset$ when $s = 7$). □

Lemma 4.3. *There exists a* $(gv, g, 3, 3)_3$-DF *over* Z_{gv} *for* $g \equiv 8 \pmod{12}$, $v \equiv 3 \pmod 6$ *and* $v > 3$.

Proof. For $g \equiv 8 \pmod{12}$ and $v = 9, 15, 21$, the conclusion follows by Lemma 4.1. For $g = 8$, $v \equiv 3 \pmod 6$ and $v > 3$, the conclusion follows by Lemma 4.1. For $g \equiv 8 \pmod{12}$, $g \geq 20$, $v \equiv 3 \pmod 6$ and $v \geq 27$, let $g = 12t + 8$ where $t \geq 1$. Let (x_l, y_l) be the ordered pairs obtained in Lemma 4.2 for $1 \leq l \leq v - 1$. The desired base blocks are as follows.
$\{0, 6, 12\}$, $2\{0, 8, (3t + 2)v + 4\}$, $2\{0, 11, (15t + 10)v/3 + 6\}$,
$\{0, 6, (3t + 2)v + 2\}$, $2\{0, 10, (3t + 2)v + 5\}$, $3\{0, 3, (15t + 10)v/3 + 2\}$,
$\{0, 8, (3t + 2)v + 5\}$, $2\{0, 12, (3t + 2)v + 6\}$, $3\{0, 7, (15t + 10)v/3 + 4\}$,
$\{0, 10, (3t + 2)v + 4\}$, $3\{0, 4, (3t + 2)v + 3\}$, $3\{0, 9, (15t + 10)v/3 + 5\}$,
$\{0, 11, (3t + 2)v + 6\}$, $3\{0, 1, (12t + 8)v/3 - 1\}$, $\{0, 2, (15t + 10)v/3 + 3\}$,
$2\{0, 5, (3t + 2)v + 2\}$, $2\{0, 2, (15t + 10)v/3\}$, $\{0, 5, (15t + 10)v/3\}$,
$3\{0, 2v/3 + 1, (2t + 2)v - 1\}$, $\{0, (6t + 4)v/3 - 4, (21t + 14)v/3 - 6\}$,
$3\{0, v - 2, (5t + 3)v - 1\}$, $2\{0, (6t + 4)v/3 - 4, (21t + 14)v/3 - 3\}$,
$3\{0, 4v/3 - 1, (6t + 4)v/3 + 1\}$, $3\{0, (6t + 4)v/3, ((42t + 25)v + 3)/6\}$,
$3\{0, (3t + 2)v - 2, (6t + 4)v - 1\}$, $3\{0, (6t + 4)v/3 + 2, ((42t + 25)v + 9)/6\}$,
$3\{0, 14 + 2j, (3t + 2)v + 7 + j\}$, $j \in [0, (3t + 2)v/3 - 10] \setminus \{tv - 6\}$, and $j \not\equiv v - 7 \pmod v$,
$3\{0, 13 + 2j, (15t + 10)v/3 + 7 + j\}$, $j \in [0, v/3 - 7]$,
$3\{0, 2v/3 + 3 + 2j, (15t + 11)v/3 + 1 + j\}$, $j \in [0, v/3 - 3] \setminus \{(v - 15)/6, (v - 9)/6\}$,
and
$3\{0, (2t - 2)v + (y_l - x_l) - 2jv, 6tv + y_l - jv\}$, where $l \in [1, v - 1]$ and $j \in [0, t - 1]$. □

By checking with Lemmas 2.18, 3.8, 3.9, 3.11 and 3.4 of [13], we have the following results which will be used later.

Lemma 4.4 (*[13]*).

(1) *There exists a* $(gv, g, 3, 1)_1$-DF *over* Z_{gv} *for* $g \equiv 4 \pmod{12}$, $v \equiv 0 \pmod 3$ *and* $v > 3$, *or* $g \equiv 10 \pmod{12}$ *and* $v \equiv 0, 9 \pmod{12}$, *or* $g \equiv 1 \pmod 6$, $v \equiv 3 \pmod 6$ *and* $(g, v) \neq (1, 9)$*;*
(2) *There exists a* $(gv, g, 3, 2)_1$-DF *over* Z_{gv} *for* $g \equiv 2 \pmod 6$ *and* $v \equiv 0 \pmod 3$, *or* $g \equiv 5 \pmod 6$, $v \equiv 0 \pmod 3$ *and* $v \not\equiv 2 \pmod 4$*;*
(3) *There exists a* $(gv, g, 3, 2)_2$-DF *over* Z_{gv} *for* $g \equiv 4 \pmod 6$ *and* $v \equiv 0 \pmod 3$, *or* $g \equiv 1 \pmod 6$, $v \equiv 0 \pmod 3$, $v \not\equiv 2 \pmod 4$ *and* $(g, v) \neq (1, 9)$*;*
(4) *There exists a* $(gv, g, 3, 4)_2$-DF *over* Z_{gv} *for* $g \equiv 2 \pmod 3$ *and* $v \equiv 0 \pmod 3$*;*
(5) *There exists a* $(6g, g, 3, 4)_1$-DF *over* Z_{6g} *for* $g \equiv 1 \pmod 6$.

Now the necessary and sufficient conditions for the existence of a $(gv, g, 3, 3)_3$-DF over Z_{gv} are determined as follows.

Lemma 4.5. *A* $(gv, g, 3, 3)_3$-DF *over* Z_{gv} *exists if and only if*

(1) $v \equiv 3 \pmod 6$ *when* $g \equiv 1, 5 \pmod 6$*;*
(2) $v \equiv 0 \pmod 3$ *and* $v > 3$ *when* $g \equiv 4, 8 \pmod{12}$*;*
(3) $v \equiv 0, 9 \pmod{12}$ *when* $g \equiv 2, 10 \pmod{12}$.

Proof. The necessity follows from Lemma 1.4. So we establish the sufficiency as follows.

(1) For $g \equiv 1 \pmod 6$ and $v = 3$, the conclusion follows by repeating the base blocks of a $(gv, g, 3, 1)_1$-DF over Z_{gv} three times from Lemma 4.4(1). For $g \equiv 5 \pmod 6$ and $v = 3$, the result follows by Lemma 4.1. For $g \equiv 1, 5 \pmod 6$, $v \equiv 3 \pmod 6$ and $v \geq 9$, let $v = 3v'$ where $v' \equiv 1 \pmod 2$ and $v' \geq 3$. Since $3g \equiv 1 \pmod 2$, there exists a $(3gv', 3g, 3, 3)_0$-DF over $Z_{3gv'}$ by Lemma 3.5. That is a $(gv, 3g, 3, 3)_0$-DF over Z_{gv} for $v \equiv 3 \pmod 6$ and $v \geq 9$. Then we apply Theorem 2.4 with a $(3g, g, 3, 3)_3$-DF over Z_{3g} mentioned above to obtain a $(gv, g, 3, 3)_3$-DF over Z_{gv}.

(2) For $g \equiv 4 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$, repeating the base blocks of a $(gv, g, 3, 1)_1$-DF over Z_{gv} three times from Lemma 4.4(1), we can draw the conclusion. For $g \equiv 8 \pmod{12}$, $v \equiv 3 \pmod 6$ and $v > 3$, the conclusion follows from Lemma 4.3. For $g \equiv 8 \pmod{12}$ and $v = 6, 12, 18$, the desired DFs come from Lemma 4.1. For $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 6$ and $v > 18$, let $v = 6v'$, where $v' > 3$. Since $6g \equiv 0 \pmod 4$, we observe that there is a $(6gv', 6g, 3, 3)_0$-DF over $Z_{6gv'}$ by Lemma 3.5. That is a $(gv, 6g, 3, 3)_0$-DF over Z_{gv} for $v \equiv 0 \pmod 6$ and $v > 18$. Hence we use Theorem 2.4 with a $(6g, g, 3, 3)_3$-DF over Z_{3g} from Lemma 4.1 to get a $(gv, g, 3, 3)_3$-DF over Z_{gv}.

(3) For $g = 2$ and $v = 12$, the base blocks are $3\{0, 1, 11\}$, $\{0, 2, 6\}$, $2\{0, 2, 7\}$, $\{0, 3, 7\}$, $2\{0, 3, 9\}$, $\{0, 4, 9\}$. For $g = 2$, $v \equiv 0 \pmod{12}$ and $v > 12$, let $v = 4v'$ where $v' \equiv 0 \pmod 3$ and $v' > 3$. Since there exists a $(8v', 8, 3, 3)_3$-DF over $Z_{8v'}$ from (2), which is a $(2v, 8, 3, 3)_3$-DF over Z_{2v}. We apply Theorem 2.5 with a $(8, 2, 3, 3)_0$-DF over Z_8 from Lemma 3.5 to get a $(2v, 2, 3, 3)_3$-DF over Z_{2v}. For $g = 2$ and $v \equiv 9 \pmod{12}$, by Lemma 4.1 there exists a $(gv, g, 3, 3)_3$-DF over Z_{gv}. For $g \equiv 2, 10 \pmod{12}$, $g \geq 10$ and $v \equiv 0, 9 \pmod{12}$, g can be written as $g = 2g'$ where $g' \equiv 1, 5 \pmod 6$ and $g' \geq 5$. Start with a $(2v, 2, 3, 3)_3$-DF over Z_{2v} mentioned above and a $(3g', g', 3, 3)_3$-DF over $Z_{3g'}$ from (1), applying Theorem 2.6 with a CDM$(3; g')$ from Lemma 2.1, we obtain a $(2g'v, 2g', 3, 3)_3$-DF over $Z_{2g'v}$, which is a $(gv, g, 3, 3)_3$-DF over Z_{gv}. □

5. Some other constructions

In this section, we need to build certain classes of DFs for later use in Section 6. We first list some direct constructions from [14].

Lemma 5.1. (1) *For* $v \equiv 0 \pmod 3$ *and* $v > 9$, *there exists a* $(v, 1, 3, 4)_1$-DF *over* Z_v.
(2) *For* $v \equiv 0 \pmod 3$ *and* $v \geq 9$, *there exists a* $(2v, 2, 3, 6)_6$-DF *over* Z_{2v}.
(3) *For* $g \equiv 4 \pmod 6$, *there exists a* $(3g, g, 3, 4)_1$-DF *over* Z_{3g}.
(4) *For* $g \equiv 2 \pmod 6$ *and* $g > 2$, *there exists a* $(3g, g, 3, 6)_6$-DF *over* Z_{3g}.
(5) *For* $g \equiv 10 \pmod{12}$, *there exists a* $(6g, g, 3, 4)_1$-DF *over* Z_{6g}.
(6) *For* $g \equiv 5 \pmod 6$, $\alpha \in \{1, 7\}$, *there exists a* $(6g, g, 3, 8)_\alpha$-DF *over* Z_{6g}.
(7) *For* $g \equiv 1 \pmod 6$ *and* $g > 1$, *there exists a* $(6g, g, 3, 4)_4$-DF *over* Z_{6g}.
(8) *For* $g \equiv 2 \pmod{12}$ *and* $g > 2$, *there exists a* $(6g, g, 3, 6)_6$-DF *over* Z_{6g}.
(9) *For* $g \equiv 5 \pmod 6$, *there exists a* $(6g, g, 3, 12)_{12}$-DF *over* Z_{6g}.

Lemma 5.2. *There exists a* $(gv, g, 3, 4)_1$-DF *over* Z_{gv} *for* $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (1, 3)$.

Proof. First we deal with the case of $v = 3, 6, 9$. For $(g, v) = (1, 9)$, the base blocks are $2\{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 4\}, \{0, 1, 5\}$. For $g \equiv 1 \pmod 6$, $v = 3$ and $(g, v) \neq (1, 3)$, or $g \equiv 4 \pmod{12}$ and $v = 6$, or $g \equiv 1 \pmod 3$, $v = 9$ and $(g, v) \neq (1, 9)$, the conclusion follows by taking together the base blocks of a $(gv, g, 3, 1)_1$-DF over Z_{gv} from Lemma 4.4(1), and a $(gv, g, 3, 3)_0$-DF over Z_{gv} from Lemma 3.5. For $g \equiv 4 \pmod 6$ and $v = 3$, or $g \equiv 1, 7, 10 \pmod{12}$ and $v = 6$, a $(gv, g, 3, 4)_1$-DF over Z_{gv} exists from Lemmas 4.4(5) and 5.1.

Then the case of $v > 9$ can be solved as follows. For $g = 1$, $v \equiv 0 \pmod 3$ and $v > 9$, the required DF comes from Lemma 5.1. For $g \equiv 1 \pmod 3$, $g > 1$, $v \equiv 0 \pmod 3$ and $v > 9$, let $v = 3v'$ where $v' > 3$. Note that $3g \equiv 0 \pmod 3$, so by Lemma 3.5 there exists a $(3gv', 3g, 3, 4)_0$-DF over $Z_{3gv'}$, which is a $(gv, 3g, 3, 4)_0$-DF over Z_{gv}. Combining a $(3g, g, 3, 4)_1$-DF over Z_{3g} mentioned above, the existence of a $(gv, g, 3, 4)_1$-DF over Z_{gv} then follows immediately by Theorem 2.4. □

Lemma 5.3. *There exists a* $(gv, g, 3, 4)_4$-DF *over* Z_{gv} *for* $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (1, 6), (1, 9)$.

Proof. For $g \equiv 1 \pmod 6$, $v = 6$ and $(g, v) \neq (1, 6)$, the conclusion follows by Lemma 5.1. For $g \equiv 4 \pmod 6$ and $v \equiv 0 \pmod 3$, or $g \equiv 1 \pmod 6$, $v = 3, 9$ and $(g, v) \neq (1, 9)$, repeating the base blocks of a $(gv, g, 3, 2)_2$-DF over Z_{gv} twice from Lemma 4.4(3), we can obtain a $(gv, g, 3, 4)_4$-DF over Z_{gv}. For $g \equiv 1 \pmod 6$, $v \equiv 0 \pmod 3$ and $v > 9$, let $v = 3v'$ where $v' > 3$. $3g \equiv 0 \pmod 3$, so by Lemma 3.5 there exists a $(3gv', 3g, 3, 4)_0$-DF over $Z_{3gv'}$, which is a $(gv, 3g, 3, 4)_0$-DF over Z_{gv}. The conclusion follows by Theorem 2.4 with a $(3g, g, 3, 4)_4$-DF over Z_{3g} mentioned above. □

Lemma 5.4. *There exists a* $(gv, g, 3, 6)_3$-DF *over* Z_{gv} *for* $g \equiv 2, 4 \pmod 6$ *and* $v \equiv 0 \pmod 3$, *or* $g \equiv 1, 5 \pmod 6$, $v \equiv 0 \pmod 3$, $v \not\equiv 2 \pmod 4$ *and* $(g, v) \neq (1, 3)$.

Proof. For $(g, v) = (1, 9)$, a $(gv, g, 3, 6)_3$-DF over Z_{gv} is obtained by taking together the base blocks of a $(gv, g, 3, 3)_0$-DF from Lemma 3.5 and a $(gv, g, 3, 3)_3$-DF over Z_{gv} from Lemma 4.5. For $g \equiv 1 \pmod 6$, $v \equiv 0 \pmod 3$, $v \not\equiv 2 \pmod 4$ and $(g, v) \neq (1, 3), (1, 9)$, or $g \equiv 4 \pmod 6$ and $v \equiv 0 \pmod 3$, by taking together the base blocks of a $(gv, g, 3, 2)_2$-DF from Lemma 4.4(3) and a $(gv, g, 3, 4)_1$-DF over Z_{gv} from Lemma 5.2, we can obtain the desired design. For $g \equiv 5 \pmod 6$, $v \equiv 0 \pmod 3$ and $v \not\equiv 2 \pmod 4$, or $g \equiv 2 \pmod 6$ and $v \equiv 0 \pmod 3$, repeat the base blocks of a $(gv, g, 3, 2)_1$-DF over Z_{gv} three times from Lemma 4.4(2) to get the result. □

Lemma 5.5. *There exists a* $(gv, g, 3, 6)_6$-DF *over* Z_{gv} *for* $g \equiv 2, 4 \pmod 6$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (2, 3), (2, 6)$, *or* $g \equiv 1, 5 \pmod 6$, $v \equiv 0 \pmod 3$ *and* $v \not\equiv 2 \pmod 4$.

Proof. Case 1: $g \equiv 2, 4 \pmod 6$ and $v = 3, 6$. For $g \equiv 4 \pmod 6$ and $v = 3, 6$, we repeat the base blocks of a $(gv, g, 3, 2)_2$-DF over Z_{gv} three times from Lemma 4.4(3) to get the required design. For $g \equiv 2 \pmod 6$, $v = 3$ and $(g, v) \neq (2, 3)$, or $g \equiv 2 \pmod{12}$, $v = 6$ and $(g, v) \neq (2, 6)$, the result follows by Lemma 5.1. For $g \equiv 8 \pmod{12}$ and $v = 6$, repeating the base blocks of a $(gv, g, 3, 3)_3$-DF over Z_{gv} twice from Lemma 4.5, we can get a $(gv, g, 3, 6)_6$-DF over Z_{gv}.

Case 2: $g \equiv 2, 4 \pmod 6$ and $v \geq 9$. For $g = 2$, $v \equiv 0 \pmod 3$ and $v \geq 9$, the conclusion follows from Lemma 5.1. For $g \equiv 2, 4 \pmod 6$, $g \geq 4$, $v \equiv 0 \pmod 3$ and $v \geq 9$, let $v = 3v'$, where $v' \geq 3$. Since $3g \equiv 0 \pmod 2$, there exists a $(3gv', 3g, 3, 6)_0$-DF over $Z_{3gv'}$ for $v' \geq 3$ by Lemma 3.5. That is a $(gv, 3g, 3, 6)_0$-DF over Z_{gv} for $v \equiv 0 \pmod 3$ and $v \geq 9$. Applying Theorem 2.4 with a $(3g, g, 3, 6)_6$-DF over Z_{3g} from Case 1, we obtain a $(gv, g, 3, 6)_6$-DF over Z_{gv}.

Case 3: $g \equiv 1, 5 \pmod 6$. For $g \equiv 1, 5 \pmod 6$ and $v = 3$, repeating the base blocks of a $(gv, g, 3, 3)_3$-DF over Z_{gv} twice from Lemma 4.5 to get the result. For $g \equiv 1, 5 \pmod 6$, $v \equiv 0 \pmod 3$, $v \not\equiv 2 \pmod 4$ and $v \geq 9$, let $v = 3v'$ where $v' \not\equiv 2 \pmod 4$ and $v' \geq 3$. Since $3g \equiv 1 \pmod 2$, there exists a $(3gv', 3g, 3, 6)_0$-DF over $Z_{3gv'}$ for $v' \not\equiv 2 \pmod 4$ and $v' \geq 3$ by Lemma 3.5. That is a $(gv, 3g, 3, 6)_0$-DF over Z_{gv} for $v \equiv 0 \pmod 3$, $v \not\equiv 2 \pmod 4$ and $v \geq 9$. We now apply Theorem 2.4 with a $(3g, g, 3, 6)_6$-DF over Z_{3g} from above to get a $(gv, g, 3, 6)_6$-DF over Z_{gv}. □

Lemma 5.6. *There exists a* $(gv, g, 3, 8)_1$-DF *over* Z_{gv} *for* $g \equiv 2 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (2, 3)$.

Proof. For $g \equiv 5 \pmod 6$ and $v = 6$, the conclusion holds by Lemma 5.1. For $g \equiv 2 \pmod 3$, $v = 3, 9$ and $(g, v) \neq (2, 3)$, or $g \equiv 2 \pmod 6$ and $v = 6$, or $g = 2$, $v \equiv 0 \pmod 3$ and $v \geq 6$, the conclusion follows by taking together the base blocks of a $(gv, g, 3, 2)_1$-DF over Z_{gv} from Lemma 4.4(2), and a $(gv, g, 3, 6)_0$-DF over Z_{gv} from Lemma 3.5. For $g \equiv 2 \pmod 3$, $g > 2$, $v \equiv 0 \pmod 3$ and $v > 9$, let $v = 3v'$ where $v' > 3$. We start with a $(3gv', 3g, 3, 8)_0$-DF over $Z_{3gv'}$ for $v' > 3$ from Lemma 3.5 since $3g \equiv 0 \pmod 3$. That is a $(gv, 3g, 3, 8)_0$-DF over Z_{gv} for $v \equiv 0 \pmod 3$ and $v > 9$. Then we use Theorem 2.4 with a $(3g, g, 3, 8)_1$-DF over Z_{3g} mentioned above to produce the desired $(gv, g, 3, 8)_1$-DF over Z_{gv}. □

Lemma 5.7. *There exists a* $(gv, g, 3, 8)_5$-DF *over* Z_{gv} *for* $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (1, 3), (1, 6)$.

Proof. For $(g, v) = (1, 9)$, put together the base blocks of a $(9, 1, 3, 5)_2$-DF over Z_9 whose base blocks are $3\{0, 1, 3\}, 2\{0, 1, 5\}, \{0, 2, 4\}$, and a $(9, 1, 3, 3)_3$-DF over Z_9 from Lemma 4.5, then we can draw the conclusion. For $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (1, 3), (1, 6), (1, 9)$, the conclusion follows by taking together the base blocks of a $(gv, g, 3, 4)_1$-DF over Z_{gv} from Lemma 5.2, and a $(gv, g, 3, 4)_4$-DF over Z_{gv} from Lemma 5.3. □

Lemma 5.8. *There exists a* $(gv, g, 3, 8)_7$-DF *over* Z_{gv} *for* $g \equiv 2 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (2, 3), (2, 6)$.

Proof. For $g \equiv 5 \pmod 6$ and $v = 6$, a $(gv, g, 3, 8)_7$-DF over Z_{gv} exists from Lemma 5.1. For $g \equiv 2 \pmod 3$, $v = 3, 9$ and $(g, v) \neq (2, 3)$, or $g \equiv 2 \pmod 6$, $v = 6$ and $(g, v) \neq (2, 6)$, or $g = 2$, $v \equiv 0 \pmod 3$ and $v \geq 9$, the conclusion follows by taking together the base blocks of a $(gv, g, 3, 2)_1$-DF from Lemma 4.4(2), and a $(gv, g, 3, 6)_6$-DF over Z_{gv} from Lemma 5.5. For $g \equiv 2 \pmod 3$, $g > 2$, $v \equiv 0 \pmod 3$ and $v > 9$, let $v = 3v'$ where $v' > 3$. Note that there is a $(3gv', 3g, 3, 8)_0$-DF over $Z_{3gv'}$ for $v' > 3$ by Lemma 3.5 since $3g \equiv 0 \pmod 3$. That is a $(gv, 3g, 3, 8)_0$-DF over Z_{gv} for $v \equiv 0 \pmod 3$ and $v > 9$. Hence we use Theorem 2.4 with a $(3g, g, 3, 8)_7$-DF over Z_{3g} mentioned above to get a $(gv, g, 3, 8)_7$-DF over Z_{gv}. □

Lemma 5.9. *There exists a* $(gv, g, 3, 12)_3$-DF *over* Z_{gv} *for* $g \equiv 1, 2 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (1, 3), (2, 3)$.

Proof. For $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (1, 3)$, we repeat the base blocks of a $(gv, g, 3, 4)_1$-DF over Z_{gv} three times from Lemma 5.2 to get the result. For $g \equiv 2 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (2, 3)$, taking together the base blocks of a $(gv, g, 3, 8)_1$-DF from Lemma 5.6 and a $(gv, g, 3, 4)_2$-DF over Z_{gv} from Lemma 4.4(4), we produce a $(gv, g, 3, 12)_3$-DF over Z_{gv}. □

Lemma 5.10. *There exists a* $(gv, g, 3, 12)_6$-DF *over* Z_{gv} *for* $g \equiv 1, 2 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (1, 3), (1, 6)$.

Proof. For $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (1, 3), (1, 6)$, the conclusion follows by taking together the base blocks of a $(gv, g, 3, 4)_1$-DF from Lemma 5.2, and a $(gv, g, 3, 8)_5$-DF over Z_{gv} from Lemma 5.7. For $g \equiv 2 \pmod 3$ and $v \equiv 0 \pmod 3$, repeating the base blocks of a $(gv, g, 3, 4)_2$-DF over Z_{gv} three times from Lemma 4.4(4), we obtain a $(gv, g, 3, 12)_6$-DF over Z_{gv}. □

Lemma 5.11. *There exists a* $(gv, g, 3, 12)_9$-DF *over* Z_{gv} *for* $g \equiv 1, 2 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (1, 3), (2, 3), (1, 6), (2, 6)$.

Proof. We deal with the problem by considering three cases, and each case is solved by a similar method. For $(g, v) = (1, 9)$, take together the base blocks of a $(gv, g, 3, 6)_3$-DF over Z_{gv} from Lemma 5.4, and a $(gv, g, 3, 6)_6$-DF over Z_{gv} from Lemma 5.5. For $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (1, 3), (1, 6), (1, 9)$, take together the base blocks of a $(gv, g, 3, 4)_4$-DF over Z_{gv} twice from Lemma 5.3, and a $(gv, g, 3, 4)_1$-DF over Z_{gv} from Lemma 5.2. For $g \equiv 2 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (2, 3), (2, 6)$, take together the base blocks of a $(gv, g, 3, 4)_2$-DF from Lemma 4.4(4), and a $(gv, g, 3, 8)_7$-DF over Z_{gv} from Lemma 5.8. The conclusion then follows. □

Lemma 5.12. *There exists a* $(gv, g, 3, 12)_{12}$-DF *over* Z_{gv} *for* $g \equiv 1, 2 \pmod 3$, $v \equiv 0 \pmod 3$ *and* $(g, v) \neq (2, 3), (1, 6), (2, 6)$.

Proof. For $g \equiv 1 \pmod 6$, $v = 6$ and $(g, v) \neq (1, 6)$, repeat the base blocks of a $(gv, g, 3, 4)_4$-DF over Z_{gv} three times from Lemma 5.3 to get the result. For $g \equiv 5 \pmod 6$ and $v = 6$, the needed DF is from Lemma 5.1. For $g \equiv 1, 2 \pmod 3$, $v = 3$ and $(g, v) \neq (2, 3)$, or $g \equiv 2, 4 \pmod 6$, $v = 6$ and $(g, v) \neq (2, 6)$, or $g = 2$, $v \equiv 0 \pmod 3$ and $v \geq 9$, the conclusion follows by repeating the base blocks of a $(gv, g, 3, 6)_6$-DF over Z_{gv} twice from Lemma 5.5.

For $g \equiv 1, 2 \pmod 3$, $g \neq 2$, $v \equiv 0 \pmod 3$ and $v \geq 9$, let $v = 3v'$ where $v' \geq 3$. Since $3g \equiv 0 \pmod 3$, there exists a $(3gv', 3g, 3, 12)_0$-DF over $Z_{3gv'}$ for $v' \geq 3$ by Lemma 3.5. That is a $(gv, 3g, 3, 12)_0$-DF over Z_{gv} for $v \equiv 0 \pmod 3$ and $v \geq 9$. Applying Theorem 2.4 with a $(3g, g, 3, 12)_{12}$-DF mentioned above gives a $(gv, g, 3, 12)_{12}$-DF over Z_{gv}. □

6. Conclusions

Theorem 6.1. *A* $(gv, g, 3, \lambda)_\alpha$-DF *over* Z_{gv} *exists if and only if*

(1) $\lambda g(v-1) - 2\alpha \equiv 0 \pmod 6$, $v \neq 2$;
(2) $v \not\equiv 2, 3 \pmod 4$ *when* $g \equiv 2 \pmod 4$ *and* $\lambda \equiv 1 \pmod 2$;

(3) $v \not\equiv 2 \pmod 4$ *when* $g \equiv 1 \pmod 2$ *and* $\lambda \equiv 2 \pmod 4$*;*
(4) $g \not\equiv 0 \pmod 3$ *and* $v \equiv 0 \pmod 3$ *when* $\alpha \neq 0$*;*
(5) $\lambda(3g-1) - 2\alpha g \equiv 0 \pmod 6$ *when* $v = 3$*;*
(6) $\lambda = \alpha$ *when* $(g, v) = (1, 3), \lambda = 2\alpha$ *when* $(g, v) = (2, 3), \lambda = 4\alpha$ *when* $(g, v) = (1, 6), \lambda \geq 2\alpha$ *when* $(g, v) = (2, 6), \lambda \equiv 0 \pmod 3$ *when* $(g, v) = (1, 9)$ *and* $\lambda = \alpha$*.*

Proof. The necessity follows by Lemma 1.4, so we establish the sufficiency as follows.

When $(g, v) = (1, 3)$ and $\lambda = \alpha$, the DF is degenerate. When $(g, v) = (2, 3)$, $(1, 6)$, or $(2, 6)$, repeat the base blocks of certain DFs over Z_{gv} as listed at the table below to obtain the required designs.

Condition	DF used	Repetition	Source
$(g, v) = (2, 3), \lambda = 2\alpha$	$(gv, g, 3, 2)_1$-DF	α	Lemma 4.4(2)
$(g, v) = (1, 6), \lambda = 4\alpha$	$(gv, g, 3, 4)_1$-DF	α	Lemma 5.2
$(g, v) = (2, 6), \lambda \geq 2\alpha$	$(gv, g, 3, 2)_1$-DF	α	Lemma 4.4(2)
	$(gv, g, 3, 6)_0$-DF	$(\lambda - 2\alpha)/6$	Lemma 3.5

For $(g, v) \neq (1, 3), (2, 3), (1, 6), (2, 6)$, the sufficiency is obtained in the following four cases.

Case 1: $\alpha = 0$: The conclusion holds by Lemma 3.5.

Case 2: $\alpha \equiv 0 \pmod 3$, $\alpha \geq 3$:

When $\lambda \equiv 3 \pmod 6$, we have (i) $g \equiv 1, 5 \pmod 6$ and $v \equiv 3 \pmod 6$, (ii) $g \equiv 2, 10 \pmod{12}$ and $v \equiv 0, 9 \pmod{12}$, (iii) $g \equiv 4, 8 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$.

Condition	DF used	Repetition	Source
$\lambda \geq \alpha$	$(gv, g, 3, 3)_3$-DF	$\alpha/3$	Lemma 4.5
	$(gv, g, 3, 3)_0$-DF	$(\lambda - \alpha)/3$	Lemma 3.5

When $\lambda \equiv 6 \pmod{12}$, we have (i) $g \equiv 1, 5 \pmod 6$, $v \equiv 0 \pmod 3$ and $v \not\equiv 2 \pmod 4$, (ii) $g \equiv 2, 4 \pmod 6$ and $v \equiv 0 \pmod 3$.

Condition	DF used	Repetition	Source
$\alpha \equiv 3 \pmod 6, \lambda \geq \alpha + 3$	$(gv, g, 3, 6)_3$-DF	1	Lemma 5.4
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 3)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 3)/6$	Lemma 3.5
$\alpha \equiv 0 \pmod 6, \lambda \geq \alpha$	$(gv, g, 3, 6)_6$-DF	$\alpha/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha)/6$	Lemma 3.5

When $\lambda \equiv 0 \pmod{12}$, we have $g \equiv 1, 2 \pmod 3$ and $v \equiv 0 \pmod 3$.

Condition	DF used	Repetition	Source
$\alpha \equiv 0 \pmod{12}, \lambda \geq \alpha$	$(gv, g, 3, 12)_{12}$-DF	$\alpha/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha)/12$	Lemma 3.5
$\alpha \equiv 3 \pmod{12}, \lambda \geq \alpha + 9$	$(gv, g, 3, 12)_3$-DF	1	Lemma 5.9
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 3)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 9)/12$	Lemma 3.5
$\alpha \equiv 6 \pmod{12}, \lambda \geq \alpha + 6$	$(gv, g, 3, 12)_6$-DF	1	Lemma 5.10
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 6)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 6)/12$	Lemma 3.5
$\alpha \equiv 9 \pmod{12}, \lambda \geq \alpha + 3$	$(gv, g, 3, 12)_9$-DF	1	Lemma 5.11
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 9)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 3)/12$	Lemma 3.5

Case 3: $\alpha \equiv 1 \pmod 3$:

When $\lambda \equiv 1 \pmod 6$, we have (i) $g \equiv 1 \pmod 6$ and $v \equiv 3 \pmod 6$, (ii) $g \equiv 10 \pmod{12}$ and $v \equiv 0, 9 \pmod{12}$, (iii) $g \equiv 4 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$, (iv) $(g, v) \neq (1, 9)$ when $\lambda = \alpha$.

Condition	DF used	Repetition	Source
$\lambda \geq \alpha + 3$	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
	$(gv, g, 3, 3)_3$-DF	$(\alpha - 1)/3$	Lemma 4.5
	$(gv, g, 3, 3)_0$-DF	$(\lambda - \alpha)/3 - 1$	Lemma 3.5
$\lambda = \alpha, (g, v) \neq (1, 9)$	$(gv, g, 3, 1)_1$-DF	1	Lemma 4.4(1)
	$(gv, g, 3, 3)_3$-DF	$(\alpha - 1)/3$	Lemma 4.5

When $\lambda \equiv 5 \pmod 6$, we have (i) $g \equiv 5 \pmod 6$ and $v \equiv 3 \pmod 6$, (ii) $g \equiv 2 \pmod{12}$ and $v \equiv 0, 9 \pmod{12}$, (iii) $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$.

Condition	DF used	Repetition	Source
	$(gv, g, 3, 2)_1$-DF	1	Lemma 4.4(2)
$\lambda \geq \alpha + 1$	$(gv, g, 3, 3)_3$-DF	$(\alpha - 1)/3$	Lemma 4.5
	$(gv, g, 3, 3)_0$-DF	$(\lambda - \alpha - 1)/3$	Lemma 3.5

When $\lambda \equiv 2 \pmod{12}$, we have (i) $g \equiv 2 \pmod 6$ and $v \equiv 0 \pmod 3$, (ii) $g \equiv 5 \pmod 6$, $v \equiv 0 \pmod 3$ and $v \not\equiv 2 \pmod 4$.

Condition	DF used	Repetition	Source
	$(gv, g, 3, 2)_1$-DF	1	Lemma 4.4(2)
$\alpha \equiv 1 \pmod 6$, $\lambda \geq \alpha + 1$	$(gv, g, 3, 6)_6$-DF	$(\alpha - 1)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 1)/6$	Lemma 3.5
	$(gv, g, 3, 2)_1$-DF	4	Lemma 4.4(2)
$\alpha \equiv 4 \pmod 6$, $\lambda \geq \alpha + 4$	$(gv, g, 3, 6)_6$-DF	$(\alpha - 4)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 4)/6$	Lemma 3.5

When $\lambda \equiv 10 \pmod{12}$, we have (i) $g \equiv 4 \pmod 6$ and $v \equiv 0 \pmod 3$, (ii) $g \equiv 1 \pmod 6$, $v \equiv 0 \pmod 3$ and $v \not\equiv 2 \pmod 4$, (iii) $(g, v) \neq (1, 9)$ when $\lambda = \alpha$.

Condition	DF used	Repetition	Source
	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
$\alpha \equiv 1 \pmod 6$, $\lambda \geq \alpha + 3$	$(gv, g, 3, 6)_6$-DF	$(\alpha - 1)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 3)/6$	Lemma 3.5
	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
$\alpha \equiv 4 \pmod 6$, $\lambda \geq \alpha + 6$	$(gv, g, 3, 6)_3$-DF	1	Lemma 5.4
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 4)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha)/6 - 1$	Lemma 3.5
$\lambda = \alpha$, $(g, v) \neq (1, 9)$	$(gv, g, 3, 2)_2$-DF	2	Lemma 4.4(3)
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 4)/6$	Lemma 5.5

When $\lambda \equiv 4 \pmod{12}$, we have (i) $g \equiv 1 \pmod 3$ and $v \equiv 0 \pmod 3$, (ii) $(g, v) \neq (1, 9)$ when $\lambda = \alpha$.

Condition	DF used	Repetition	Source
	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
$\alpha \equiv 1 \pmod{12}$, $\lambda \geq \alpha + 3$	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 1)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 3)/12$	Lemma 3.5
	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
$\alpha \equiv 4 \pmod{12}$, $\lambda \geq \alpha + 12$	$(gv, g, 3, 12)_3$-DF	1	Lemma 5.9
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 4)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha)/12 - 1$	Lemma 3.5
$\lambda = \alpha$, $(g, v) \neq (1, 9)$	$(gv, g, 3, 4)_4$-DF	1	Lemma 5.3
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 4)/12$	Lemma 5.12
	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
$\alpha \equiv 7 \pmod{12}$, $\lambda \geq \alpha + 9$	$(gv, g, 3, 12)_6$-DF	1	Lemma 5.10
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 7)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 9)/12$	Lemma 3.5
	$(gv, g, 3, 4)_1$-DF	1	Lemma 5.2
$\alpha \equiv 10 \pmod{12}$, $\lambda \geq \alpha + 6$	$(gv, g, 3, 12)_9$-DF	1	Lemma 5.11
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 10)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 6)/12$	Lemma 3.5

When $\lambda \equiv 8 \pmod{12}$, we have $g \equiv 2 \pmod 3$ and $v \equiv 0 \pmod 3$.

Condition	DF used	Repetition	Source
$\alpha \equiv 1 \pmod{12}$, $\lambda \geq \alpha + 7$	$(gv, g, 3, 8)_1$-DF	1	Lemma 5.6
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 1)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 7)/12$	Lemma 3.5
$\alpha \equiv 4 \pmod{12}$, $\lambda \geq \alpha + 4$	$(gv, g, 3, 4)_2$-DF	2	Lemma 4.4(4)
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 4)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 4)/12$	Lemma 3.5
$\alpha \equiv 7 \pmod{12}$, $\lambda \geq \alpha + 1$	$(gv, g, 3, 8)_7$-DF	1	Lemma 5.8
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 7)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 1)/12$	Lemma 3.5
$\alpha \equiv 10 \pmod{12}$, $\lambda \geq \alpha + 10$	$(gv, g, 3, 8)_1$-DF	1	Lemma 5.6
	$(gv, g, 3, 12)_9$-DF	1	Lemma 5.11
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 10)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 10)/12$	Lemma 3.5

Case 4: $\alpha \equiv 2 \pmod 3$:

When $\lambda \equiv 1 \pmod 6$, we have (i) $g \equiv 5 \pmod 6$ and $v \equiv 3 \pmod 6$, (ii) $g \equiv 2 \pmod{12}$ and $v \equiv 0, 9 \pmod{12}$, (iii) $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$.

Condition	DF used	Repetition	Source
$\lambda \geq \alpha + 2$	$(gv, g, 3, 4)_2$-DF	1	Lemma 4.4(4)
	$(gv, g, 3, 3)_3$-DF	$(\alpha - 2)/3$	Lemma 4.5
	$(gv, g, 3, 3)_0$-DF	$(\lambda - \alpha - 2)/3$	Lemma 3.5

When $\lambda \equiv 5 \pmod 6$, we have (i) $g \equiv 1 \pmod 6$ and $v \equiv 3 \pmod 6$, (ii) $g \equiv 10 \pmod{12}$ and $v \equiv 0, 9 \pmod{12}$, (iii) $g \equiv 4 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$, (iv) $(g, v) \neq (1, 9)$ when $\lambda = \alpha$.

Condition	DF used	Repetition	Source
$\lambda \geq \alpha$, $(g, v) \neq (1, 9)$	$(gv, g, 3, 1)_1$-DF	2	Lemma 4.4(1)
	$(gv, g, 3, 3)_3$-DF	$(\alpha - 2)/3$	Lemma 4.5
	$(gv, g, 3, 3)_0$-DF	$(\lambda - \alpha)/3$	Lemma 3.5
$\lambda \geq \alpha + 3$, $(g, v) = (1, 9)$	$(gv, g, 3, 5)_2$-DF	1	Lemma 5.7
	$(gv, g, 3, 3)_3$-DF	$(\alpha - 2)/3$	Lemma 4.5
	$(gv, g, 3, 3)_0$-DF	$(\lambda - \alpha)/3 - 1$	Lemma 3.5

When $\lambda \equiv 2 \pmod{12}$, we have (i) $g \equiv 4 \pmod 6$ and $v \equiv 0 \pmod 3$, (ii) $g \equiv 1 \pmod 6$, $v \equiv 0 \pmod 3$ and $v \not\equiv 2 \pmod 4$, (iii) $(g, v) \neq (1, 9)$ when $\lambda = \alpha$.

Condition	DF used	Repetition	Source
$\alpha \equiv 5 \pmod 6$, $\lambda \geq \alpha + 3$	$(gv, g, 3, 8)_5$-DF	1	Lemma 5.7
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 5)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 3)/6$	Lemma 3.5
$\alpha \equiv 2 \pmod 6$, $\lambda \geq \alpha + 6$	$(gv, g, 3, 4)_1$-DF	2	Lemma 5.2
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 2)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha)/6 - 1$	Lemma 3.5
$\lambda = \alpha$, $(g, v) \neq (1, 9)$	$(gv, g, 3, 2)_2$-DF	1	Lemma 4.4(3)
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 2)/6$	Lemma 5.5

When $\lambda \equiv 10 \pmod{12}$, we have (i) $g \equiv 2 \pmod 6$ and $v \equiv 0 \pmod 3$, (ii) $g \equiv 5 \pmod 6$, $v \equiv 0 \pmod 3$ and $v \not\equiv 2 \pmod 4$.

Condition	DF used	Repetition	Source
$\alpha \equiv 2 \pmod 6$, $\lambda \geq \alpha + 2$	$(gv, g, 3, 2)_1$-DF	2	Lemma 4.4(2)
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 2)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 2)/6$	Lemma 3.5
$\alpha \equiv 5 \pmod 6$, $\lambda \geq \alpha + 5$	$(gv, g, 3, 2)_1$-DF	5	Lemma 4.4(2)
	$(gv, g, 3, 6)_6$-DF	$(\alpha - 5)/6$	Lemma 5.5
	$(gv, g, 3, 6)_0$-DF	$(\lambda - \alpha - 5)/6$	Lemma 3.5

When $\lambda \equiv 4 \pmod{12}$, we have $g \equiv 2 \pmod 3$ and $v \equiv 0 \pmod 3$.

Condition	DF used	Repetition	Source
$\alpha \equiv 2 \pmod{12}, \lambda \geq \alpha + 2$	$(gv, g, 3, 4)_2$-DF	1	Lemma 4.4(4)
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 2)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 2)/12$	Lemma 3.5
$\alpha \equiv 5 \pmod{12}, \lambda \geq \alpha + 11$	$(gv, g, 3, 4)_2$-DF	1	Lemma 4.4(4)
	$(gv, g, 3, 12)_3$-DF	1	Lemma 5.9
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 5)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 11)/12$	Lemma 3.5
$\alpha \equiv 8 \pmod{12}, \lambda \geq \alpha + 8$	$(gv, g, 3, 4)_2$-DF	4	Lemma 4.4(4)
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 8)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 8)/12$	Lemma 3.5
$\alpha \equiv 11 \pmod{12}, \lambda \geq \alpha + 5$	$(gv, g, 3, 4)_2$-DF	1	Lemma 4.4(4)
	$(gv, g, 3, 12)_9$-DF	1	Lemma 5.11
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 11)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 5)/12$	Lemma 3.5

When $\lambda \equiv 8 \pmod{12}$, we have (i) $g \equiv 1 \pmod 3$ and $v \equiv 0 \pmod 3$, (ii) $(g, v) \neq (1, 9)$ when $\lambda = \alpha$.

Condition	DF used	Repetition	Source
$\alpha \equiv 2 \pmod{12}, \lambda \geq \alpha + 6$	$(gv, g, 3, 4)_1$-DF	2	Lemma 5.2
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 2)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 6)/12$	Lemma 3.5
$\alpha \equiv 5 \pmod{12}, \lambda \geq \alpha + 3$	$(gv, g, 3, 8)_5$-DF	1	Lemma 5.7
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 5)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 3)/12$	Lemma 3.5
$\alpha \equiv 8 \pmod{12}, \lambda \geq \alpha + 12$	$(gv, g, 3, 4)_1$-DF	2	Lemma 5.2
	$(gv, g, 3, 12)_6$-DF	1	Lemma 5.10
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 8)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha)/12 - 1$	Lemma 3.5
$\lambda = \alpha, (g, v) \neq (1, 9)$	$(gv, g, 3, 4)_4$-DF	2	Lemma 5.3
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 8)/12$	Lemma 5.12
$\alpha \equiv 11 \pmod{12}, \lambda \geq \alpha + 9$	$(gv, g, 3, 4)_1$-DF	2	Lemma 5.2
	$(gv, g, 3, 12)_9$-DF	1	Lemma 5.11
	$(gv, g, 3, 12)_{12}$-DF	$(\alpha - 11)/12$	Lemma 5.12
	$(gv, g, 3, 12)_0$-DF	$(\lambda - \alpha - 9)/12$	Lemma 3.5

This completes the proof of Theorem 6.1. □

Now we are in the position to establish the following main result.

Theorem 6.2. *A cyclic $(3, \lambda)$-GDD of type g^v having α short orbits exists if and only if*

(1) $\lambda g(v - 1) - 2\alpha \equiv 0 \pmod 6$, $\alpha \leq \lambda$, $v \geq 3$;
(2) $v \not\equiv 2, 3 \pmod 4$ *when* $g \equiv 2 \pmod 4$ *and* $\lambda \equiv 1 \pmod 2$;
(3) $v \not\equiv 2 \pmod 4$ *when* $g \equiv 1 \pmod 2$ *and* $\lambda \equiv 2 \pmod 4$;
(4) $g \not\equiv 0 \pmod 3$ *and* $v \equiv 0 \pmod 3$ *when* $\alpha \neq 0$;
(5) $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod 6$ *when* $v = 3$;
(6) $\lambda = \alpha$ *when* $(g, v) = (1, 3)$, $\lambda = 2\alpha$ *when* $(g, v) = (2, 3)$, $\lambda = 4\alpha$ *when* $(g, v) = (1, 6)$, $\lambda \geq 2\alpha$ *when* $(g, v) = (2, 6)$, $\lambda \equiv 0 \pmod 3$ *when* $(g, v) = (1, 9)$ *and* $\lambda = \alpha$.

Proof. Suppose that there exists a cyclic $(3, \lambda)$-GDD of type g^v, in which a is the number of full orbits. A simple counting shows that $6a + 2\alpha = \lambda g(v - 1)$, that is $\lambda g(v - 1) - 2\alpha \equiv 0 \pmod 6$. Condition (1) of Theorem 6.2 then follows. Conditions (2) and (3) are obtained with similar arguments as Lemma 1.3. It is easy to see that $g \not\equiv 0 \pmod 3$ and $v \equiv 0 \pmod 3$ when $\alpha \neq 0$. So Condition (4) follows. With a similar proof to that of Lemma 1.2, we can get $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod 6$ when $v = 3$, but here we consider the differences covered by a full orbits. Therefore, Condition (5) follows. Condition (6) follows with similar arguments as Lemma 1.1.

Now we are going to prove the sufficiency. Note that a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} generates a cyclic $(3, \lambda)$-GDD of type g^v having α short orbits. By Theorem 6.1 the conclusion then holds. □

It should be pointed out that a cyclic $(3, 1)$-GDD of type 3^{2n+1} and of type 2^{3n+1} are equivalent, respectively, to the well-known cyclic and 1-rotational $(6n + 3, 3, 1)$-BIBD (see [11,9]). Their existence is contained in Theorem 6.2 as special cases.

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