Journal of Pure and Applied Algebra 215 (2011) 2847–2852

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On the homology of elementary Abelian groups as modules over the Steenrod algebra

Shaun V. Ault, William Singer *

Department of Mathematics, Fordham University, Bronx, NY 10458, USA

ARTICLE INFO

Article history:
Received 11 January 2011
Available online 10 May 2011
Communicated by E.M. Friedlander

MSC: 55Q10; 55Q45; 55S05; 55S10; 55T15

ABSTRACT

We examine the dual of the so-called "hit problem", the latter being the problem of determining a minimal generating set for the cohomology of products of infinite projective spaces as a module over the Steenrod Algebra $\mathcal{A}$ at the prime 2. The dual problem is to determine the set of $\mathcal{A}$-annihilated elements in homology. The set of $\mathcal{A}$-annihilateds has been shown by David Anick to be a free associative algebra. In this note we prove that, for each $k \geq 0$, the set of k *partially* $\mathcal{A}$*-annihilateds*, the set of elements that are annihilated by Sq^i for each $i \leq 2^k$, itself forms a free associative algebra.

1. Introduction and notations

Let $\mathbb{F}_2$ be the field of 2 elements and $\Gamma = \{\Gamma_{s,*}\}_{s\geq 0}$ be the bigraded $\mathbb{F}_2$-space defined by

$$\Gamma_{s,*} = H_*(B(\mathbb{Z}/2)^{\times s}, \mathbb{F}_2), \quad \text{for each } s \geq 0.$$

The bigrading (s, d) is by the number of direct product factors of $B(\mathbb{Z}/2)$, and by homological degree. We shall say that an element x of bidegree (s, d) has *length* s and *degree* d. This paper studies Γ with its canonical structure as a right module over the Steenrod algebra $\mathcal{A}$. We are interested in particular in the problem of determining the graded vector spaces

$$\Gamma^{\mathcal{A}}_{s,*} = \{a \in \Gamma_{s,*} \mid (a)Sq^k = 0, \forall k > 0\}$$

consisting of all elements of Γ that are annihilated by the Steenrod operations of positive degree. This problem and its dual (finding a minimal generating set for the cohomology of $B(\mathbb{Z}/2)^{\times s}$ as a left $\mathcal{A}$-module) have been much studied in recent years. The reader can find comprehensive bibliographies covering the work done through 2000 in [7,8], and will find recent work in [4–6]. Much progress has been made, but the general problem remains unsolved.

We will work in terms of the reduced homology groups of the smash products:

$$\widetilde{\Gamma}_{s,*} = \widetilde{H}_*(B(\mathbb{Z}/2)^{\wedge s}, \mathbb{F}_2), \quad \text{for each } s \geq 1.$$

We adopt also the convention $\widetilde{\Gamma}_{0,*} = H_*(*, \mathbb{F}_2)$, the homology of a point. We assemble the spaces $\widetilde{\Gamma}_{s,*}$ into a bigraded vector space

$$\widetilde{\Gamma} = \{\widetilde{\Gamma}_{s,*}\}_{s\geq 0}$$

and write the associated vector spaces of $\mathcal{A}$-annihilated elements $\widetilde{\Gamma}^{\mathcal{A}}_{s,*} \subseteq \widetilde{\Gamma}_{s,*}$. The vector spaces $\Gamma^{\mathcal{A}}_{s,*}$ are easily expressed in terms of the spaces $\widetilde{\Gamma}^{\mathcal{A}}_{p,*}$ for $p \leq s$, so a study of the smash products is sufficient. The natural mappings:

$$B(\mathbb{Z}/2)^{\wedge p} \times B(\mathbb{Z}/2)^{\wedge q} \longrightarrow B(\mathbb{Z}/2)^{\wedge(p+q)}$$

* Corresponding author.
E-mail address: singer@fordham.edu (W. Singer).

0022-4049/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2011.04.004

induce pairings of vector spaces:

$$\widetilde{\Gamma}_{p,*} \otimes \widetilde{\Gamma}_{q,*} \longrightarrow \widetilde{\Gamma}_{p+q,*}$$

for all $p, q \geq 0$, which make $\widetilde{\Gamma}$ into a connected bigraded algebra. By the Künneth theorem, $\widetilde{\Gamma}$ is a free associative $\mathbb{F}_2$-algebra. For each $k \geq 1$, it is convenient to represent the canonical generators,

$$\gamma_k \in \widetilde{\Gamma}_{1,k} = \widetilde{H}_k(B(\mathbb{Z}/2)).$$

Then we have $\widetilde{\Gamma} = \mathbb{F}_2\langle\{\gamma_1, \gamma_2, \gamma_3, \ldots\}\rangle$ (we use the notation of Cohn [2]: for a field $\Bbbk$ and set X, $\Bbbk\langle X\rangle$ is the free associative $\Bbbk$-algebra generated by X). The Cartan formula implies that the bigraded vector space $\widetilde{\Gamma}^{\mathcal{A}} = \{\widetilde{\Gamma}^{\mathcal{A}}_{s,*}\}_{s\geq 0}$ is a subalgebra of $\widetilde{\Gamma}$. Anick proves in [1] that this subalgebra is itself free. Now for each $k \geq 0$, and $s, d \geq 0$, define:

$$\Delta(k)_{s,d} = \bigcap_{i=0}^{k} \ker(Sq^{2^i} : \widetilde{\Gamma}_{s,d} \to \widetilde{\Gamma}_{s,d-2^i}),$$

and set $\Delta(k) = \{\Delta(k)_{s,d}\}_{s,d\geq 0}$, a bigraded space called the "k partially $\mathcal{A}$-annihilateds." Using a variant of Anick's argument, we will show that:

Theorem 1. *For each $k \geq 0$, $\Delta(k)$ is a free subalgebra of $\widetilde{\Gamma}$.*

Note that if k is chosen so that $d < 2^{k+2}$, then we have for any $s \geq 0$:

$$\widetilde{\Gamma}^{\mathcal{A}}_{s,d} = \Delta(k)_{s,d}.$$

Thus, determining the sets S_k such that $\Delta(k) = \mathbb{F}_2\langle S_k\rangle$ would solve the "$\mathcal{A}$-annihilated problem", and the solution would be in terms of explicitly-given algebra generators. Furthermore, partial progress is meaningful, as the determination of the set S_k would determine all $\mathcal{A}$-annihilateds of degree $d < 2^{k+2}$.

In this note, $\ker Sq^p$ and $\mathrm{im} Sq^p$ will be understood to involve the restricted maps $Sq^p : \widetilde{\Gamma} \to \widetilde{\Gamma}$.

2. Proof of the main theorem

In this section we prove that $\Delta(k)$ is a tensor algebra, using a remarkable lemma of Anick [1], here stated for the case of $\mathbb{Z}^t$-graded algebras. Let $\Bbbk$ be a field. For $t \geq 1$, a $\Bbbk$-algebra A is *$\mathbb{Z}^t$-graded* if $A = \{A_I\}_{I\in\mathbb{Z}^t}$, and multiplication in A is a family of maps,

$$A_I \otimes A_J \to A_{I+J}.$$

If $x \in A_I$, we say the degree of x is $I = (i_1, i_2, \ldots, i_t)$. Introduce a lexicographic ordering of degree as follows: $I < J$ if and only if there is an integer r with $1 \leq r \leq t$ such that $i_p = j_p$ if $p < r$, and $i_r < j_r$. The algebra A is said to be connected if $A_I = 0$ whenever I contains a negative entry, and $A_{(0,\ldots,0)} \cong \Bbbk$. The *positively-graded* elements of A are the elements of the set

$$A^+ = \bigcup\{A_I \mid \text{all entries of } I \text{ are non-negative and at least one entry is positive}\}.$$

Definition 2. Let A be a connected $\mathbb{Z}^t$-graded algebra ($t \geq 1$) over a field $\Bbbk$. A is said to satisfy *Anick's Condition* if whenever a relation,

$$\sum_{i=1}^{n} a_i b_i = 0, \qquad (1)$$

holds in A, where each $b_i \neq 0$, then there is a j such that

$$a_j \in \sum_{i\neq j} a_i A. \qquad (2)$$

Lemma 3 (*Anick [1]*)**.** *Let A be a connected $\mathbb{Z}^t$-graded algebra over a field $\Bbbk$. Then A is a tensor algebra, $A = \Bbbk\langle X\rangle$, for some set of positively-graded elements $X \subset A^+$, if and only if A satisfies Anick's Condition.*

Anick's proof makes use of the work of Cohn [2] on so-called free ideal rings (firs). For completeness, we shall provide a proof that avoids as much of this machinery as possible. Furthermore, our working in the graded case allows us to simplify some of Cohn's arguments.

Proof. The backward direction is the easier of the two. The proof is Anick's [1]. Suppose that A is not a tensor algebra. Choose a minimal set X of generators for A, and write $A = \Bbbk\langle X\rangle/R$ where R is the non-zero ideal of relations. Choose a non-zero $\alpha \in R$ of minimal degree. Then α can be expanded uniquely in the form:

$$\alpha = \sum_{i=1}^{m} x_i Y_i,$$

where the x_i are distinct elements of the generating set X, and $Y_i \neq 0$ for each i. For each $x \in \Bbbk\langle X\rangle$, write $\bar{x}$ for the corresponding element of A. Since $\alpha \in R$, we have:

$$\sum_{i=1}^{m} \overline{x_i}\,\overline{Y_i} = 0. \tag{3}$$

Since α is of minimal degree in R, we have $\overline{Y_i} \neq 0$ in A, for each value of i. So Eq. (3) is a relation of the form (1). But if there were a j with $1 \leq j \leq m$ such that

$$\overline{x_j} \in \sum_{i\neq j} \overline{x_i} A, \tag{4}$$

then the generating set X would not be minimal, contradicting our assumption. Thus A cannot satisfy Anick's Condition.

For the forward direction, assume that A is a connected graded tensor algebra $\Bbbk\langle X\rangle$ on a generating set $X \subset A^+$. Suppose now that there is a relation,

$$\sum_{i=1}^{n} a_i b_i = 0, \tag{5}$$

for $a_i, b_i \in A$, and each $b_i \neq 0$, as in the premise of Anick's Condition. We may assume the summands are ordered so that $deg(b_1) \geq deg(b_2) \geq \cdots \geq deg(b_n)$. Let $I = deg(b_n)$, and let $c\mu = cx_1 \cdots x_s$ be a term of degree I occurring in b_n ($c \in \Bbbk$, $x_i \in X$). For any element $a \in A$, we may write $a = a_0 + a^*\mu$ for some $a_0, a^* \in A$ such that μ does not right-divide any term of a_0. Moreover, both a_0 and a^* are uniquely-determined since A is free. Observe, the function $a \mapsto a^*$ is $\Bbbk$-linear of degree $-deg(\mu) = -I$ (this mapping is known as left transduction for μ, and a^* is known as the left cofactor of μ in a, see [3]).

Suppose $b \in A$ is any single term. Then either μ does not right-divide b, in which case $b^* = 0$, or μ does, and $b^*\mu = b$. Thus, if $deg(b) \geq I$, then for any $a \in A$, $(ab)^* = ab^*$. By linearity of transduction, we have:

$$(ab)^* = ab^*, \quad \text{for any } a, b \in A \text{ such that } deg(b) \geq I.$$

Applying transduction for μ to Eq. (5), we have, since each b_i has degree at least I,

$$0 = \left(\sum_{i=1}^{n} a_i b_i\right)^* = \sum_{i=1}^{n} a_i b_i^*.$$

Finally, since $b_n^* \neq 0$ has degree $(0, \ldots, 0)$, and A is connected, we have in fact shown that $b_n^* \in \Bbbk^\times$. We obtain a relation of the form:

$$a_n = \left(-\sum_{i=1}^{n-1} a_i b_i^*\right)(b_n^*)^{-1} = \sum_{i=1}^{n-1} a_i(-b_n^*)^{-1} b_i^*.$$

Therefore, $a_n \in \sum_{i\neq n} a_i A$, as desired. □

The following is a useful application of Lemma 3.

Lemma 4. *Let A be a connected $\mathbb{Z}^t$-graded algebra over a field $\Bbbk$, and suppose that A is a tensor algebra, $A = \Bbbk\langle X\rangle$, on some set of positively-graded elements $X \subset A^+$. Let S be any set of positively graded elements that form a minimal generating set for A. Then A is the tensor algebra on S.*

Proof. Consider the canonical algebra mapping $\Bbbk\langle S\rangle \to A$. We must show that the kernel is zero. Suppose to the contrary there is a non-zero element of the kernel. We choose one of least degree; say $\sum_{i=1}^{n} s_i Y_i$, where the elements s_i are distinct members of the set S, and each Y_i is a non-zero element of $\Bbbk\langle S\rangle$. Then we have in A the relation:

$$\sum_{i=1}^{n} \overline{s_i}\,\overline{Y_i} = 0. \tag{6}$$

Our assumption that (6) is a relation of least degree assures that each $\overline{Y_i}$ is a non-zero element of A . But by Lemma 3, A satisfies Anick's condition. So there must be an index j with $1 \leq j \leq n$ and elements $c_i \in A$ for $i \neq j$ such that:

$$\overline{s_j} = \sum_{i\neq j} \overline{s_i} c_i. \tag{7}$$

Now for each index $i \neq j$, the element c_i must be expressible as a non-commutative polynomial in the elements of S. Further, since A is graded-connected, and each $\overline{s_i}$ has positive degree, none of these polynomials can involve the element $\overline{s_j}$. Hence, Eq. (7) expresses $\overline{s_j}$ in terms of the other generators. This dependence would contradict the assumed minimality of the generating set S. Thus there can be no relation of the form Eq. (6), and the result is proved. □

Now we come to our main result.

Theorem 5. *For $k \geq 0$, $\Delta(k)$ is a free associative $\mathbb{F}_2$-algebra.*

Proof. We will show that $\Delta(k)$ satisfies Anick's Condition. Suppose there is a relation in $\Delta(k)$,

$$\sum_{i=1}^{n} a_i b_i = 0, \tag{8}$$

where each $b_i \neq 0$. We want to show that there is an index j such that

$$a_j \in \sum_{i \neq j} a_i \Delta(k).$$

This will surely be the case if the elements $a_1, \ldots, a_n$ were not distinct, so we may assume that the a_i are distinct. Now Eq. (8) can be read as a relation in the connected tensor algebra $\widetilde{\Gamma}$. The fact that one such relation among the elements of $\{a_i\}$ exists means that we can find one for which n is minimal. In other words, let

$$\sum_{i=1}^{p} a_i c_i = 0, \tag{9}$$

be a relation with a minimal number of summands in $\widetilde{\Gamma}$, involving elements from the set $\{a_i\}$, with each $c_i \neq 0$. Since $\widetilde{\Gamma}$ satisfies Anick's Condition (by Lemma 3), there is an index j such that

$$a_j = \sum_{1 \leq i \leq p,\, i \neq j} a_i d_i, \tag{10}$$

for some $d_i \in \widetilde{\Gamma}$. We shall show that every d_i is in fact a member of $\Delta(k)$. Let ℓ be an integer, $0 \leq \ell \leq k$. Apply Sq^{2^ℓ} to both sides of Eq. (10). Note, $a_i Sq^q = 0$ for each q satisfying $0 < q \leq 2^\ell$ and every i, since $a_i \in \Delta(k)$. Hence, by the Cartan formula,

$$0 = \sum_{i \neq j} a_i \left(d_i Sq^{2^\ell} \right). \tag{11}$$

If there are any indices i such that $d_i Sq^{2^\ell} \neq 0$, then Eq. (11) would represent a non-trivial relation among the elements of $\{a_i\}$, of strictly fewer number of terms than the supposed minimal one. Therefore, $d_i Sq^{2^\ell} = 0$ for each i. But this is true for any $0 \leq \ell \leq k$, so each $d_i \in \Delta(k)$. This shows that $a_j \in \sum_{i \neq j} a_i \Delta(k)$, and so $\Delta(k)$ satisfies Anick's Condition. Hence $\Delta(k)$ is a tensor algebra on a positively-graded generating set. □

3. Analysis of $\Delta(\mathbf{0})$

In an effort to de-clutter our formulas, we use the notation:

$$\begin{aligned} [i_1, i_2, \ldots, i_s] &= \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_s} \in \widetilde{\Gamma}_{s,*}, \quad s \geq 1 \\ [\,] &= 1 \in \widetilde{\Gamma}_{0,0}. \end{aligned}$$

For $s \geq 1$, and integers $m_i \geq 0$, we define special elements of $\widetilde{\Gamma}$:

$$\sigma(m_1, m_2, \ldots, m_s) \stackrel{def}{=} [2m_1 + 2, 2m_2 + 2, \ldots, 2m_s + 2] Sq^1,$$

and we let S_0 be the set:

$$S_0 = \{\sigma(m_1, m_2, \ldots, m_s) \mid s \geq 1, m_1 \geq 0, \ldots m_s \geq 0\}.$$

Our goal in this section is to prove that $\Delta(0) = \ker Sq^1$ is the free algebra on the set S_0.

Lemma 6. *For each $s \geq 1$ one has in $\widetilde{\Gamma}_{s,*}$:*

$$\ker Sq^1 = \operatorname{im} Sq^1.$$

Proof. Sq^1 acts as a differential on $\widetilde{\Gamma}_{s,*}$; and the isomorphism:

$$\widetilde{\Gamma}_{s,*} = (\widetilde{\Gamma}_{1,*})^{\otimes s}$$

is an isomorphism of chain complexes. Since $\widetilde{\Gamma}_{1,*}$ is acyclic, our results follows from the Künneth theorem. □

Given a monomial $\mu = [i_1, i_2, \ldots, i_s]$ in $\widetilde{\Gamma}_{s,*}$ we define its weight to be the number of the indices $i_1, i_2, \ldots i_s$ that are odd.

Lemma 7. *Let $\mu \in \widetilde{\Gamma}$ be any monomial. Then $(\mu)Sq^1$ lies in the algebra generated by S_0.*

Proof. We will prove the lemma by induction on t, the weight of μ. The case $t = 0$ is tautological. Now suppose that $t \geq 1$ and that the lemma has been proved for all monomials μ of weights less than t. Let $\mu = [i_1, i_2, \ldots, i_s]$ be a given monomial of weight t. Choose an index i_k that is odd; say, $i_k = 2m - 1$. Then $[i_k] = [2m]Sq^1$ and $[i_k]Sq^1 = 0$. Let $\alpha = [i_1, \ldots, i_{k-1}]$ and $\beta = [i_{k+1}, \ldots, i_s]$ so that using the product in $\widetilde{\Gamma}$ we may write: $\mu = \alpha \cdot [i_k] \cdot \beta$. Then,

$$\begin{aligned}(\mu)Sq^1 &= (\alpha \cdot [i_k] \cdot \beta)Sq^1 \\ &= (\alpha)Sq^1 \cdot [i_k] \cdot \beta + 0 + \alpha \cdot [i_k] \cdot (\beta)Sq^1 \\ &= (\alpha)Sq^1 \cdot [i_k] \cdot \beta + (\alpha)Sq^1 \cdot [2m] \cdot (\beta)Sq^1 + (\alpha)Sq^1 \cdot [2m] \cdot (\beta)Sq^1 + \alpha \cdot [i_k] \cdot (\beta)Sq^1 \\ &= (\alpha)Sq^1 \cdot ([2m] \cdot \beta)Sq^1 + (\alpha \cdot [2m])Sq^1 \cdot (\beta)Sq^1.\end{aligned}$$

But the right hand side of this equation is a sum of products of elements of the form $(\gamma)Sq^1$, where in each case, γ is a monomial of weight less than t. So the inductive hypothesis implies that $(\mu)Sq^1$ lies in the algebra generated by S_0, and our inductive proof is complete. □

Combining Lemmas 6 and 7, we find:

Lemma 8. *$\Delta(0)$ is generated as an algebra by the set S_0.*

The next lemma will be useful in proving that the set S_0 is algebraically independent. In what follows, when we write "α expanded in terms of $\widetilde{\Gamma}$", we mean to express α as a sum of monomials $[i_1, i_2, \ldots, i_s] \in \widetilde{\Gamma}_{s,*}$. By abuse of notation, we say that i_j is a factor of the term $[i_1, i_2, \ldots, i_s]$, and so we may speak of the odd or even factors of such a term.

Lemma 9. *S_0 is a linearly independent subset of $\widetilde{\Gamma}$.*

Proof. Since $\widetilde{\Gamma}$ is graded on length, any potential linear dependence will be of the form

$$\sum_j \sigma(m_{j,1}, m_{j,2}, \ldots, m_{j,s}) = 0 \tag{12}$$

where the sum is over a positive number of indices j and the value of s is the same for all terms. We wish to show that such a relation is impossible if the sequences $\{m_{j,1}, m_{j,2}, \ldots, m_{j,s}\}$ are all distinct. But when a term $\sigma(m_{j,1}, m_{j,2}, \ldots, m_{j,s})$ is expanded in terms of $\widetilde{\Gamma}$, the monomial $[2m_{j,1} + 1, 2m_{j,2} + 2, \ldots, 2m_{j,s} + 2]$ will appear, and this monomial cannot appear in any of the other expansions. So a relation of the form (12) is impossible. □

Preparing the proof of the next result, we define $W_k\widetilde{\Gamma}$ to be the subspace of $\widetilde{\Gamma}$ that is spanned by all the monomials $[i_1, i_2, \ldots, i_s]$ of weight k. Then as a bigraded vector space, $\widetilde{\Gamma}$ splits as a direct sum:

$$\widetilde{\Gamma} = \bigoplus_{k \geq 0} W_k\widetilde{\Gamma}. \tag{13}$$

The product on $\widetilde{\Gamma}$ is compatible with this direct sum decomposition, in the sense that:

$$W_k\widetilde{\Gamma} \cdot W_\ell\widetilde{\Gamma} \subseteq W_{k+\ell}\widetilde{\Gamma}. \tag{14}$$

Proposition 10. $\Delta(0) = \mathbb{F}_2\langle S_0\rangle$.

Proof. We have already shown that the set of all products of elements from S_0 generates $\ker Sq^1$ as a vector space. All that remains is to prove that S_0 is an algebraically independent set. According to Lemma 4, it will be enough to prove that S_0 is a *minimal* generating set for the algebra $\ker Sq^1$. Suppose the contrary. Then it would be possible to express one particular generator, say, $\sigma(m_1, m_2, \ldots, m_s)$, as a (non-commutative) polynomial in the other generators. But each generator $\sigma(n_1, n_2, \ldots, n_l)$ lies in $W_1\widetilde{\Gamma}$, so Eqs. (13) and (14) imply that our expression of $\sigma(m_1, m_2, \ldots, m_s)$ in terms of the other generators would reduce to an expression of $\sigma(m_1, m_2, \ldots, m_s)$ as a linear combination of the others. According to Lemma 9, this is not possible. So S_0 must be a minimal generating set for the algebra $\ker Sq^1$, and our proof is complete. □

Using Proposition 10, we can determine a formula for the dimension of the homogeneous components of $\Delta(0)$.

Proposition 11. *Let $c_{s,d}$ be the dimension of the component of $\Delta(0)$ in bidegree (s, d). Let $\eta_{s,d}$ be defined by*

$$\eta_{s,d} = \begin{cases} 0, & \text{if } d \text{ is even} \\ \begin{pmatrix} \frac{d-1}{2} \\ s-1 \end{pmatrix}, & \text{if } d \text{ is odd.} \end{cases}$$

There is a recurrence relation,

$$\begin{aligned} c_{0,0} &= 1 \\ c_{s,0} &= 0, \quad \text{if } s > 0 \\ c_{0,d} &= 0, \quad \text{if } d > 0 \\ c_{s,d} &= \sum_{r=1}^{s} \sum_{a=1}^{d} \eta_{r,a} c_{s-r,d-a}, \quad \text{if } s, d \geq 1. \end{aligned}$$

Proof. Since $\Delta(0)$ is a tensor algebra, $c_{s,d}$ counts the number of terms of length s and degree d. Partition the terms of $\Delta(0)_{s,d}$ by length and degree of the first factor in the term. In what follows, "σ" always represents an algebra generator, such as $\sigma(m_1, m_2, \ldots, m_k)$, while "$\tau$" represents a (possibly empty) product of algebra generators.

$$\Delta(0)_{s,d} = \text{span}\left(\coprod_{r,a} \{\sigma \cdot \tau \mid \ell(\sigma) = r, deg(\sigma) = a, \ell(\tau) = s - r, deg(\tau) = d - a\}\right).$$

Now since τ is an arbitrary term in $\Delta(0)_{s-r,d-a}$, we obtain the formula,

$$c_{s,d} = \sum_{r,a} (\text{number of algebra generators } \sigma \text{ in bidegree } (r, a)) \cdot c_{s-r,d-a}.$$

For a typical σ in bidegree (r, a), we have $\sigma = \sigma(m_1, \ldots, m_r)$ such that $2(m_1 + \cdots + m_r) + 2r - 1 = a$. Thus $m_1 + \cdots + m_r = (a + 1)/2 - r$. So the number of algebra generators σ in bidegree (r, a) is found by counting the number of ordered partitions of $(a + 1)/2 - r$ into r parts. Elementary combinatorics tells us that the number of such partitions is exactly $\eta_{r,a}$ as defined above.

The base cases for the recurrence are easily verified. □

As examples, we find closed formulas for $c_{s,d}$ for small s:

- $c_{1,d} = \begin{cases} 0, & d \text{ even} \\ 1, & d \text{ odd} \end{cases}$
- $c_{2,d} = \begin{cases} \frac{d}{2}, & d \text{ even} \\ \frac{d-1}{2}, & d \text{ odd} \end{cases}$
- $c_{3,d} = \begin{cases} \frac{d(d-2)}{4}, & d \text{ even} \\ \frac{(d-1)^2}{4}, & d \text{ odd.} \end{cases}$

It was pointed out by the referee that the exactness of Sq^1 on $\widetilde{\Gamma}_{s,*}$ implies a nice reduction formula:

$$c_{s,d} + c_{s,d+1} = dim\left(\widetilde{\Gamma}_{s,d+1}\right) = \binom{d}{s-1}. \tag{15}$$

References

[1] D.J. Anick, On the homogeneous invariants of a tensor algebra, in: Algebraic Topology, Proc. Int. Conf. (Evanston 1988), in: Contemp. Math., vol. 96, American Mathematical Society, Providence, RI, 1989, pp. 15–17.
[2] P.M. Cohn, Free Rings and Their Ideals, second ed., Academic Press, New York, 1985.
[3] P.M. Cohn, An Introduction to Ring Theory, Springer, London, 2000.
[4] M. Kameko, Generators of the cohomology of bv_4. preprint, Toyama University, 2003.
[5] Tran Ngoc Nam, $\mathcal{A}$-générateurs génériques pour algèbre polynomiale, Adv. Math. 186 (2004) 334–362.
[6] N. Sum, On the hit problem for the polynomial algebra in four variables. preprint, University of Quynhon, Vietnam, 2007.
[7] R.M.W. Wood, Problems in the Steenrod algebra, Bull. Lond. Math. Soc. 30 (1998) 194–220.
[8] R.M.W. Wood, Hit problems and the Steenrod algebra, in: Proceedings of the Summer School 'Interactions between Algebraic Topology and Invariant Theory', A Satellite Conference of the Third European Congress of Mathematics, Ioannina University, Greece, pp. 65–103, 2000.