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Randomized Kolmogorov and Linear Widths on Generalized Besov Classes with Mixed Smoothness

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Abstract

In this paper, we study the Kolmogorov and the linear widths on the generalized Besov classes $B^{\Omega}_{p,\theta}$ with mixed smoothness in the Monte Carlo setting. Applying the discretization technique and some properties of pseudo-s-scale, we determine the exact asymptotic orders of the Kolmogorov and the linear widths for some values of the parameters p,q,θ.

Keywords: Kolmogorov width; linear width; generalized Besov class; Monte Carlo method

1. Notation and main result

Let X,Y be Banach spaces and X_0 be the unit ball of X. Let S be a continuous operator from X_0 to Y. We seek to approximate S by mappings of the form $u=\varphi\circ N$, where $N:X_0\to\mathbb{R}^n$, $\varphi:N(X_0)\to Y$. N and φ describe a numerical method. We mainly consider the following classes of methods. For fixed $k\in\mathbb{N}$, a rule $u:X_0\to Y$ of the form $u=\varphi\circ N$ is said to be a *Kolmogorov method*, if the information operator N is an arbitrary mapping from X_0 to $\mathbb{R}^k$ and φ extends to a linear mapping from $\mathbb{R}^k$ to Y; a *linear method*, if the information operator N is the restriction of a continuous

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linear mapping from from X_0 to $\mathbb{R}^k$ and φ extends to a linear mapping from $\mathbb{R}^k$ to Y. Let $D_k(X_0,Y), A_k(X_0,Y)$ denote the sets of all Kolmogorov and linear methods which have cardinality equal to k, respectively and put $\mathfrak{D}^n(X_0,Y) := \bigcup_{k=0}^{n} D_k(X_0,Y); \mathfrak{A}^n(X_0,Y) := \bigcup_{k=0}^{n} A_k(X_0,Y)$, such that $\mathfrak{D}(X_0,Y) := \bigcup_{n\in\mathbb{N}} \mathfrak{D}^n(X_0,Y); \mathfrak{A}(X_0,Y) := \bigcup_{n\in\mathbb{N}} \mathfrak{A}^n(X_0,Y)$ give rise to the respective classes of Kolmogorov and linear methods. Denote by $\mathfrak{M}(X_0,Y)$ any of the classes of Kolmogorov and linear methods in this paper.

The worst case error of any method $u \in \mathfrak{M}(X_0,Y)$ is measured by

$$e(S,u) := \sup\{\| S(f) - u(f) \|_Y, f \in X_0\}.$$

Minimizing the errors with respect to the choice of methods within the given class, we get the n-th minimal error defined by

$$e_n(S,\mathfrak{M},X,Y) := \inf\{e(S,u), u \in \mathfrak{M}^{n-1}(X_0,Y)\}.$$

Denote $d_n(S,X,Y) := e_n(S,\mathfrak{D},X,Y)$; $a_n(S,X,Y) := e_n(S,\mathfrak{A},X,Y)$.

Next we pass to the randomized setting. We assume that both X_0 and Y are equipped with their respective Borel σ-algebras $\mathfrak{B}(X_0)$ and $\mathfrak{B}(Y)$, i.e., the σ-algebras generated by the open sets.

Definition 1[3]. Given a class of methods $\mathfrak{M}(X_0,Y)$, a triple $\mathrm{P}_{\mathfrak{M}} := ([\Omega,F,P],u,k)$ is called an $\mathfrak{M}$-Monte Carlo method, if

(1) $[\Omega,F,P]$ is a probability space;

(2) $u:\Omega \to \mathfrak{M}(X_0,Y)$ is such that the mapping $\Phi: X_0 \times \Omega \to Y$ defined by

$$\Phi(f,\omega) := (u(\omega))(f), \quad f \in X_0, \quad \omega \in \Omega,$$

is product measurable into Y and the set $\{(u(\omega))(f), f \in X_0, \omega \in \Omega\}$ is a separable subset in Y;

(3) The cardinality function $k:\Omega \to \mathbb{N}$ is a measurable natural number, for which

$$u_\omega := u(\omega) \in \mathfrak{M}^{k(\omega)}(X_0,Y), \quad \omega \in \Omega.$$

The error of a Monte Carlo method $\mathrm{P}_{\mathfrak{M}}$ is defined as

$$e(S,\mathrm{P}_{\mathfrak{M}}) := \sup\left\{\left(\int_\Omega \| S(f) - u_\omega(f) \|_Y^2 \, dP(\omega)\right)^{1/2}, f \in X_0\right\},$$

with the cardinality $\mathrm{M}C - card(\mathrm{P}_{\mathfrak{M}}) := \int_\Omega k(\omega) dP(\omega)$. The n-th Monte Carlo error is defined as

$$e_n^{MC}(S,\mathfrak{M},X,Y) := \inf\{e(S,\mathrm{P}_{\mathfrak{M}}), \mathrm{M}C - card(\mathrm{P}_{\mathfrak{M}}) \le n-1\}.$$

Denote $d_n^{MC}(S,X,Y) := e_n^{MC}(S,\mathfrak{D},X,Y)$; $a_n^{MC}(S,X,Y) := e_n^{MC}(S,\mathfrak{A},X,Y)$. It is obvious that

$$d_n^{MC}(S,X,Y) \le d_n(S,X,Y); \quad a_n^{MC}(S,X,Y) \le a_n(S,X,Y). \tag{1}$$

Now we introduce the generalized Besov classes. Denote by $L_q(\mathrm{T}^d), 1 < q < \infty$, the space of q-th powers Lebesgue integrable functions defined on the d-dimensional torus $\mathrm{T}^d := [0,2\pi)^d$. Let

$e_d := \{1,\ldots,d\}$, $e \subset e_d := \{1,\ldots,d\}$. If $e = \{j_1,\ldots,j_m\}, j_1 < j_2 < \ldots < j_m$, then we write $t^e := (t_{j_1},\ldots,t_{j_m}),(t^e,1^{\hat{e}}) := (\bar{t}_1,\ldots,\bar{t}_d)$, where $\bar{t}_i = t_i$ for $i \in e$, $\bar{t}_i = 1$ for $i \in \hat{e} = e_d \setminus e$.

Definition 2[4]. For $\Omega(t) = \Psi_l^*$, we write $f \in B^{\Omega}_{p,\theta}(\mathrm{T}^d)$ if it satisfies

(1) $f \in L_p^0(\mathrm{T}^d)$;

(2) for any non-empty $e \subset e_d$,

$$\left\{\int_0^{2\pi}\cdots\int_0^{2\pi}\left(\frac{\Omega^{l^e}(f,t^e)_p}{\Omega(t^e,1^{\hat{e}})}\right)^{\theta}\prod_{j\in e}\frac{dt_j}{t_j}\right\}^{1/\theta} < \infty, 1 \le \theta < \infty, \text{ and } \sup_{t^e>0}\frac{\Omega^{l^e}(f,t^e)_p}{\Omega(t^e,1^{\hat{e}})} < \infty, \theta = \infty ,$$

where $\Omega^{l^e}(f,t^e)_p := \sup_{|h^e|\le t^e}\left\|\Delta_{h^e}^{l^e}(f,x)\right\|_p$, $h^e := (h_{j_1},\ldots,h_{j_m})$ is the mixed modulus of smoothness.

Let I denote the identical imbedding operator from the unit ball of $B^{\Omega}_{p,\theta}(\mathrm{T}^d)$ to $L_q(\mathrm{T}^d)$. We first recall some results on the Kolmogorov and the linear widths on $B^{\Omega}_{p,\theta}(\mathrm{T}^d)$ in the deterministic setting.

Theorem 1[1]. Let $\Omega(t) = \omega(t_1 \ldots t_d)$, where $\omega(t) \in \Psi_l^*$ for some $\alpha > 0$. Then for any natural numbers M and n such that $M \asymp 2^n n^{d-1}$, we have

$$a_M(I, B^{\Omega}_{p,\theta}, L_q) \asymp \begin{cases} \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, & 1 < q \le 2 < p < \infty, 2 \le \theta \le \infty; \\ \omega(2^{-n})n^{(d-1)(1/2-1/\theta)_+}, & 2 < q \le p < \infty, 1 \le \theta \le \infty; \\ \omega(2^{-n})2^{n(1/p-1/q)}n^{(d-1)(1/q-1/\theta)_+}, & 1 < p \le q \le 2, 1 \le \theta \le \infty, \alpha > 1/p - 1/q; \\ \omega(2^{-n})2^{n(1/2-1/q)}, & 1 < p \le 2, p' < q < \infty, 2 \le \theta \le q, \alpha > 1 - 1/q; \\ \omega(2^{-n})2^{n(1/p-1/q)}, & 2 \le p \le q < \infty, 2 \le \theta \le q, \alpha > 1/p - 1/q, \end{cases}$$

where $1/p' + 1/p = 1$.

Our main result is the following theorem.

Theorem 2. Under the assumption of Theorem 1, we have

$$d_M^{MC}(I, B^{\Omega}_{p,\theta}, L_q) \asymp a_M^{MC}(I, B^{\Omega}_{p,\theta}, L_q) \asymp$$
$$\asymp \begin{cases} \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, & 1 < q \le 2 < p < \infty, 2 \le \theta \le \infty; \\ \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, & 2 < q \le p < \infty, 2 \le \theta \le \infty; \\ \omega(2^{-n})2^{n(1/p-1/q)}, & 1 < p \le q \le 2, 1 \le \theta \le q, \alpha > 1/p - 1/q; \\ \omega(2^{-n})2^{n(1/p-1/2)}, & 1 < p \le 2 \le q < \infty, 1 \le \theta \le 2, \alpha > 1/p; \\ \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, & 2 \le p \le q < \infty, 2 \le \theta \le \infty, \alpha > 1/2. \end{cases}$$

Comparing Theorem 1 with Theorem 2, one can see that the randomized methods lead to considerably better rates than those of the deterministic ones for $2 \le p \le q < \infty$. Quantitatively, the gain can reach a factor $(M^{-1}\log^{d-1}M)^{1/p-1/q}$ roughly.

2. Proof of main result

To prove the main result, we use the discretization technique due to Maiorov[2] to reduce the approximation of Besov embedding to those of identity mappings between finite-dimensional spaces. For this purpose, we need some auxiliary notations and lemmas.

We associate every vector $s=(s_1,\dots,s_d)$ whose coordinates are nonnegative integers with the set

$$\rho(s)=\left\{k\in\mathbb{Z}^d:\left\lfloor 2^{s_j-1}\right\rfloor\le\left|k_j\right|<2^{s_j},\ j=1,\dots,d\right\},$$

and for $n\in\mathbb{N}$, let

$$Q_n=\left\{k:k\in\bigcup_{(s,1)\le n}\rho(s)\right\}$$

be a step hyperbolic cross. Consider the Fourier partial sum operators $S_{Q_n}(f)=f*D_{Q_n}$ (see [5]) where $D_{Q_n}(x)=\sum_{k\in Q_n}e^{i(k,x)}$ and a sequence of operators T_n from $B^{\Omega}_{p,\theta}$ to L_q,

$$T_0=S_{Q_0}=0,\quad T_n=S_{Q_n}-S_{Q_{n-1}},\qquad \text{for } n\ge1.$$

We set $S_n=\{s=(s_1,\dots,s_d)\in\mathbb{N}^d:(s,1)=n\}$ and $F_{S_n}=span\{e^{i(k,x)}:k\in\rho(s),s\in S_n\}$.

In what follows, we will give the discretization inequalities which are important for the estimates of widths. These inequalities can be obtained following the idea in Mathe [3]. Here we omit the proofs.

Lemma 1. Let s_M denote any of d_M^{MC},a_M^{MC}. For $M,n\in\mathbb{N}$ and $1\le\theta\le p$, then we have

$$s_M(I,B^{\Omega}_{p,\theta},L_q)\ \square\ \omega(2^{-n})\,|S_n|^{(1/p-1/\theta)}|S_n|^{(1/2-1/q)_-}\,2^{n(1/p-1/q)}s_M(I_{p,q},\ell_p^{\|S_n\|},\ell_q^{\|S_n\|}).$$

Lemma 2. Let s_M denote any of d_M^{MC},a_M^{MC}. For $1<p,q<\infty$, and $M,j_k\in\mathbb{N}$ with $\sum_{k=0}^{\infty}j_k\le M$, then the following inequality holds

$$s_M(I,B^{\Omega}_{p,\theta},L_q)\ \square\ \sum_{k=0}^{\infty}2^{n(1/p-1/q)}\,|S_k|^{-(1/2-1/p)_-}\,|S_k|^{(1/2-1/q)_+}\left\|\tilde{T}_k\right\|s_{j_k}(I_{p,q},\ell_p^{\|S_k\|},\ell_q^{\|S_k\|}),$$

where the operators $\tilde{T}_k$ from $B^{\Omega}_{p,\theta}$ to $F_{S_k}\cap L_p$ have the same definitions as the operators T_k.

Proof of Theorem 2. By the definitions, we have $d_M^{MC}(I,B^{\Omega}_{p,\theta},L_q)\le a_M^{MC}(I,B^{\Omega}_{p,\theta},L_q)$. We only need to estimate the upper bounds for $a_M^{MC}(I,B^{\Omega}_{p,\theta},L_q)$ and the lower bounds for $d_M^{MC}(I,B^{\Omega}_{p,\theta},L_q)$.

We start with the estimates of the lower bounds and divide our consideration into the two cases according to p. First, for $\max\{2,q\}<p<\infty$ and $1<p<\infty$, it is sufficient to estimate the lower bounds for $1<q\le2\le p<\infty$. In this case, we first deal with $2\le\theta\le p$. By Lemma 1 and the width of finite-dimensional ball [3], for any M, choosing a natural number n satisfying $M\ \square\ \|S_n\|$ and $r'M\le\|S_n\|$, we have

$$\begin{aligned} d_M^{MC}(I,B^{\Omega}_{p,\theta},L_q)\ &\square\ \omega(2^{-n})\,|S_n|^{(1/p-1/\theta)}|S_n|^{(1/2-1/q)}\,2^{n(1/p-1/q)}d_M^{MC}(I_{p,q},\ell_p^{\|S_n\|},\ell_q^{\|S_n\|})\\ &\square\ \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}. \end{aligned}\qquad(2)$$

For $p<\theta$, by (2) we can obtain

$$d_M^{MC}(I,B_{p,\theta}^{\Omega},L_q)\ge d_M^{MC}(I,B_{\theta,\theta}^{\Omega},L_q)\asymp \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}.$$

Next, for $1<p\le\min\{2,q\}$ and $1<q<\infty$, by the monotonicity of L_q norm, we only need to estimate the lower bounds for $1<p\le q\le 2$. For $p=\theta$ and $q=2$, by Lemma 1 and the widths of finite-dimensional ball[3], we get

$$d_M^{MC}(I,B_{p,p}^{\Omega},L_2)\asymp \omega(2^{-n})2^{n(1/p-1/2)}.$$

Further, for $p\le\theta\le q$, we have

$$d_M^{MC}(I,B_{p,\theta}^{\Omega},L_2)\ge d_M^{MC}(I,B_{p,p}^{\Omega},L_2)\asymp \omega(2^{-n})2^{n(1/p-1/2)}. \tag{3}$$

According to the above estimate (3), we obtain

$$d_M^{MC}(I,B_{p,\theta}^{\Omega},L_q)\asymp 2^{n(1/2-1/q)}d_M^{MC}(I,B_{p,\theta}^{\Omega}\cap F_{S_n},L_2\cap F_{S_n})\asymp \omega(2^{-n})2^{n(1/p-1/q)}. \tag{4}$$

For $p>\theta$, by the embedding relation $B_{\theta,\theta}^{\Omega_1}\subset B_{p,\theta}^{\Omega}$, where $\Omega_1(t)=\Omega(t)t^{\beta},\beta=1/\theta-1/p$ and the estimate (4) , we can still get the required lower bounds.

Now we pass to the estimates of the upper bounds. By the relation (1) and Theorem 1, it remains to prove the upper bounds for $2\le p<q<\infty$ and $1<p<2\le q<\infty$. In fact, we only need deal with the case $2\le p<q<\infty$. By the embedding theorem the latter can also be solved. Similar to the lower estimates, by Lemma 2 and the widths of finite-dimensional ball [3], we can obtain the required upper bounds. Here we omit the details of proof.

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