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Full Low-Frequency Asymptotics for the Reduced Wave Equation

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Abstract—A new and simple procedure for determining the full low-frequency expansions of solutions of the exterior Dirichlet and Neumann boundary value problem for the Helmholtz equation with variable coefficients in two- and three-dimensional exterior domains is presented.

Keywords—Low frequency asymptotics, Reduced wave equation, Helmholtz equation, Exterior boundary value problems.

1. INTRODUCTION

The scattering of acoustic waves at low frequencies has been the subject of considerable study. In the case of a bounded domain, it is just an exercise to construct complete asymptotic expansions with respect to the frequency of solutions to the reduced wave equation. The case of an unbounded domain presents some specific difficulties related to the radiation condition. We refer to the papers by Werner [1], Weck and Witsch [2,3], and Kleinman and Vainberg [4] and the references cited therein, which are representative of the work conducted in this area. These investigations were motivated not only by the problem in its own right, but also by its application to the large time behavior of solutions to initial boundary value problems for the wave equation [5] and to the existence proofs for nonlinear wave equation.

In the case of Maxwell's equations, the present authors investigated in [6] a new and simple way to attack this problem, and thereby obtained the complete asymptotic expansions of the electric and magnetic fields, as well as a complete characterization of their dependence on the topological properties of the domains under consideration [7].

In the present paper, we apply this new approach to the study of the reduced wave equation in exterior domains at low frequencies which allows us to treat in a simple way nonsmooth variable coefficients and nonsmooth boundaries.

The main steps of our method are as follows. We first consider the scattering for a TM plane wave (i.e., the Dirichlet problem) impinging on dielectric material containing a conducting body. We formulate this problem equivalently on a bounded domain by introducing an adequate Dirichlet-Neumann operator. The expression of the Dirichlet-Neumann operator is derived from our knowledge of the explicit form of the outgoing solution of the Helmholtz equation outside a ball. The total field, as well as the Dirichlet-Neumann operator, is expanded with respect to the

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frequency. The full asymptotic expansions of solutions are established and error estimates are proved by combining a stability result for the Dirichlet-Neumann operator with respect to the frequency with a variational method. Similar techniques also work in the case of TE incident plane wave (i.e., the Neumann problem), but have to be refined due to the fact that the corresponding static problem has in two dimensions an "eigensolution". The generalization of this approach to the acoustic scattering problem from periodic structures, as well as the exterior Robin problem for the reduced wave equation with variable, possibly nonsmooth coefficients is immediate.

2. ASYMPTOTICS

Let Ω_c be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3), and let Ω_e be the complement of $\overline{\Omega_c}$ in $\mathbb{R}^N$. The boundary Γ_c of the conducting body Ω_c is assumed to be of class C^1. By n, we denote the unit normal to Γ_c directed into Ω_e. The conductor Ω_c is surrounded by a bounded dielectric material denoted by Ω_d. We finally set the truncated domain $\Omega_R = \Omega_e \cap B_R$, where B_R is a ball of radius R containing $\overline{\Omega_d \cup \Omega_c}$ and introduce the functional spaces $\mathcal{V}_{TE} = H^1(\Omega_R)$, $\mathcal{V}_{TM} = \{u \in H^1(\Omega_R), u = 0 \text{ on } \Gamma_c\}$, and $H^s(S_R)$ is the Sobolev space of order $s \in \mathbb{R}$ on $S_R = \{r = |x| = R\}$. Let an incident plane wave $u_{in}^{(\omega)}$ impinge on the dielectric material Ω_d. The total wave $u^{(\omega)}$ is the solution of the Helmholtz equation

$$\begin{aligned} \operatorname{div}\frac{1}{\mu}\operatorname{grad} u^{(\omega)} + \omega^2 \varepsilon u^{(\omega)} &= 0, \qquad \text{in } \Omega_e, \\ u^{(\omega)} = 0 \text{ or } \partial_n u^{(\omega)} &= 0, \qquad \text{on } \Gamma_c, \end{aligned}$$

$$u^{(\omega)} - u_{in}^{(\omega)} \text{ satisfies the classical radiation condition,}$$

where the electric permittivity ε and the magnetic permeability μ are assumed to be bounded functions in the dielectric material Ω_d and constant in the exterior domain $\Omega_e \setminus \overline{\Omega_d \cup \Omega_c}$. This paper is devoted to the study of the asymptotic behavior of the wave $u^{(\omega)}$ as the frequency ω goes to zero.

It is now well known that we can reduce the exterior Helmholtz equation to a boundary value problem set in the truncated domain Ω_R for R large enough by making use of the Dirichlet-Neumann operator $T^{(\omega)}$ on S_R defined by

$$T^{(\omega)}\left(\varphi = \begin{cases} \sum_{m\in\mathbb{Z}} \varphi_m e^{im\theta}, \\ \sum_{m=0}^{\infty}\sum_{l=-m}^{m} \varphi_m^l Y_m^l, \end{cases}\right) \mapsto \begin{cases} \sum_{m\in\mathbb{Z}} z_m(\omega,R)\varphi_m e^{im\theta}, \\ \sum_{m=0}^{\infty} z_m(\omega,R) \sum_{l=-m}^{m} \varphi_m^l Y_m^l, \end{cases}$$

where

$$z_m(\omega,R) = \begin{cases} \dfrac{\omega\left(H_m^{(1)}\right)'(\omega R)}{H_m^{(1)}(\omega R)}, & \text{if } N = 2, \\[2ex] \dfrac{\omega\left(h_m^{(1)}\right)'(\omega R)}{h_m^{(1)}(\omega R)}, & \text{if } N = 3. \end{cases}$$

Here, θ is the angular variable, $(Y_m^l)_m$ is an orthonormal sequence of spherical harmonics of order m on the unit sphere, and $H_m^{(1)}$ (respectively, $h_m^{(1)}$) the Hankel function of integer order (respectively, half-integer order). The asymptotic behavior of $u^{(\omega)}$ when ω goes to zero will be deduced from the asymptotic of the pseudo-differential operator $T^{(\omega)}$. We first recall that the inequality $|z_m(\omega,R)| \leq C|m|$ for $|m|$ large enough holds uniformly in a neighborhood of $\omega = 0$ (see, for instance, [6] for a proof). Using the fact that in 3D (respectively, 2D), the map

$\omega \mapsto z_m(\omega, R)$ is analytic with respect to the variable ω in a neighborhood of zero for $m \in \mathbb{Z}$ (respectively, $m \in \mathbb{Z} \setminus \{0\}$) together with the well-known asymptotic expansions

$$H_0^{(1)}(\omega R) = \frac{2i}{\pi} \sum_{m=0}^{+\infty} (-1)^m \frac{\omega^{2m} R^{2m}}{2^{2m}(m!)^2} \left(\ln(\omega\gamma) + \ln R - \sum_{j=1}^{m} \frac{1}{j} \right),$$

and

$$-\frac{i\omega}{2} \left(H_0^{(1)}\right)'(\omega R) = \frac{1}{\pi} \sum_{m=1}^{+\infty} (-1)^m \frac{\omega^{2m} R^{2m-1}}{2^{2m-1} m!(m-1)!} \left(\ln(\omega\gamma) + \ln R + \frac{1}{2m} - \sum_{j=1}^{m} \frac{1}{j} \right) + \frac{1}{\pi R},$$

where $2\gamma = e^{\tilde{\gamma} - i\pi/2}$ and $\tilde{\gamma}$ is Euler's constant, allows us to prove the following key lemma.

LEMMA 2.1. *The Dirichlet-Neumann operator $T^{(\omega)}$ on S_R admits the following asymptotic expansions with respect to the frequency ω:*

$$\begin{aligned} T^{(\omega)} &= T^{(0)} + \frac{1}{\ln(\omega\gamma) + \ln R} T_1^{(0)} + \omega T^{(1)} + \omega^2 T^{(2)} \\ &\quad + \frac{\omega^2}{\ln(\omega\gamma) + \ln R} T_1^{(2)} + \mathcal{R}_3^{(\omega)}, \qquad \text{if } N = 2, \\ T^{(\omega)} &= T^{(0)} + \omega T^{(1)} + \omega^2 T^{(2)} + \omega^3 T^{(3)} + \mathcal{R}_3^{(\omega)}, \qquad \text{if } N = 3, \end{aligned}$$

where the operators $T^{(j)}, T_i^{(j)} : H^{1/2}(S_R) \mapsto H^{-1/2}(S_R)$ are continuous and

$$\left\| \mathcal{R}_3^{(\omega)} \right\|_{\mathcal{L}(H^{1/2}(S_R), H^{-1/2}(S_R))} = \begin{cases} o\left(\dfrac{\omega^2}{\ln(\omega\gamma) + \ln R} \right), & \text{if } N = 2, \\ o\left(\omega^3\right), & \text{if } N = 3. \end{cases}$$

Let us note that the operator $T^{(0)}$ is the Dirichlet-Neumann operator associated with the Laplace equation outside the ball B_R and $T^{(0)}$ (constant) $= T^{(1)}$(constant)$= 0$ in two dimensions. It is also well known that the Dirichlet (respectively, Neumann) boundary value problem for the Helmholtz equation is equivalent to the variational equation:

$$\begin{aligned} a^{(\omega)}(u, v) &= \int_{\Omega_R} \text{grad } u.\ \text{grad } v - \omega^2 \int_{\Omega_R} u, v - \left\langle T^{(\omega)}(u), v \right\rangle, \\ &= \left\langle g_{in}^{(\omega)}, v \right\rangle, \qquad \forall v \in \mathcal{V}_{TM} \text{ (respectively, } \mathcal{V}_{TE}), \end{aligned}$$

where the bracket $\langle , \rangle$ denotes the duality between $H^{-1/2}(S_R)$ and $H^{1/2}(S_R)$ and $g_{in}^{(\omega)} = T^{(\omega)}$ $(u_{in}^{(\omega)}) - \partial_r u_{in}^{(\omega)}$. Lemma 2.1 shows immediately that

$$\begin{aligned} a^{(\omega)}(u, v) &= a^{(0)}(u, v) + \frac{1}{\ln(\omega\gamma) + \ln R} a_1^{(0)}(u, v) + \omega a^{(1)}(u, v) \\ &\quad + \omega^2 a^{(2)}(u, v) + \mathcal{A}_2^{(\omega)}(u, v), \qquad \text{if } N = 2, \\ a^{(\omega)}(u, v) &= a^{(0)}(u, v) + \omega a^{(1)}(u, v) + \omega^2 a^{(2)}(u, v) \\ &\quad + \mathcal{A}_2^{(\omega)}(u, v), \qquad \text{if } N = 3, \end{aligned}$$

where the sesquilinear forms $a^{(i)}$ and $a_j^{(i)}$ are continuous on $H^1(\Omega_R) \times H^1(\Omega_R)$ and

$$\left| \mathcal{A}_2^{(\omega)}(u, v) \right| = \begin{cases} 0\left(\dfrac{\omega^2}{\ln(\omega\gamma) + \ln R} \right) \|u\|_{H^1(\Omega_R)} \|v\|_{H^1(\Omega_R)}, & \text{if } N = 2, \\ 0\left(\omega^3\right) \|u\|_{H^1(\Omega_R)} \|v\|_{H^1(\Omega_R)}, & \text{if } N = 3. \end{cases}$$

Since the incident wave $u_{in}^{(\omega)}$ is a plane wave, the function $g_{in}^{(\omega)}$ admits in two dimensions the following asymptotic expansion with respect to ω:

$$g_{in}^{(\omega)} = g^{(0)} + \frac{1}{\ln(\omega\gamma) + \ln R} g_1^{(0)} + \omega g^{(1)} + \frac{\omega}{\ln(\omega\gamma) + \ln R} g_1^{(1)} + \omega^2 g^{(2)} + \frac{\omega^2}{\ln(\omega\gamma) + \ln R} g_1^{(2)} + o\left(\frac{\omega^2}{\ln(\omega\gamma) + \ln R}\right),$$

where $\langle g^{(0)}, 1\rangle = \langle g^{(1)}, 1\rangle = 0$. Now, if we assume that the wave $u^{(\omega)}$ admits in the truncated domain Ω_R the following asymptotic expansion in two dimensions:

$$u^{(\omega)} = u^{(0)} + \frac{1}{\ln(\omega\gamma) + \ln R} u_1^{(0)} + \omega u^{(1)} + \frac{\omega}{\ln(\omega\gamma) + \ln R} u_1^{(1)} + \omega^2(\ln(\omega\gamma) + \ln R) u_{-1}^{(2)} + \omega^2 u^{(2)} + \cdots,$$

we obtain

$$\begin{aligned}
a^{(0)}\left(u^{(0)}, v\right) &= \left\langle g^{(0)}, v\right\rangle, \\
a^{(0)}\left(u_1^{(0)}, v\right) &= \left\langle g_1^{(0)} + T_1^{(0)}(u^{(0)}), v\right\rangle, \\
a^{(0)}\left(u^{(1)}, v\right) &= \left\langle g^{(1)} + T^{(1)}\left(u^{(0)}\right), v\right\rangle, \\
a^{(0)}\left(u_1^{(1)}, v\right) &= \left\langle g_1^{(1)} + T^{(1)}\left(u_1^{(0)}\right) + T_1^{(0)}\left(u^{(1)}\right), v\right\rangle, \\
a^{(0)}\left(u_{-1}^{(2)}, v\right) &= 0, \\
a^{(0)}\left(u^{(2)}, v\right) &= -\int_{\Omega_R} \varepsilon u^{(0)} v + \left\langle g^{(2)} + T^{(1)}\left(u^{(1)}\right) + T^{(2)}\left(u^{(0)}\right) + T_1^{(0)}\left(u_{-1}^{(2)}\right), v\right\rangle,
\end{aligned}$$

where the sesquilinear form $a^{(0)}$ is defined by

$$a^{(0)}(u, v) = \int_{\Omega_R} \operatorname{grad} u . \operatorname{grad} v - \left\langle T^{(0)}(u), v\right\rangle.$$

It is then immediately obvious from Lemma 2.1 that in the TM case, the terms $u^{(0)}$, $u_1^{(0)}$, $u^{(1)}$, $u_1^{(1)}$, $u_{-1}^{(2)}$, $u^{(2)}$ are well defined by the above variational equations. In the TE case, it is easy to see from the explicit expression of the operator $T^{(0)}$ that the constants are solutions in $\mathcal{V}_{TE}$ of the variational equation $a^{(0)}(u, v) = 0$, $\forall v \in \mathcal{V}_{TE}$ in two dimensions. If we wish to solve these equations in this case, we have to impose that the terms $u^{(0)}, u_1^{(0)}, \ldots, u^{(2)}$ satisfy the additional conditions:

$$\left\langle g_1^{(0)} + T_1^{(0)}\left(u^{(0)}\right), 1\right\rangle = 0, \qquad \left\langle T_1^{(0)}\left(u_1^{(0)}\right), 1\right\rangle = 0, \qquad \left\langle g_1^{(1)} + T_1^{(0)}\left(u^{(1)}\right), 1\right\rangle = 0,$$
$$\left\langle T_1^{(0)}\left(u_1^{(1)}\right), 1\right\rangle = 0, \qquad \left\langle g^{(2)} + T_1^{(0)}\left(u_{-1}^{(2)}\right), 1\right\rangle = \int_{\Omega_R} \varepsilon u^{(0)},$$
$$\left\langle g_1^{(2)} + T^{(2)}\left(u_1^{(0)}\right) + T_1^{(2)}\left(u^{(0)}\right) + T_1^{(0)}\left(u^{(2)}\right), 1\right\rangle = 0.$$

The terms $u^{(0)}, u_1^{(0)}, \ldots, u^{(2)}$ satisfying these additional conditions in the TE case are then well defined by the above variational equations. The additional condition (for instance) on $u^{(0)}$ which gives the unique solvability of $a^{(0)}(u^{(0)}, v) = \langle g^{(0)}, v\rangle$ is the compatibility relation needed to solve $a^{(0)}(u_1^{(0)}, v) = \langle g_1^{(0)} + T_1^{(0)}(u^{(0)}), v\rangle$. Let us also observe that $a^{(0)}(u_{-1}^{(2)}, v) = 0$ implies that $u_{-1}^{(2)} \equiv$ constant (but depends on the radius R) determined by the additional condition

$$\left\langle g^{(2)} + T_1^{(0)}\left(u_{-1}^{(2)}\right), 1\right\rangle = \int_{\Omega_R} \varepsilon u^{(0)}.$$

We are now ready for the formulation of our main result. Let $r^{(\omega)}$ be defined by

$$r^{(\omega)} = \begin{cases} u^{(\omega)} - \left(u^{(0)} + \dfrac{1}{\ln(\omega\gamma) + \ln R} u_1^{(0)} + \omega u^{(1)} \right. \\ \left. + \dfrac{\omega}{\ln(\omega\gamma) + \ln R} u_1^{(1)} + \omega^2 (\ln(\omega\gamma) + \ln R) u_{-1}^{(2)} + \omega^2 u^{(2)} \right), & \text{if } N = 2, \\ u^{(\omega)} - \left(u^{(0)} + \omega u^{(1)} + \omega^2 u^{(2)} \right), & \text{if } N = 3. \end{cases}$$

The stability result for the operator $T^{(\omega)}$ formulated in Lemma 2.1, together with the coerciveness properties of the sesquilinear form $a^{(0)}$ allows us to prove the following theorem.

THEOREM 2.1. *There exist $\omega_0 > 0$ and $C > 0$ such that the following inequality:*

$$\left\| r^{(\omega)} \right\|_{H^1(\Omega_R)} \leq \begin{cases} C \dfrac{\omega^2}{|\ln \omega|}, & \textit{if } N = 2, \\ C\omega^3, & \textit{if } N = 3 \end{cases}$$

holds for any $\omega \in]0, \omega_0[$.

The dependence of the constant C on the radius R can be explicitly characterized.

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