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Symmetric spaces of Hermitian type

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Abstract: Let $M = G/H$ be a semisimple symmetric space, τ the corresponding involution and $D = G/K$ the Riemannian symmetric space. Then we show that the following are equivalent: M is of Hermitian type; τ induces a conjugation on D; there exists an open regular H-invariant cone Ω in $\mathbf{q} = \mathbf{h}^{\perp}$ such that $\mathbf{k} \cap \Omega \neq \emptyset$. We relate the spaces of Hermitian type to the regular and parahermitian symmetric spaces, analyze the fine structure of D under τ and construct an equivariant Cayley transform. We collect also some results on the classification of invariant cones in $\mathbf{q}$. Finally we point out some applications in representations theory.

Keywords: Symmetric spaces, semisimple Lie groups, invariant convex cones, causal orientation, ordering, convexity theorem.

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Introduction

Bounded symmetric domains and their unbounded counterparts, the Siegel domains, have long been an important part of different fields of mathematics, e.g., number theory, algebraic geometry, harmonic analysis and representations theory. So the holomorphic discrete series and other interesting representations of a group live in spaces of holomorphic functions on such domains. In the last years some interplays with harmonic analysis on affine symmetric spaces have also become apparent, e.g., a construction of non-zero harmonic forms related to the discrete series of such spaces (see [44] and the literature there). Also in [31,32] and [11] the notion of holomorphic discrete series and Hardy spaces was generalized to affine symmetric spaces of *Hermitian type*. The intertwining operators into spaces of holomorphic functions on the associated bounded symmetric domain were explicitly written down as well as it was proved, that the analytic continuation of the corresponding functions on the symmetric space was given by an integral operator. But for further work it is necessary to analyze how the involution acts on the fine structure of the domain and describe the geometry of the symmetric spaces of Hermitian type. In particular this holds for a maximal set of strongly orthogonal roots as they contain so many geometrical information.

In the first part of this paper we characterize those spaces in terms of an *infinitesimal causal ordering* [42, 30], the operation of the involution as *conjugation* on the associated bounded domain and in terms of the c-dual respectively dual symmetric space. We describe how the involution acts on geometric datas as strongly orthogonal roots and

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Cayley transforms and then we collect some results from [30] about the classification of H-invariant cones in the tangent space.

Let $M = G/H$ be a semisimple symmetric space, where G is a connected semisimple Lie group and H an open subgroup of the fixpoint group G^τ of some non-trivial involution τ of G. We assume that H contains no non-compact normal subgroup of G. For simplicity we also assume that there is no non-trivial connected compact normal subgroup in G. Let θ be a Cartan involution commuting with τ. Denote by $\mathbf{k}$ the $+1$ eigenspace of θ and $\mathbf{q}$ the -1 eigenspace of τ. That M is of Hermitian type was defined in [31] as $\mathbf{k} \cap \mathbf{q}$ having non-trivial center $\mathbf{c}$ and $\mathbf{z}_{\mathbf{q}}(\mathbf{c}) = \mathbf{q} \cap \mathbf{k}$. It turns out that this definition is exactly the right one to provide the existence of Hardy spaces and holomorphic discrete series associated to M [11]. It also implies that G/K is a bounded symmetric domain and in the case G simple, the above means that the one dimensional center of $\mathbf{k}$ is contained in $\mathbf{q}$. So τ anticommutes with the complex structure on D, i.e., τ defines a complex conjugation on D. It is shown that in general this is equivalent to M being of Hermitian type. Two consequences are:

1. The points fixed under τ may be characterized as the set of real points of D and if φ is another involution of Hermitian type, then the corresponding fixpoint set D^φ is diffeomorphic to D^τ via a diffeomorphism explicitly constructed in terms of the complex structure and the exponential map.

2. In the special case that $D \simeq \mathbb{R}^n + i\Omega$ is a tube domain the Cayley transform leads to an involution whose fixpoint set is exactly the cone Ω. Thus in case that D is a tube domain D^τ is-(up to the above diffeomorphism)-always a self dual proper cone.

This also leads to a classification of all symmetric spaces of Hermitian type using the work of H. Jaffee [13] and [14] where he classifies all non-conjugate complex conjugations on D or equivalently the non-conjugate real forms of D and $\Gamma\backslash D$. In those papers all possible $\mathbf{h}$'s can also be found.

The second point above relates now the holomorphic discrete series of M and its realization on D (see [31,32]) to the work of H. Rossi and M. Vergne [39,40] on the analytic continuation of the holomorphic discrete series of G realizing them as $\mathbf{L}^2$-spaces on the cone for regular parameters or its boundary in the singular cases. As it is possible to write down how τ permutes the strongly orthogonal roots, we know how τ acts on the different boundary components and the associated partial Cayley transforms [19,20], this observation leads to the conclusion that the holomorphic discrete series of M may be realized in general in some $\mathbf{L}^2$-spaces on H-orbits on D or its boundaries, giving some hope for an 'orbit-picture' for this representations [33] but it should be underlined, that this is not *geometric* at all, except for some special cases.

Cones and semigroups have turned up in different fields and problems in harmonic analysis and physics [3,6,10,11,31,35,36,42,43,46] where they are e.g., used for constructing Hardy spaces and defining orderings in symmetric spaces and groups as well as for generalizing the notions of Laplace transformation, Volterra algebra and some special functions to infinitesimal causal spaces. To all the spaces of Hermitian type there is associated a proper H-invariant cone through the element in the center of $\mathbf{k}$ defining the complex structure of D. But there are also other classes of spaces containing proper H-invariant cones but *not* of Hermitian type, the simplest example being the complexified group G_c with the complex conjugation as an involution. If G/K is Hermitian, then $i\mathbf{g}$ always contains G-invariant cones, but G_c/G is *never* of Hermitian type. The characterization of spaces of Hermitian type is now, that they are exactly those spaces having proper H-invariant open cones in $\mathbf{q}$ with

$$\Omega \cap \mathbf{k} \neq \emptyset.$$

The center of $\mathbf{k} \cap \mathbf{q}$ is then the vector space generated by $\Omega^{H \cap K}$.

Now if Ω is an open proper H-invariant cone in $\mathbf{q}$ then it may be shown that either $\Omega \cap \mathbf{k} \neq \emptyset$ or $\Omega \cap \mathbf{p} \neq \emptyset$. The later case corresponds to the *regular* symmetric spaces first introduced by Ol'shanskii in [35] and [36]. We also show that those two types of spaces are *c-dual* to each other in the following sense

$$\mathbf{g} \longleftrightarrow \mathbf{g}^c := \mathbf{g}_c^\eta, \quad \mathbf{h} + \mathbf{q} \longleftrightarrow \mathbf{h} + i\mathbf{q}$$

where $\eta : \mathbf{g}_c \to \mathbf{g}_c$ is the conjugate linear extension of τ to $\mathbf{g}_c = \mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}$. Now the regular symmetric space $M^c = G^c/H$ is an *ordered* space by

$$\{x \in M^c | x \geqslant x_o\} = \Gamma^c(C) \cdot x_o,$$

where C is a closed proper cone in $\mathbf{q}$ such that $C^o \cap \mathbf{k} \neq \emptyset$, $x_o = 1/H$, and $\Gamma^c(C)$ is the closed semigroup $\Gamma^c(C) := \exp(iC)H$.

The functions in the holomorphic discrete series of M extend to analytic functions on the causal interval $\Gamma^c(C)^o$ for 'positive' cones and furthermore $\Gamma^c(C)^{-1}$ is contained in a minimal parabolic subgroup of G^c that is also minimal in the sense of [28]. The corresponding H-invariant *Poisson kernel* [28] is given by $x \mapsto \Phi(x^{-1})$, where Φ is the analytic continuation of the Flensted-Jensen function. This now relates the results of [31, 32 and 11] to *spherical functions* and harmonic analysis on the ordered space M^c. This may be particularly interesting for finding the reproducing H-invariant distribution corresponding to the holomorphic discrete series.

One of the possibilities to generalize the algebra of complex numbers and complex spaces is to introduce the unit j such that $j^2 = 1$ instead of -1. This leads to the *paracomplex numbers* and to *paracomplex spaces* analyzed by Libermann and Fréchet in couple of papers around 1951/1952 [21,22], and 1954 [5], respectively, and to the affine analogue of a Hermitian symmetric space of non-compact type, the *parahermitian symmetric spaces* and *algebras* classfied by S. Kaneyuki [15] by reducing it to the classification of graded Lie algebras of the first kind done by S. Kobayashi and T. Nagano in [18]. Those spaces have been the object of growing interest in the last years [15, 16], and in particular they are shown to have a nice compactification as being the unique open dense orbit of the diagonal action of G on the compact space $K/K \cap H \times K/K \cap H$ (assuming that G is contained in a simply connected complex Lie group), [16]. Furthermore they are symplectic manifolds and may be realized as the cotangent bundle $T^*(K/K \cap H)$ opening the way for constructing representations via polarisation, [25].

Those parahermitian spaces are related to the bounded symmetric domains and the spaces of Hermitian type by the following *dual* (*Riemannian*) construction [4] that is fundamental in the construction of the discrete series of M:

$$\mathbf{g} \leftrightarrow \mathbf{g}^r := \mathbf{g}_c^{\eta\theta},$$

$$\mathbf{h}\cap\mathbf{k}\oplus\mathbf{h}\cap\mathbf{p}\oplus\mathbf{q}\cap\mathbf{k}\oplus\mathbf{q}\cap\mathbf{p} \leftrightarrow \mathbf{h}\cap\mathbf{k}\oplus i\mathbf{h}\cap\mathbf{p}\oplus i\mathbf{q}\cap\mathbf{k}\oplus\mathbf{q}\cap\mathbf{p},$$

where the superscript ^{r} stands for "Riemannian" and $\mathbf{p}$ is the -1 eigenspace of θ. We show that $(\mathbf{g},\tau)$ is of Hermitian type if and only if $(\mathbf{g}^r,\theta)$ is parahermitian. As it is always possible to find to a given parahermitian symmetric algebra a dual Hermitian algebra such that $\mathbf{h}$ goes into the maximal compactly imbedded subalgebra, much of the structure theory of parahermitian spaces is contained in the classical theory of bounded symmetric domains. This also gives a third way of classifying the symmetric spaces of Hermitian type by using the classification in [18].

At this point we know that the following are equivalent:

1. $(\mathbf{g},\tau)$ is of Hermitian type,
2. $(\mathbf{g}^c,\tau)$ is regular,
3. $(\mathbf{g}^r,\theta)$ is parahermitian,
4. There exists an open proper H-invariant cone Ω in $\mathbf{q}$ such that $\Omega\cap\mathbf{k}\neq\emptyset$,
5. τ defines a conjugation on D.

By this it becomes clear that there is an interesting subclass of spaces consisting of all those spaces of Hermitian type that are also regular *or* parahermitian. It is shown that then the space is also parahermitian resp. regular and that the spaces 1.–3. are in fact all isomorphic via a natural Cayley transform that we construct. We show that those are exactly the spaces, where D is a tube domain and τ a square of a classical Cayley transform or equivalently that G/G^τ is an orbit through an hyperbolic element X_o in the Lie algebra such that $\operatorname{ad} X_o$ has only the eigenvalues 0, $+1$, -1. For those spaces we state now the following problems and facts:

I) By [27] and [17] the manifold is given (up to a covering) as $T^*(K/K\cap H)\simeq K\times_{H\cap K}\mathbf{q}\cap\mathbf{k}$.

II) By Lemma 5.4 there exists a group isomorphism $\psi: G\to G^r$ such that $\psi(K)=H^r$ thus inducing a diffeomorphism $\psi: G/K\to G^r/H^r$. This and the construction of Flensted-Jensen [4] gives an intertwining operator from the principal series of G into $\mathrm{L}^2(M)$ in the following way. First the discrete series of M is constructed via Poisson transformation and analytic continuation from the principal series representation of G^r (see [4]). Using the homomorphism ψ to hdentify the principal series of G^r and G, an intertwining operator is produced. Via the Flensted-Jensen isomorphism and boundary-value maps it is also possible to go another way round. By [28] we also have an intertwining operator constructed via Poisson integrals and its analytic continuation in the ν-parameter, which may be zero or having singularities at the points at interest. The problem is then to relate this different operators by some 'regularization'.

III) By Theorem 6.1 and Theorem 5.6 we can find a closed cone C in $\mathbf{q}$ such that $C^o\cap\mathbf{k}=\emptyset$. We may then define a semigroup $\Gamma(C):=\exp(C)H$ and an ordering in

M by $x \geqslant x_0 \Leftrightarrow \exists g \in \Gamma(C) : x = gx_0$ as before (see [3,30,35,36]) such that all finite causal intervals are compact [30], and in fact M is hyperbolic. Thus we can define Volterra kernels, spherical functions and spherical Laplace transform (with respect to the above semigroup) as in [3]. Thus there is a natural problem to classify/construct the spherical functions and invert the Laplace transform.

IV) For general M we determine in [11] the Hardy spaces of M and show that the functions in the holomorphic discrete series extend as holomorphic functions to a complex domain $\Xi(C_g) := G\exp(iC_g)/H_c \simeq G \times_H iC_g \cap \mathbf{q}$, where C_g is a G-invariant cone in $\mathbf{g}$. Now the regular H-invariant cones in $\mathbf{q}$ were classified in [30], where it was also proved that every such cone with $C^o \cap \mathbf{k}$ extends to a G-invariant cone in $\mathbf{g}$. In the above special case, the cone C is unique up to a sign and the domain $\Xi(C_g)$ may also be viewed (up to a singular set) as $G^\tau/H^\tau \times G^\tau/H^\tau$ which, via a Cayley transform, lives in $\mathbf{q}$. This relates harmonic analysis on M to that on $K/K \cap H \times K/K \cap H$ and tube domains over $\mathbf{q}$. Notice that in this case the compactification of M is actually the Shylov boundary of $G^\tau/H^\tau \times G^\tau/H^\tau$ and in fact the 'classic' Hardy spaces on this tube domain can be shown to be naturally isomorphic to the Hardy space on M, [34].

V) Those are the spaces where the H-orbit in D through 0 is the cone C turning up in the realization of D as a tube. Hence we have by [39] and [40] a geometric realization of the holomorphic discrete series via Fourier-Laplace transform on $H/H \cap H \simeq D^\tau$.

As those spaces have so many nice properties they are a natural object for further investigations. Because of their relations to Cayley transforms we call them *symmetric spaces of Cayley type.* They will be introduced and classified in Section 5 where we also give some further examples of special involutions.

One of the main tools in the geometry of D and its boundary components as well as in the theory of holomorphic functions on D, the compactification of $\Gamma\backslash D$ and in the classification of invariant cones in $\mathbf{g}$, is the maximal set of strongly orthogonal roots, the dual vectors in the Cartan subalgebra and root vectors. Those objects are e.g., used to construct Cayley transforms, to analyze boundary components and to construct maximal abelian subalgebras of $\mathbf{p}$. They are also used for constructing imbeddings of $\mathbf{sl}_2$ into $\mathbf{g}$ and for writing down concrete coordinates to estimate the behaviour of functions at infinity and so proving $\mathbf{L}^2$-estimates, [19, 20, 31, 32, 40].

As it is also possible — and in fact natural — to replace the usual constructions using Cartan subalgebra in the group case by constructions build up from a compact *Cartan subspace* in $\mathbf{q} \cap \mathbf{k}$ in the case of symmetric spaces [31], it is necessary to know how τ and the antiholomorphic extension η operate on all of the above mentioned objects and do all the relevant constructions in a η-equivariant fashion to have an overview of the projections onto $\mathbf{h}$ and $\mathbf{q}$ and for describing the H-orbits in different realizations. How the involution acts on root and roots vectors is also important for describing the set of invariant cones and how they extend to cones in $\mathbf{g}$.

In Theorem 3.4 we prove that there exist two disjoint sets $\mathcal{M}$ and $\mathcal{N}$ in $\{1, \ldots, r\}$, where r is the rank of D, such that τ permutes the set of strongly orthogonal roots $\{\gamma_1, \ldots, \gamma_r\}$, enumerated in the usual way, by:

1. If $j \in \mathcal{M}$ then $-\tau\gamma_j = \gamma_j$,
2. if $j \in \mathcal{N}$ then $-\tau\gamma_j = \gamma_{j-1}$

and furthermore $\{1, \ldots, r\} = \mathcal{M} \dot{\cup} \{j, j-1 \mid j \in \mathcal{N}\}$. By this it follows that the maximal set of strongly orthogonal roots relative to a Cartan subspace is given by

$$\{\hat{\gamma}_j = \gamma_j \mid j \in \mathcal{M}\} \dot{\cup} \{\hat{\gamma}_j = \tfrac{1}{2}(\gamma_j - \tau\gamma_j) \mid j \in \mathcal{N}\},$$

that the co-roots and root vectors are related by $\hat{H}_{\hat{\gamma}_j} = H_{\gamma_j}$, $\hat{H}_{\hat{\gamma}_j} = H_{\gamma_j} - \tau H_{-\gamma_j}$ and $\hat{E}_{\hat{\gamma}_j} = E_{\gamma_j}$, $\hat{E}_{\hat{\gamma}_j} = E_{\gamma_j} \pm \tau E_{-\gamma_j}$ for $j \in \mathcal{M}$ resp. $\mathcal{N}$.

We apply this to construct a *η-equivariant* Cayley transform allowing us to describe the H-orbit in the unbounded picture as the set of real points in $D(\Omega, Q)$ independently of τ via an explicit diffeomorphism. As mentioned before this is useful for the representation theory of M and also for generalizing classical theorems such as that of Moore [26] to the context of symmetric spaces of Hermitian type [11].

In the last part we recall some results from [30] about the classification of invariant cones in $\mathbf{q}$. First of all the invariant cones are determined by their projection onto/intersection with a Cartan subspace $\mathbf{a}$ of $\mathbf{q}$, $C = \overline{\mathrm{Ad}(H)(C \cap \mathbf{a})}$. Furthermore $C \cap \mathbf{a} = \mathrm{pr}(C)$ and $C^* \cap \mathbf{a} = (C \cap \mathbf{a})^*$, where $\mathrm{pr} : \mathbf{q} \to \mathbf{a}$ is the orthogonal projection. The main tool in the proof is the generalization of the convexity theorem of Paneitz:

$$\mathrm{pr}(\mathrm{Ad}(h)X) \in \mathrm{con}(W_H \cdot X) + c_{\min}$$

for all $X \in c_{\max}$, to semisimple symmetric pairs. Here W_H is the Weyl group of $\mathbf{a}$ in H, $c_{\min}$ is a minimal W_H- (and $\mathrm{pr}(\mathrm{Ad}(H))$) invariant cone in $\mathbf{a}$ and $c_{\max}$ its dual cone. Finally, we also have, that every cone with $C^o \cap \mathbf{k} \neq \emptyset$ can be extended to a G-invariant cone in $\mathbf{g}$.

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1. Symmetric spaces of Hermitian type

In this section we shall introduce some notations that we will use throughout the paper. Then we recall the definition of a symmetric space M to be of Hermitian type and collect some results from [31] and [32]. We then give a characterization of M to be of Hermitian type in terms of the corresponding involution on the associated Riemannian symmetric space D. We show that up to diffeomorphisms the set of fixpoints on D of τ is independent of τ. If not otherwise stated G will denote a connected semisimple Lie group although most of the results also hold for reductive groups in the Harish-Chandra class. The Lie algebra of G will be denoted by $\mathbf{g}$ and its complexification by $\mathbf{g}_c$. As we are mostly interested in the pair $(\mathbf{g}, \tau)$ we will for simplicity assume if nothing else is said, that G is contained in the simply connected Lie group G_c

with the Lie algebra $\mathbf{g}_c$. Analogous notation will be used for other Lie groups and for vector spaces. In particular, if $\mathbf{q}$ is a subspace of $\mathbf{g}$ we will usually identify $\mathbf{q}_c$ with the complex subspace of $\mathbf{g}_c$ generated by $\mathbf{q}$. Let τ be a non-trivial involution of G commuting with the Cartan involution θ. We denote also by τ respectively θ the corresponding involution on $\mathbf{g}$, $\mathbf{g}_c$, $\mathbf{g}^*$, $\mathbf{g}_c^*$, where the superscript * denotes the dual space. Let $K = G^\theta$ be the fixpoint group of θ in G and let H be an open subgroup of G^τ. Then we have an orthogonal, with respect to the inner product $X, Y \mapsto (X \mid Y)_\theta := -\operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(\theta Y))$, direct sum decomposition

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{q} = \mathbf{k} \oplus \mathbf{p} = \mathbf{h}_k \oplus \mathbf{h}_p \oplus \mathbf{q}_k \oplus \mathbf{q}_p$$

where $\mathbf{h} = \mathbf{g}^\tau$ is the Lie algebra of H, $\mathbf{k} = \mathbf{g}^\theta$ is the Lie algebra of K, $\mathbf{q} := \mathbf{h}^\perp = \{X \in \mathbf{g} \mid \tau(X) = -X\}$, $\mathbf{p} := \mathbf{k}^\perp = \{X \in \mathbf{g} \mid \theta(X) = -X\}$ and the subscript k resp. p denotes the intersection with $\mathbf{k}_c$ resp. $\mathbf{p}_c$. Let $D := G/K$ and $M := G/H$. Then D is a Riemannian symmetric space and M is a pseudo Riemannian symmetric space. Let $\mathbf{c}$ be the *center of* $\mathbf{q}$, i.e.,

$$\mathbf{c} = \{X \in \mathbf{q}_k \mid \forall Y \in \mathbf{q}_k \ : \ [X, Y] = 0\}.$$

Definition 1.1. The pair $(\mathbf{g}, \tau)$ is called of *Hermitian type* if $\mathbf{z}_{\mathbf{q}}(\mathbf{c}) = \mathbf{q}_k$ and there is no non-trivial, non-compact ideal of $\mathbf{g}$ contained in $\mathbf{h}$. We call M and τ of Hermitian type if $(\mathbf{g}, \mathbf{h})$ is of Hermitian type.

From now on we will always assume, that M is of Hermitian type. For an abelian Lie algebra $\mathbf{b}$ and a finite dimensional semisimple $\mathbf{b}$-module $\mathbf{V}$ we use the following notation:

$$\mathbf{V}_\alpha := \{v \in \mathbf{V} \mid \forall X \in \mathbf{b} : Xv = \alpha(X)v\}, \qquad \alpha \in \mathbf{b}^*,$$

$$\Delta(\mathbf{V}, \mathbf{b}) := \{\alpha \in \mathbf{b}^* \mid \alpha \neq 0, \ \mathbf{V}_\alpha \neq 0\},$$

$$\rho(\Gamma) := \frac{1}{2}\sum_{\alpha \in \Gamma}(\dim \mathbf{V}_\alpha)\alpha, \quad \mathbf{V}(\Gamma) := \bigoplus_{\alpha \in \Gamma} \mathbf{V}_\alpha, \qquad \emptyset \neq \Gamma \subset \Delta(\mathbf{V}, \mathbf{b}).$$

Lemma 1.2. *Let* $X \in \mathbf{c}$ *then* $[X, \mathbf{k}] = 0$, *i.e.*, $\mathbf{c} \subset \mathbf{c}_{\mathbf{k}}$ = *the center of* $\mathbf{k}$.

Proof. As $\mathbf{k} = \mathbf{h}_k \oplus \mathbf{q}_k$ we only have to show that $[X, \mathbf{h}_k] = 0$. Let $Y \in \mathbf{h}_k$. Then $([X,Y] | [X,Y])_\theta = -(Y | [X, [X,Y]])_\theta = 0$ as $[X, Y] \in \mathbf{q}_k$. Thus $[X, Y] = 0$ and the claim follows. □

In particular it follows that $\mathbf{z}_{\mathbf{g}_c}(\mathbf{c}_c) = \mathbf{k}_c$. Let $\mathbf{a}$ be a maximal abelian subalgebra of $\mathbf{q}$ containing $\mathbf{c}$. Then it now follows easily (see also [31]):

(i) $\mathbf{a} \subset \mathbf{q}_k$, $\mathbf{g}_c = \mathbf{z}_{\mathbf{g}_c}(\mathbf{a}_c) \oplus \bigoplus_{\alpha \in \Delta} \mathbf{g}_{c\alpha}$, $\Delta := \Delta(\mathbf{g}_c, \mathbf{a}_c)$.

(ii) Let $\alpha \in \Delta$ then $\mathbf{g}_{c\alpha} \cap \mathbf{k}_c \neq 0$ if and only if $\mathbf{g}_{c\alpha} \subset \mathbf{k}_c$ and this is equivalent to $\alpha \mid \mathbf{c}_c = 0$. Hence $\mathbf{z}_{\mathbf{g}_c}(\mathbf{a}_c) \subset \mathbf{k}_c$.

(iii) Let $\Delta_k := \{\alpha \in \Delta \mid \mathbf{g}_{c\alpha} \subset \mathbf{k}_c\}$ and $\Delta_p := \{\alpha \in \Delta \mid \mathbf{g}_{c\alpha} \subset \mathbf{p}_c\}$. Then Δ is the disjoint union of Δ_k and Δ_p, $\mathbf{k}_c = \mathbf{z}_{\mathbf{g}_c}(\mathbf{a}_c) \oplus \mathbf{g}_c(\Delta_k)$, and $\mathbf{p}_c = \mathbf{g}_c(\Delta_p)$.

From now on we keep $\mathbf{a} \subset \mathbf{q}_k$ fixed and use the notations above. We choose the ordering in $i\mathbf{a}^*$ such that $i\mathbf{c}^*$ comes first. Denote by the superscript $^+$ the corresponding positive system. Let $\mathbf{p}_c^+ := \mathbf{p}_c(\Delta_p^+)$ and $\mathbf{p}_c^- := \mathbf{p}_c(\Delta_p^-)$, where $\Delta_p^+ := \Delta_p \cap \Delta^+$ and $\Delta_p^- = -\Delta_p^+$. As $\tau|_{\mathbf{a}} = -1$ it follows that $\tau(\Delta^+) = \Delta^-$ and $\tau(\mathbf{p}_c^+) = \mathbf{p}_c^-$. $\mathbf{p}_c^+$ and $\mathbf{p}_c^-$ are abelian subalgebras of $\mathbf{p}_c$ and $\mathbf{p}_c = \mathbf{p}_c^+ \oplus \mathbf{p}_c^-$.

We recall now Harish-Chandra's realization of D as a bounded symmetric domain in $\mathbf{p}_c^+$. Let K_c, H_c, P^+ and P^- be the analytic subgroups of G_c corresponding to $\mathbf{k}_c$, $\mathbf{h}_c$, $\mathbf{p}_c^+$ and $\mathbf{p}_c^-$, respectively. Let σ be the conjugation of $\mathbf{g}_c$ relative to $\mathbf{g}$. As G_c is simply connected the involutions τ, θ, and σ are defined on G_c and will be denoted by the same letters. Then τ and θ are holomorphic, $G_c^\theta = K_c$, $G_c^\tau = H_c$ and $G_c^\sigma = G$. P^+ and P^- are simply connected and $\exp : \mathbf{p}_c^+ \to P^+$ is a holomorphic diffeomorphism. The set $P^+K_cP^-$ is open (and dense) in G_c and $G \subset P^+K_cP^-$. For $x \in G$ there are unique $p_+(x) \in P^+$, $k_c(x) \in K_c$ and $p_-(x) \in P^-$ such that

$$x = p_+(x)k_c(x)p_-(x).$$

This decomposition induces a bi-holomorphic map

$$D \to D_p,\ xK \mapsto z(xK) := (\exp|_{\mathbf{p}_c^+})^{-1}(p_+(x))$$

of D into a bounded, open and symmetric domain D_p in $\mathbf{p}_c^+$. This is Harish-Chandra's bounded realization of D. Let $\mathbf{c}_{\mathbf{k}_c}$ be the center of $\mathbf{k}_c$. As our subalgebras $\mathbf{p}_c^\pm$ are the same as those of Harish-Chandra it is well known (see [7, p. 393]) that there exists a $Z_0 \in \mathbf{c}_{\mathbf{k}_c} \cap \mathbf{k}$ such that for $J := \operatorname{ad} Z_0|_{\mathbf{p}_c}$:

$$\mathbf{p}_c^\pm = \{Z \in \mathbf{p}_c \mid JZ = \pm iZ\},$$

and J restricted to $\mathbf{p}$ gives the almost complex structure on $\mathbf{p}$ ($\simeq T_{d_0}D$, $d_0 = 1K \in D$), given by the multiplication on D by i. Furthermore, J commutes with $\operatorname{Ad}(K)$ and induces a complex structure on D. Notice that

$$\theta = \operatorname{Ad} k_o^2 \quad \text{and} \quad J = \operatorname{Ad}(k_o)|_{\mathbf{p}}.$$

with $k_o := \exp \frac{1}{2}\pi Z_0 \in K$, see [7, Chapter 8] for details and further references.

Definition 1.3. Let φ be an involution on $\mathbf{p}$ and let J be an almost complex structure on $\mathbf{p}$ commuting with $\operatorname{Ad}(K)$. Then J is called *φ-compatible* (and φ is called *J-compatible*) if $J \circ \varphi = -\varphi \circ J$.

The set $H_cK_cP^-$ is also open (and dense) in G_c and by [31, Theorem 2.4], $G \subset H_cK_cP^-$. This inclusion gives a bi-holomorphic map

$$D \to D_h, \quad xK \mapsto h_c(x)K_c \cap H_c,$$

where D_h is an open simply connected symmetric subset of $H_c/K_c \cap H_c$ and $h_c(x)$ is determined by $x \in h_c(x)K_cP^+$. Define a conjugate linear involution η on $\mathbf{g}_c$ (and on

G_c as antiholomorphic involution) by $\eta := \tau \circ \sigma = \sigma \circ \tau$. Then η leaves the subgroups K_c, H_c, P^+, and P^- stable (this follows from the fact, that θ and τ commutes with η and $\eta|_{i\mathbf{c}} = 1$) and $\eta|_G = \tau$. Via the above identifications τ induces an involution τ_p on D_p and τ_h on D_h. As η leaves P^- and K_c invariant the first part of the next lemma follows.

Lemma 1.4. *Let the notation be as above. Then the following holds:*

(1) $\tau_p = \eta|_{D_p}$ *and* $\tau_h = \eta|_{D_h} = \sigma|_{D_h}$. *Thus* τ *defines a conjugation on* D.

(2) *Let* Z_0 *and* J *be as above. Then* $Z_0 \in \mathbf{c}$ *and* J *is a* τ*-compatible almost complex structure on* $\mathbf{p}$. *Also* $J(\mathbf{h}_p) \subset \mathbf{q}_p$ *and* $J|_{\mathbf{h}_p} : \mathbf{h}_p \to \mathbf{q}_p$ *is an* $\mathbb{R}$*-linear isometric isomorphism. In particular*

$$\dim_{\mathbb{R}} \mathbf{h}_p = \dim_{\mathbb{R}} \mathbf{q}_p = \tfrac{1}{2} \dim_{\mathbb{R}} \mathbf{p}.$$

(3) *Let* $\varphi : D \to D$ *be an antiholomorphic involution with* $\varphi(d_o) = d_o$. *Then there exists an involution* τ *on* $\mathbf{g}$ *commuting with* θ *and of Hermitian type, such that the induced involution on* D *coincides with* φ.

Proof. (2) As $\tau(\mathbf{p}_c^+) = \mathbf{p}_c^-$ we have for all $Z \in \mathbf{p}_c^+$:

$$[-\tau Z_0, Z] = -\tau[Z_0, \tau Z] = -\tau(-i\tau Z) = iZ = [Z_0, Z].$$

In the same way it follows that $\mathrm{ad}(-\tau Z_0)|_{\mathbf{p}_c^-} = \mathrm{ad}(Z_0)|_{\mathbf{p}_c^-}$. As $\tau(\mathbf{c}_{\mathbf{k}_c}) = \mathbf{c}_{\mathbf{k}_c}$ we get $\mathrm{ad}(-\tau Z_0) = \mathrm{ad}\, Z_0$ and thus $-\tau Z_0 = Z_0$. Hence $Z_0 \in \mathbf{c}_{\mathbf{k}_c} \cap \mathbf{q}_{ck} = \mathbf{c}_c$. Now $\tau \circ J = \tau \circ \mathrm{ad}\, Z_0 = \mathrm{ad}(\tau Z_0) \circ \tau = -\mathrm{ad}\, Z_0 \circ \tau = -J \circ \tau$. Thus $J(\mathbf{h}_p) \subset \mathbf{q}_p$ and $J(\mathbf{q}_p) \subset \mathbf{h}_p$. As $J^2 = -1$ the lemma follows.

For the last part we let $\mathbf{l}$ be the maximal compact ideal in $\mathbf{g}$ and define $\mathbf{g}_1$ to be the orthogonal complement of $\mathbf{l}$. Then $\mathbf{g}_1$ is a semisimple ideal. Let G_1 be the analytic subgroup of G and $K_1 = G_1 \cap K$. Then $G_1/K_1 \simeq G/K$ and thus φ defines an antiholomorphic involution on G_1/K_1. If we can prove the claim for $\mathbf{g}_1$ then the lemma follows by extending the corresponding involution to be the identity on $\mathbf{l}$. Thus we may assume that $\mathbf{g}$ is without compact ideals. Let $H(D)$ be the group of holomorphic diffeomorphisms of D. Then by [7, p. 374], $H(D)_o$ is locally isomorphic to G. In particular the Lie algebra of $H(D)_o$ is $\mathbf{g}$. Hence we only have to define τ on $H(D)_o$ and this can be done by

$$\tau(f) := \varphi \circ f \circ \varphi = \mathrm{Int}(\varphi)(f).$$

Let $d \in D$ and choose $f \in H(D)_o$ such that $f(d_o) = d$. Then $\varphi(d) = \varphi(f(d_o)) = [\tau(f)](d_o)$ as $\varphi(d_o) = d_o$. Thus τ induces φ on D. □

The simplest way to see the conjugation is to use the realization of D as D_h. For D_p we notice, that η is an involution of $\mathbf{p}_c^+$. Let $\mathbf{p}_c^+ = \mathbf{p}_c^+(+) \oplus \mathbf{p}_c^+(-)$ be the decomposition of $\mathbf{p}_c^+$ into ± 1-eigenspaces of η. As η is conjugate linear, multiplication by i is an $\mathbb{R}$-linear isomorphism of $\mathbf{p}_c^+(+)$ onto $\mathbf{p}_c^+(-)$. Thus an $\mathbb{R}$-basis $E_1, \ldots, E_n$, $n := \dim_{\mathbb{R}} \mathbf{p}_c^+(+)$, of $\mathbf{p}_c^+(+)$ is also a $\mathbb{C}$ -basis of $\mathbf{p}_c^+$. In the corresponding coordinates $\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto \sum_{j=1}^n z_j E_j \in \mathbf{p}_c^+$ on $\mathbf{p}_c^+$ η is given by $\eta(z) = \bar{z}$, $z \in \mathbb{C}^n$.

Definition 1.5. The pair $(\mathbf{g}, \tau)$ is *irreducible* if $\mathbf{g}$ does not contain any τ-stable ideal.

A list of all possible types of irreducible pairs can be found in e.g., [4, p. 4] or [7, p. 379]. In the case of $(\mathbf{g}, \tau)$ of Hermitian type there are only the possibilities $\mathbf{g}$ simple and Hermitian, e.g., $\mathbf{g}$ is one of the spaces

$$\mathbf{su}(p,q),\ \mathbf{sp}(n,\mathbb{R}),\ \mathbf{so}^*(2n),\ \mathbf{so}(2,n),\ \mathbf{e}_{6(-14)} \quad \text{or} \quad \mathbf{e}_{7(-25)}$$

and τ an involution that is -1 on the one-dimensional center of $\mathbf{k}$ or $\mathbf{g} = \mathbf{g}_1 \times \mathbf{g}_1$ where $\mathbf{g}_1$ is one of the Lie algebras above and $\tau(X,Y) = (Y,X)$. For examples see [29,30,31]. If we assume $\mathbf{g}$ simple, then $\mathbf{c} = \mathbf{c_k} = \mathbb{R}Z_0$ and part (2) is just a reformulation of $\tau|_{\mathbf{c}} = -1$. For $\mathbf{g} = \mathbf{g}_1 \times \mathbf{g}_1$, $\tau(X,Y) = (Y,X)$, we have $\mathbf{p}_c^+ = \mathbf{p}_{c,1}^+ \times \mathbf{p}_{c,1}^-$, [32, Chapter 6], $J = (J_1, -J_1)$ and $Z_0 = (Z_{1,0}, -Z_{1,0})$.

Define a diffeomorphism $\Phi = \Phi_\tau : \mathbf{p} \to D$ (see [23, p. 161]) by

$$(X,Y) \mapsto \exp(X)\exp(Y)d_0, \qquad X \in \mathbf{q}_p,\ Y \in \mathbf{h}_p.$$

(It was pointed out to me by M. Flensted-Jensen, that it was first proved by C.C. Moore, that the above map is a diffeomorphism.) Let φ be another involution of Hermitian type commuting with θ. Let $\tilde{\mathbf{h}}_c := \mathbf{g}^\varphi$, $\tilde{\mathbf{q}} = (-1)$-eigenspace of φ, and let J_φ be a φ-compatible almost complex structure on $\mathbf{p}$ contained in $\mathbf{c_k}$. By Lemma 1.4 there exists an isometry $T : \mathbf{h}_p \to \tilde{\mathbf{h}}_p$. Define $\Psi^* : \mathbf{p} \to \mathbf{p}$ by

$$\Psi^*(JX + Y) := J_\varphi(TX) + TY, \qquad X, Y \in \mathbf{h}_p.$$

Then $\Psi := \Phi_\varphi \circ \Psi^* \circ \Phi_\tau^{-1} : D \to D$ is a diffeomorphism, $\Psi \circ \tau = \varphi \circ \Psi$ and $\Psi(D^\tau) = D^\varphi$. If τ and φ commute then $\mathbf{p} = \mathbf{p}^{-\tau\varphi} \oplus \mathbf{p}^{\tau\varphi}$, where $\mathbf{p}^{-\tau\varphi} = \{X\mathbf{p} \mid \tau\varphi X = -X\}$. If J can be chosen τ- and φ-compatible then Ψ^* may be taken as

$$\Psi^*(X+Y) = JX + Y, \quad X \in \mathbf{p}^{-\tau\varphi},\ Y \in \mathbf{p}^{\tau\varphi}.$$

We notice that this is always the case if $\mathbf{g}$ is simple.

Theorem 1.6. *Let τ and φ be two commuting involutions of Hermitian type commuting with the Cartan involution θ.*

(1) *The map*

$$\Phi_{\tau,\varphi} : \mathbf{q}_p \cap \tilde{\mathbf{q}}_p \oplus \mathbf{q}_p \cap \tilde{\mathbf{h}}_p \oplus \mathbf{h}_p \cap \tilde{\mathbf{q}}_p \oplus \mathbf{h}_p \cap \tilde{\mathbf{h}}_p \to D$$

$$(X_{q\tilde{q}}, X_{q\tilde{h}}, X_{h\tilde{q}}, X_{h\tilde{h}}) \mapsto \exp X_{q\tilde{q}} \exp X_{q\tilde{h}} \exp X_{h\tilde{q}} \exp X_{h\tilde{h}} \cdot d_0$$

is a diffeomorphism.

(2) *Define $\Psi := \Phi_{\varphi,\tau} \circ \Psi^* \circ \Phi_{\tau,\varphi}^{-1}$. Then Ψ is a diffeomorphism, $\Psi \circ \tau = \varphi \circ \Psi$ and $\Psi(D^\tau) = D^\varphi$.*

Proof. (2) follows easily from (1). Let $d \in D$. According to [23, p. 161] there are unique $X \in \mathbf{q}_p$, $Y \in \mathbf{h}_p$ such that $d = \exp X \exp Y \cdot d_0$. By the same fact applied to the triple $(G^{\theta\tau}, \varphi|_{G^{\theta\tau}}, \theta|_{G^{\theta\tau}})$ instead of (G, τ, θ) there are unique $X_{q\tilde{q}} \in \mathbf{q}_p \cap \tilde{\mathbf{q}}_p$, $X_{q\tilde{h}} \in \mathbf{q}_p \cap \tilde{\mathbf{h}}_p$ and $k \in K^\tau$ such that $\exp X = \exp X_{q\tilde{q}} \exp X_{q\tilde{h}} k$. As $k \in K^\tau$ it follows, that $\mathrm{Ad}(k)Y \in \mathbf{h}_p$ and thus, again by [23] now used for $(H, \varphi|_H, \theta|_H)$, there are unique $X_{h\tilde{q}} \in \mathbf{h}_p \cap \tilde{\mathbf{q}}_p$, $X_{h\tilde{h}} \in \mathbf{h}_p \cap \tilde{\mathbf{h}}_p$ and $h \in K \cap H$ with $\exp \mathrm{Ad}(k)Y = \exp X_{h\tilde{q}} \exp X_{h\tilde{h}} h$. It follows that

$$d = \exp X_{q\tilde{q}} \exp X_{q\tilde{h}} \exp X_{h\tilde{q}} \exp X_{h\tilde{h}} \cdot d_0$$

and, as the elements in every step above are unique, this decomposition is unique. It is also clear that this map is differentiable and, as the construction above depends differentiably on $\exp X$ and $\exp Y$ which in turn depends differentiably on d, the lemma follows. □

2. Other classes of symmetric pairs

In this section we introduce two other classes of symmetric pairs $(\mathbf{g}, \tau)$ that are closely related with symmetric spaces of Hermitian type. The first class is the class of *parahermitian symmetric spaces*. We will only formulate the definition for the pair $(\mathbf{g}, \tau)$. For the general case see [27, 15, 16, 30] and the literature there. The other class is related to the *regular real forms of* $\mathbf{g}_c$ introduced by Ol'shanskii for irreducible Lie algebras in his papers [35] and [36] on invariant cones.

Let in this section $(\mathbf{g}, \tau)$ be an arbitary semisimple symmetric pair. Otherwise we keep the notation from the previous sections. Notice that an ideal $\mathbf{m}$ of $\mathbf{h}$ is an ideal of $\mathbf{g}$ if and only if $\mathbf{m} \subset \{X \in \mathbf{h} \mid \mathrm{ad}\, X|_{\mathbf{q}} = 0\}$. This follows from $[\mathbf{m}, \mathbf{q}] \subset \mathbf{m} \cap \mathbf{q} \subset \mathbf{h} \cap \mathbf{q} = \{0\}$. To simplify notation we define the $\mathbf{h}$-representation $\mathrm{ad}_{\mathbf{q}} : \mathbf{h} \to \mathrm{End}(\mathbf{q})$ by

$$\mathrm{ad}_{\mathbf{q}} X := \mathrm{ad}(X)|_{\mathbf{q}}.$$

Definition 2.1. (1) $(\mathbf{g}, \tau)$ is called *effective* if the representation $\mathrm{ad}_{\mathbf{q}}$ of $\mathbf{h}$ is faithful.

(2) $(\mathbf{g}, \tau)$ is called *parahermitian* if there exists a linear endomorphism I_o on $\mathbf{q}$ and a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbf{q}$ such that

(a) $I_o^2 = \mathrm{id}$,

(b) $[I_o, \mathrm{ad}_{\mathbf{q}}\, \mathbf{h}] = 0$,

(c) $\langle I_o X, Y \rangle + \langle X, I_o Y \rangle = 0$ for all $X, Y \in \mathbf{q}$,

(d) $\langle \mathrm{ad}(X)Y, Z \rangle + \langle Y, \mathrm{ad}(X)Z \rangle = 0$ for all $X \in \mathbf{h}$ and $Y, Z \in \mathbf{q}$.

In that case $\{\mathbf{g}, \mathbf{h}, I_o, \langle \cdot, \cdot \rangle\}$ is called a *parahermitian symmetric system.*

(3) Let $Z^o \in \mathbf{g}$. Then $(\mathbf{g}, Z^o)$ is called *graded* (*of the first kind*) if

$$\mathbf{g} = \mathbf{g}(-1) \oplus \mathbf{g}(0) \oplus \mathbf{g}(+1)$$

where $\mathbf{g}(\lambda) = \{X \in \mathbf{g} \mid \mathrm{ad}(Z^o)X = \lambda X\}$.

If Z^o is as in (3), then we can find a Cartan involution θ of $\mathbf{g}$ such that $\theta Z^o = -Z^o$. Furthermore $\mathbf{u} := \mathbf{z}_{\mathbf{g}}(Z_0) \oplus \mathbf{g}(1)$ is a parabolic subalgebra with Levi-decomposition as indicated and $\mathbf{g}(1)$ abelian. Now one of the main results in [17] can be formulated as

Lemma 2.2. *The semisimple symmetric pair* $(\mathbf{g}, \tau)$ *is effective and parahermitian if and only if there exists a* $Z \in \mathbf{g}$ *such that* $(\mathbf{g}, Z)$ *is graded and* $\mathbf{z}_{\mathbf{g}}(Z) = \mathbf{h}$. *In particular,* $Z \in \mathbf{h}$, $\operatorname{ad} Z$ *has the eigenvalues* $0, 1, -1$ *and on* $\mathbf{g}_c$ *the involution* τ *is given by*

$$\tau = \exp(\pi i \operatorname{ad} Z).$$

The idea of the proof is to show, that the linear map $D : \mathbf{g} \to \mathbf{g}$ $D|_{\mathbf{h}} = 0$, $D|_{\mathbf{q}} = I_o$ is a derivation. As $\mathbf{g}$ is semisimple then there exists a $Z \in \mathbf{g}$ such that $D = \operatorname{ad} Z$. The other direction is: define $I_o = \operatorname{ad}_{\mathbf{q}} Z$, and let $\langle \cdot , \cdot \rangle$ be the restriction of the Killing-form to $\mathbf{q}$.

Remark 2.3. Let $(\mathbf{g}, \tau)$ be a semisimple symmetric pair associated with the symmetric space G/H, then $(\mathbf{g}, \tau)$ is parahermitian if and only if G/H is parahermitian and $H \subset C_G(Z) := \{a \in G \mid \operatorname{Ad}(a)Z = Z\}$ where $Z \in \mathbf{h}_p$ is given by $I_o = \operatorname{ad}_{\mathbf{q}} Z$. If G is contained in G_c as we are assuming in this paper, then we always have $C_G(Z) = G^\tau$ as this obviously holds for the simply connected group G_c (see [17, Lemma 3.5, p. 91)].

We introduce now the regular spaces by interchangig the rôle of the compact and non-compact part of $\mathbf{q}$. For $\mathbf{g}$ simple those spaces were first introduced by Ol'shanskii in [35,36]

Definition 2.4. The semisimple symmetric pair $(\mathbf{g}, \tau)$ is called *regular* if $\mathbf{z}_{\mathbf{q}}(\mathbf{c}_p) = \mathbf{q}_p$ where $\mathbf{c}_p$ is the center of $\mathbf{q}_p$.

View $\mathbf{g} \subset \mathbf{g}_c$ and let σ be the conjuagtion of $\mathbf{g}_c$ relative to $\mathbf{g}$. Define

$$\mathbf{g}^c := \mathbf{g}_c^\eta = \mathbf{h} \oplus i\mathbf{q} \quad \text{and} \quad \mathbf{g}^r := \mathbf{g}_c^{\theta\eta} = \mathbf{h}_k \oplus i\mathbf{h}_p \oplus \mathbf{q}_p \oplus i\mathbf{q}_k.$$

If we want to keep in mind the involution we use to construct $\mathbf{g}^r$ or $\mathbf{g}^c$ we write $(\mathbf{g}, \tau)^c := \mathbf{g}^c$ and $(\mathbf{g}, \tau)^r := \mathbf{g}^r$. By holomorphic extension and then restriction τ and θ define involutions on $\mathbf{g}^c$ and $\mathbf{g}^r$. We denote those involutions by the same letters or with the superscript ^{c} respectively ^{r}. Then $\tau^c = \sigma|_{\mathbf{g}^c}$ and $\tau^c\theta^c =: \theta_0$ is a Cartan involution of $\mathbf{g}^c$. τ^r is a Cartan involution of $\mathbf{g}^r$ and $\theta\tau|_{\mathbf{g}^r} = \sigma|_{\mathbf{g}^r}$. To simplify notation we define the *associated pair* by $(\mathbf{g}, \tau)^a := (\mathbf{g}, \tau \circ \theta)$ and $\tau^a = \tau \circ \theta$. Notice that ^{r} and ^{c} are related by

$$(\mathbf{g}, \tau, \theta)^r = (\mathbf{g}, \tau^a, \theta)^c.$$

with the obvious notation. The pair $(\mathbf{g}, \tau)^c$ is called the *c-dual* of $(\mathbf{g}, \tau)$ and $(\mathbf{g}, \tau)^r$ is the *dual* or *Riemannian dual* of $(\mathbf{g}, \tau)$.

Definition 2.5. Let $(\mathbf{g}, \tau)$ and $(\mathbf{l}, \varphi)$ be two symmetric pairs. Then $(\mathbf{g}, \tau)$ and $(\mathbf{l}, \varphi)$ are *isomorphic*, $(\mathbf{g}, \tau) \simeq (\mathbf{l}, \varphi)$, if there exists an isomorphism of Lie algebras $\lambda : \mathbf{g} \to \mathbf{l}$ such that

$$\lambda \circ \tau = \varphi \circ \lambda.$$

Lemma 2.6. *Assume that* $(\mathbf{g}, \tau)$ *is effective. Then*

(1) $(\mathbf{g}, \tau)$ *is of Hermitian type, respectively regular if and only if the same holds for each irreducible factor.*

(2) $(\mathbf{g}, \tau)$ *is regular if and only if* $(\mathbf{g}, \tau^a)$ *is is parahermitian.*

(3) *Let* $(\mathbf{g}, \tau)$ *be of Hermitian type. Choose* $Z_o \in \mathbf{c}$ *defining a complex structure on* $\mathbf{p}$ *(and* $\theta = \exp \pi Z_o$*). Define* $\varphi := \exp \frac{1}{2}\pi Z_o$*. Then* $\tau \circ \varphi = \varphi \circ \tau^a$ *and* φ *induces isomorphisms*

$$(\mathbf{g}, \tau) \simeq (\mathbf{g}, \tau)^a, \quad (\mathbf{g}^c, \tau) \simeq (\mathbf{g}^r, \tau^a) \quad \text{and} \quad (\mathbf{g}^c, \theta) \simeq (\mathbf{g}^r, \theta).$$

Proof. The first part is obvious. For the second claim we notice first that $(\mathbf{g}, \tau^a)$ parahermitian $\Rightarrow$ $(\mathbf{g}, \tau)$ regular by Lemma 2.2. For the other direction we go over to the Riemannian dual of $(\mathbf{g}, \tau^a)$. As the maximal compact subalgebra of that algebra has a non-trivial center, it follows by [7, Chapter 7] that there exists an X such that $\mathbf{h}^a = \mathbf{z_g}(X)$ and $(\mathbf{g}, X)$ is graded of the first category. For the last claim we know that $\theta = \mathrm{Ad}(\exp \pi Z_o)$. Thus $\varphi^4 = \mathrm{id}$, $\varphi^2 = \theta$ and

$$\varphi \circ \tau = \tau \circ \mathrm{Ad}\left(\exp\left(-\frac{\pi}{2} Z_o\right)\right) = \tau \circ \varphi^3 = \tau \circ \theta \circ \varphi.$$

Therefore $\varphi \circ \tau = \tau^a \circ \varphi$ and (in the same way) $\tau \circ \varphi = \varphi \circ \tau^a$. As $\theta \circ \varphi = \varphi \circ \theta$ and $\sigma \circ \varphi = \varphi \circ \sigma$, as well as $\mathbf{g}^r = \mathbf{g}_c^{\theta\eta}$ and $\mathbf{g}^c = \mathbf{g}_c^{\eta}$ the second part follows. □

Theorem 2.7. *Let* $(\mathbf{g}, \tau)$ *be an effective symmetric pair such that* $\mathbf{g}$ *has no compact ideals. Then the following are equivalent.*

(1) $(\mathbf{g}, \tau)$ *is of Hermitian type*;

(2) $(\mathbf{g}^c, \tau)$ *is regular*;

(3) $(\mathbf{g}^c, \theta)$ *is effective and parahermitian*;

(4) $(\mathbf{g}, \theta\tau)$ *is of Hermitian type*;

(5) $(\mathbf{g}^r, \theta\tau)$ *is regular*;

(6) $(\mathbf{g}^r, \theta)$ *is effective and parahermitian.*

Proof. As $\mathbf{q}^c = i\mathbf{q}$ and $\mathbf{q}_p^c = i\mathbf{q}_k$, it follows that $\mathbf{c}_p^c = i\mathbf{c}$. Thus (1) and (2) are obviously equivalent. Assume (1) and choose $Z_o \in \mathbf{c}$ as before. Then $\mathrm{ad}(Z_o)$ has in $\mathbf{g}_c$ the eigenvalues $0, i, -i$ and

$$\mathbf{g}_c(0) = \mathbf{k}_c, \quad \mathbf{g}_c(i) = \mathbf{p}_c^+ \quad \text{and} \quad \mathbf{g}_c(-i) = \mathbf{p}_c^-.$$

It follows that $-iZ_o \in \{X \in \mathbf{g}^c \mid \theta(X) = X\}_p = \mathbf{k}_c \cap \mathbf{p}^c$, that $(\mathbf{g}^c, -iZ_o)$ is graded (of the first kind) and $\mathbf{z}_{\mathbf{g}^c}(-iZ_o) = \mathbf{g}^c \cap \mathbf{k}_c$. That $(\mathbf{g}, \theta)$ is effective is just the assumption that $\mathbf{g}$ has no compact ideals. Thus (3) holds. By reversing the arguments (1) follows from (3). The theorem now follows from Lemma 2.6. □

We give now a list of the spaces occurring in Theorem 2.7 for $\mathbf{g}$ simple. In the first column we list the simple Lie algebras such that $(\mathbf{g}, \mathbf{h})$ is of Hermitian type. In the second column we list the c-dual regular Lie algebras $\mathbf{g}^c$. By the above theorem we have

$$\text{Hermitian type } (\mathbf{g}, \tau) \Leftrightarrow (\mathbf{g}^c, \tau) \text{ regular}$$

where the correspondence is a bijection. In the third column we list the fixpoint algebra $\mathbf{h}$ and then we give the subalgebra $\mathbf{k}_c \cap \mathbf{g}^c$ occurring as the fixpoint algebra in the parahermitian case. In the last one we list rank $\mathbf{g}$, rank G^c/K^c = rank G/H, and rank $\mathbf{h}$, in this ordering. Here $n = p + q$ if p and q are given. We always assume that $0 \leqslant p \leqslant q$ and not both equal 0. The group case is listed in the second table. The list is taken from [2, 17, 27].

$\mathbf{g}$	$\mathbf{g}^c$	$\mathbf{h}$	$\mathbf{k}_c \cap \mathbf{g}^c$	Rank: $\mathbf{g}$, M, $\mathbf{h}$
$\mathbf{su}(p,q)$	$\mathbf{sl}(p+q,\mathbb{R})$	$\mathbf{so}(p,q)$	$\mathbf{sl}(p,\mathbb{R}) \times \mathbf{sl}(q,\mathbb{R}) \times \mathbb{R}$	$n-1, n-1, [\frac{n}{2}]$
$\mathbf{su}(n,n)$	$\mathbf{su}(n,n)$	$\mathbf{sl}(n,\mathbb{C}) \times \mathbb{R}$	$\mathbf{sl}(n,\mathbb{C}) \times \mathbb{R}$	$2n-1, n, 2n-1$
$\mathbf{su}(2p,2q)$	$\mathbf{su}^*(2(p+q))$	$\mathbf{sp}(p,q)$	$\mathbf{su}^*(2p) \times \mathbf{su}^*(2q) \times \mathbb{R}$	$2n-1, n-1, n$
$\mathbf{so}^*(2n)$	$\mathbf{so}(n,n)$	$\mathbf{so}(n,\mathbb{C})$	$\mathbf{sl}(n,\mathbb{R}) \times \mathbb{R}$	$n, n, 2[\frac{n}{2}]$
$\mathbf{so}^*(4n)$	$\mathbf{so}^*(4n)$	$\mathbf{su}^*(2n) \times \mathbb{R}$	$\mathbf{su}^*(2n) \times \mathbb{R}$	$2n, n, 2n$
$\mathbf{so}(2,p+q)$	$\mathbf{so}(p+1,q+1)$	$\mathbf{so}(p,1) \times \mathbf{so}(1,q)$	$\mathbf{so}(p,q) \times \mathbb{R}$	$[\frac{n+2}{2}], p+1,$ $[\frac{p+1}{2}] + [\frac{q+1}{2}]$
$\mathbf{sp}(n,\mathbb{R})$	$\mathbf{sp}(n,\mathbb{R})$	$\mathbf{sl}(n,\mathbb{R}) \times \mathbb{R}$	$\mathbf{sl}(n,\mathbb{R}) \times \mathbb{R}$	n, n, n
$\mathbf{sp}(2n,\mathbb{R})$	$\mathbf{sp}(n,n)$	$\mathbf{sp}(n,\mathbb{C})$	$\mathbf{su}^*(2n) \times \mathbb{R}$	$2n, n, 2n$
$\mathbf{e}_{6(-14)}$	$\mathbf{e}_{6(6)}$	$\mathbf{sp}(2,2)$	$\mathbf{so}(5,5) \times \mathbb{R}$	$6, 6, 2$
$\mathbf{e}_{6(-14)}$	$\mathbf{e}_{6(-26)}$	$\mathbf{f}_{4(-20)}$	$\mathbf{so}(9,1) \times \mathbb{R}$	$6, 2, 4$
$\mathbf{e}_{7(-25)}$	$\mathbf{e}_{7(-25)}$	$\mathbf{e}_{6(-26)} \times \mathbb{R}$	$\mathbf{e}_{6(-26)} \times \mathbb{R}$	$7, 3, 7$
$\mathbf{e}_{7(-25)}$	$\mathbf{e}_{7(7)}$	$\mathbf{su}^*(8)$	$\mathbf{e}_{6(6)} \times \mathbb{R}$	$7, 7, 7$

$\mathbf{g}$: Hermitian type	$\mathbf{g}^c$: Regular	$\mathbf{h}$
$\mathbf{su}(p,q) \times \mathbf{su}(p,q)$	$\mathbf{sl}(p+q,\mathbb{C})$	$\mathbf{su}(p,q)$
$\mathbf{so}^*(2n) \times \mathbf{so}^*(2n)$	$\mathbf{so}(2n,\mathbb{C})$	$\mathbf{so}^*(2n)$
$\mathbf{so}(2,n) \times \mathbf{so}(2,n)$	$\mathbf{so}(n+2,\mathbb{C})$	$\mathbf{so}(2,n)$
$\mathbf{sp}(n,\mathbb{R}) \times \mathbf{sp}(n,\mathbb{R})$	$\mathbf{sp}(n,\mathbb{C})$	$\mathbf{sp}(n,\mathbb{R})$
$\mathbf{e}_{6(-14)} \times \mathbf{e}_{6(-14)}$	$\mathbf{e}_6$	$\mathbf{e}_{6(-14)}$
$\mathbf{e}_{7(-25)} \times \mathbf{e}_{7(-25)}$	$\mathbf{e}_7$	$\mathbf{e}_{7(-25)}$

Up to now we only have looked at the infinitesimal situation. We describe now shortly how to construct the corresponding spaces. Remember that we are assuming that $G \subset G_c$ where G_c is simply connected. Thus we can define the involutins $\theta, \tau, \tau^a, \eta, \sigma$ etc. on G_c, and as G_c is simply connected the fixpoint groups of those involutions are all connected. For H we have the Cartan decomposition $H = (H \cap K)\exp(\mathbf{h}_p)$. Define $(G^{\tau^a})_o \subset H^a := (H \cap K)\exp(\mathbf{q}_p) \subset G^{\tau^a}$, $G^c := G_c^\eta$ and $H^c := H \subset G^c$. As $\tau \circ \theta$ is a

Cartan involution on $\mathbf{g}^c$ we see that $(\tau^c)^a = \theta$ and thus $(H^c)^a$ is well defined. Thus we can now define

$$M^a := G/H^a, \quad M^c := G^c/H^c \quad \text{and} \quad M^{ca} = G^c/H^{ca}.$$

Analogously we can define the correspodning spaces for the Riemannian dual, herby using the relation between ^{c} and ^{r}. In that case $H^r = (G^r)^\tau$ is a maximal compact subgroup of G^r and thus $M^r := G^r/H^r$ is of non-compact type and independent of the covering group we use.

3. The strongly orthogonal roots

In this section we show that τ (τ of Hermitian type) permutes the maximal set of strongly orthogonal roots of $\Delta(\mathbf{p}_c^+, \mathbf{t}_c)$ ($\mathbf{t}_c$ a compact Cartan subalgebra) in a very simple way, and henceforth, that the constructions in [19,20,40] can be done in τ-equivariant fashion. Then we relate this to the root system Δ.

Let $\mathbf{t}$ be a Cartan subalgebra of $\mathbf{k}$ (and $\mathbf{g}$) containing $\mathbf{a}$. Then $\mathbf{t} = \mathbf{a} \oplus \mathbf{t} \cap \mathbf{h}$ is τ-stable ($X \in \mathbf{t} \Rightarrow X - \tau X \in \mathbf{q} \cap \mathbf{z_g}(\mathbf{a}) = \mathbf{a} \Rightarrow \tau X \in X + \mathbf{a} \subset \mathbf{t}$). Choose an ordering in $i\mathbf{t}^*$ such that $i\mathbf{a}^*$ comes first. Denote again the corresponding set of positive roots by the superscript $^+$. Choose some τ- and (Weyl group)-invariant inner product $(\cdot \mid \cdot)$ on $i\mathbf{t}^*$ (e.g., that coming from the Killing form of $\mathbf{g}_c$, [31, p. 135]). We recall the following definition:

Definition 3.1. Let Σ denote one of the sets of roots. Then $\alpha, \beta \in \Sigma$ are called *strongly orthogonal* if $\alpha \neq \pm\beta$ and $\alpha \pm \beta \notin \Sigma$.

Notice, that α, β strongly orthogonal implies α, β orthogonal. Assume for the moment, that $\mathbf{g}$ is simple. Let r be the real rank of $\mathbf{g}$ and let $\Gamma_r := \Delta(\mathbf{p}_c^+, \mathbf{t}_c)$. Let γ_r be the highest root in Γ_r. If we have defined $\Gamma_r \supset \Gamma_{r-1} \supset \cdots \supset \Gamma_k \neq \emptyset$ and $\gamma_j \in \Gamma_j$, $j = k, \ldots r$, we define Γ_{k-1} to be the set of all γ in Γ_k that are strongly orthogonal to γ_k. If Γ_{k-1} is not empty (or equivalent $k > 1$) we let γ_{k-1} be the highest root in Γ_{k-1}. Set $\Gamma := \{\gamma_1, \ldots, \gamma_r\}$. If $\mathbf{g}$ is of the form $\mathbf{g}_1 \times \mathbf{g}_1$, let $\Gamma_0 = \{\gamma_1^0, \ldots, \gamma_s^0\}$, $s = r/2$ be the above constructed set for $\mathbf{g}_1$. Let

$$\gamma_{2j} := (\gamma_j^0, 0), \quad \gamma_{2j-1} := (0, -\gamma_j^0), \qquad j = 1, \ldots, s.$$

Then $\Gamma := \{\gamma_j \mid j = 1, ..., r\}$ is a maximal set of strongly orthogonal roots in $\Delta(\mathbf{p}_c^+, \mathbf{t}_c)$ and

$$-\tau\gamma_{2j} = \gamma_{2j-1}, \quad -\tau\gamma_{2j-1} = \gamma_{2j}, \qquad j = 1, \ldots, s.$$

We will now generalize this and describe how τ permutes the strongly orthogonal roots in general. For that we need first the following lemma, that can also be found in [24, p. 65] but with a different proof.

Lemma 3.2. *Let $\alpha \in \Delta(\mathbf{p}_c, \mathbf{t}_c)$. If $\tau\alpha \neq \pm\alpha$ then α and $\tau\alpha$ are strongly orthogonal.*

Proof. Assume as we may that α is positive. Then $-\tau\alpha$ is positive too and thus $\alpha - \tau\alpha$ is not a root. Let $X \in \mathbf{p}_{c\alpha}$. Then $\tau X \in \mathbf{p}_{c\tau\alpha}$ and $[X, \tau X] \in \mathbf{k}_{c(\alpha+\tau\alpha)} \cap \mathbf{q}_c \subset \mathbf{q}_{ck} \cap \mathbf{z}_{\mathbf{g}_c}(\mathbf{a}_c) = \mathbf{a}_c \subset \mathbf{t}_c$ and thus $\alpha + \tau\alpha$ cannot be a root. □

For a linear form $\lambda \in \mathbf{t}_c^*$ we set

$$\hat{\lambda} := \lambda|_{\mathbf{a}_c} = \tfrac{1}{2}(\lambda - \tau\lambda),$$
$$\tilde{\lambda} := \lambda|_{\mathbf{t}_c \cap \mathbf{h}_c} = \tfrac{1}{2}(\lambda + \tau\lambda),$$
$$\lambda^\vee = 2\|\lambda\|^{-2}\lambda.$$

Corollary 3.3. *Let $\alpha \in \Delta(\mathbf{p}_c, \mathbf{t}_c)$. If $\tau\alpha \neq -\alpha$ then $\|\hat{\alpha}\| = \|\tilde{\alpha}\|$ and $\|\alpha\|^2 = 2\|\hat{\alpha}\|^2$.*

Proof. As $\alpha \in \Delta(\mathbf{p}_c, \mathbf{t}_c)$, $\alpha|_{\mathbf{a}} \neq 0$, thus $\tau\alpha \neq \pm\alpha$ and by the above lemma α and $\tau\alpha$ are strongly orthogonal and thus orthogonal. Hence

$$0 = (\alpha \mid -\tau\alpha) = (\hat{\alpha} \mid \hat{\alpha}) - (\tilde{\alpha} \mid \tilde{\alpha}).$$

As $(\alpha \mid \alpha) = (\hat{\alpha} \mid \hat{\alpha}) + (\tilde{\alpha} \mid \tilde{\alpha})$ the claim follows. □

Theorem 3.4. *Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ be the maximal set of strongly orthogonal roots enumerated as above. Then there exist two disjoint sets $\mathcal{M}$ and $\mathcal{N}$ in $\{1, \ldots, r\}$ such that*

(1) $\{1, \ldots, r\} = \mathcal{M} \dot{\cup} \{j, j-1 \mid j \in \mathcal{N}\}$,
(2) *if* $j \in \mathcal{N}$ *then* $-\tau\gamma_j = \gamma_{j-1}$,
(3) *for* $j \in \mathcal{M}$, $-\tau\gamma_j = \gamma_j$.

Proof. By the definition of Γ we can assume that $(\mathbf{g}, \tau)$ is irreducible. By the above the theorem holds for $\mathbf{g} = \mathbf{g}_1 \times \mathbf{g}_1$. Thus we may assume, that $\mathbf{g}$ is simple. We then prove the theorem by induction on r. If $r = 1$ we set $\mathcal{M} := \{1\}$ and $\mathcal{N} := \emptyset$.

Assume then that the theorem holds for all $s < r$. Let γ_r and Γ_r be as before. If $-\tau\gamma_r = \gamma_r$ we set $\Lambda := \Gamma_{r-1}$ otherwise γ_r and $\delta := -\tau\gamma_r$ are strongly orthogonal, e.g., $\delta \in \Gamma_{r-1}$. Assume, that δ is *not* the highest root in Γ_{r-1}. Then we can find a $\gamma \in \Gamma_{r-1}$, some $\alpha \in \Delta^+(\mathbf{g}_c, \mathbf{t}_c)$ and natural numbers $n_\alpha > 0$ such that

$$\gamma = \delta + \sum_\alpha n_\alpha \alpha.$$

Let $Z_0 \in \mathbf{c}$ be as in Section 1. Then $\alpha(Z_0) = 0$ or i according to α compact or non-compact. Thus $\sum n_\alpha \alpha(Z_0) = 0$ and it follows that all the α's are compact.

We now claim that $(\alpha \mid \gamma_r) = 0$ for all α. As δ and γ are both orthogonal to γ_r it follows that

$$\sum_\alpha n_\alpha (\alpha \mid \gamma_r) = 0.$$

As $n_\alpha > 0$ and $(\alpha \mid \gamma_r) \geqslant 0$ (otherwise $\gamma_r + \alpha$ would be a positve non-compact root greater than γ_r) the claim follows.

Let $\beta := \sum n_\alpha \alpha$. Then $(\gamma \mid \delta^\vee) = 2 + (\beta \mid \delta^\vee)$ and $(\gamma \mid \delta) \geqslant 0$ (otherwise $\gamma + \delta$ would be a root). As $(\gamma - 3\delta)(Z_0) = -2i$, $\gamma - 3\delta$ is not a root and $(\gamma \mid \delta^\vee) \leqslant 2$. Hence $(\beta \mid \delta^\vee) \leqslant 0$. If we assume $(\alpha \mid \delta) < 0$ then $0 > (\alpha \mid \delta) = (-\tau\alpha \mid \gamma_r)$ and so $\gamma_r + (-\tau\alpha) \in \Delta(\mathbf{p}_c^+, \mathbf{t}_c)$. But then $-\tau\alpha$ is negative. As $-\tau$ leaves the set $\{\alpha \in \Delta^+(\mathbf{g}_c, \mathbf{t}_c) \mid \hat{\alpha} \neq 0\}$ stable, we have $\alpha|_{\mathbf{a}} = 0$. But then $\tau\alpha = \alpha$ and $0 = (\gamma_r \mid \alpha) = (\gamma_r \mid -\tau\alpha)$, a contradiction. Thus $(\beta \mid \delta) = 0$ and $(\gamma \mid \delta^\vee) = 2$. Hence $\beta = \gamma - \delta \in \Delta(\mathbf{k}_c, \mathbf{t}_c)$ and β is orthogonal to γ_r and δ. From this it follows, that

$$-\tau\gamma = \gamma_r + (-\tau\beta) \in \Delta(\mathbf{p}_c^+, \mathbf{t}_c).$$

As the Weyl group $W_{\mathbf{k}}$ of $\Delta(\mathbf{k}_c, \mathbf{t}_c)$ leaves $\Delta(\mathbf{p}_c^+, \mathbf{t}_c)$ stable and $-\tau\beta \perp \gamma_r$ it follows, that

$$\gamma_r - (-\tau\beta) = s_{-\tau\beta}(\gamma_r - \tau\beta) \in \Delta(\mathbf{p}_c^+, \mathbf{t}_c),$$

where $s_{-\tau\beta} \in W_{\mathbf{k}}$ is, as usually, the reflection in the hyperplane orthogonal to $-\tau\beta$. As γ_r is maximal in $\Delta(\mathbf{p}_c^+, \mathbf{t}_c)$, $-\tau\beta$ can neither be positive nor negative, which is a contradiction, and $-\tau\gamma_r$ is in fact the maximal root in Γ_{r-1}.

If $\Gamma_{r-2} = \emptyset$ we put $\mathcal{N} := \{1\}$ and $\mathcal{M} := \emptyset$; otherwise we now define $\Lambda := \Gamma_{r-2}$. Having now defined Λ we see that $-\tau\Lambda = \Lambda$. Let $\mathbf{g}_{c\Lambda}$ be the Lie algebra generated by the root spaces $\mathbf{g}_{c(\pm\gamma)}$, $\gamma \in \Lambda$. Then $\mathbf{g}_{c\Lambda}$ is a τ- and σ-stable semisimple subalgebra of $\mathbf{g}_c$ and $\mathbf{g}_{c\Lambda} \cap \mathbf{g} = \mathbf{g}_\Lambda$ has smaller real rank than $\mathbf{g}$. As $\tau|_{\gamma_\Lambda}$ anticommutes with the almost complex structure $J|_{\mathbf{p}\cap\mathbf{g}_\Lambda}$, the pair $(\mathbf{g}_\Lambda, \tau|_{\mathbf{g}_\Lambda})$ is of Hermitian type and our induction hypothesis works and the theorem is proved. □

Now we can do the same construction with $\Delta(\mathbf{p}_c^+, \mathbf{t}_c)$ replaced by Δ_p^+. In particular let s be the real-rank of M, and let

$$\Delta_p^+ = \hat{\Gamma}_s \supset \hat{\Gamma}_{s-1} \supset \cdots \supset \hat{\Gamma}_1$$

be constructed in the same way as $\Gamma_r \supset \cdots \supset \Gamma_1$ (see [31]) and let $\hat{\Gamma} = \{\lambda_1, \ldots \lambda_s\}$ be the corresponding set of strongly orthogonal roots.

Theorem 3.5. *Let the notation be as above. Then*

$$\hat{\Gamma} := \{\hat{\gamma}_j \mid j \in \mathcal{N} \cup \mathcal{M}\}.$$

In particular $s = |\mathcal{N}| + |\mathcal{M}|$. *Let* $\pi : \{1, \ldots, s\} \to \mathcal{M} \cup \mathcal{N}$ *be the bijection such that* $\pi(i) < \pi(j)$ *for* $i < j$. *Then* $\lambda_j = \hat{\gamma}_{\pi(j)}$, *for all* $j = 1, \ldots s$.

Proof. First we show that the set $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_s\}$, $s := |\mathcal{N}| + |\mathcal{M}|$, is strongly orthogonal. For that assume that $\lambda, \mu \in \{\hat{\gamma}_j \mid j \in \mathcal{N} \cup \mathcal{M}\}$, $\lambda \neq \mu$ and $\lambda - \mu \in \Delta$. Choose j and k such that $\lambda = \hat{\gamma}_j$ and $\mu = \hat{\gamma}_k$. Then $\gamma_j \perp \gamma_k, -\tau\gamma_k$ and thus $\lambda \perp \mu$. It follows that $(\lambda - \mu \mid \mu^\vee) = -2$ and thus $s_\mu(\lambda - \mu) = \lambda + \mu \in \Delta$, a contradiction. Let now $\delta \in \Delta_p^+$ be strongly orthogonal to all $\hat{\gamma}_j$'s. Let $\alpha \in \Delta(\mathbf{p}_c^+, \mathbf{t}_c)$ be such that $\hat{\alpha} = \delta$. If $\gamma_j - \alpha$ is a root for some j, then obviously $\hat{\gamma}_j - \delta \in \Delta$ and that is impossible. Thus α is strongly orthogonal to all γ_j in contradiction to the maximality of the set Γ. In the same way it follows, that $\hat{\gamma}_r$ is maximal in $\hat{\Gamma}_s$ and the last assertion follows by induction. □

Example 3.6. In Lemma 4.3 we will see that $s = |\mathcal{M}| + |\mathcal{N}|$ equals to the rank of $H/H \cap K$. As $r = |\mathcal{M}| + 2|\mathcal{N}|$ we have $|\mathcal{N}| = r - s = \operatorname{rank}(G/K) - \operatorname{rank}(H/H \cap K)$. Hence the only irreducible pairs with $\mathcal{N} \neq \emptyset$ are

1. $(\mathbf{g} \times \mathbf{g}, \text{diagonal}), s = r/2$.
2. $(\mathbf{su}(2p, 2q), \mathbf{sp}(p, q)), r = \min(2p, 2q), s = \min(q, p)$.
3. $(\mathbf{e}_{6(-14)}, \mathbf{f}_{4(-20)}), r = 2, s = 1$.

4. The Cayley transform

In this section we use the results of the last section to relate root vectors in $\mathbf{p}_{c\alpha}$ and $\mathbf{p}_{c\hat{\alpha}}$, $\alpha \in \Delta(\mathbf{p}_c, \mathbf{t}_c)$. We then use that to construct maximal abelian subalgebras $\mathbf{b}$ and $\mathbf{b}^q$ of $\mathbf{p}$ such that $\mathbf{b} \cap \mathbf{h}$ (resp. $\mathbf{b}^q \cap \mathbf{q}$) is a maximal abelian subalgebra of $\mathbf{h}_p$ (resp. $\mathbf{q}_p$). This relates our construction in [31] to 'classical' constructions based on $\mathbf{t}_c$. We also recall the construction of the Cayley transform and show how this construction can be done τ- or η-equivariant [19,40]. We will restrict ourself to the Cayley transform although this may be applied as well to the boundary components and the partial Cayley transforms by replacing the set Γ by $\hat{\Gamma}$ and X_0^q by the partial sums $\sum_{k=j}^{s} \hat{X}^q_{\pi(k)}$.

For $\alpha \in \Delta(\mathbf{g}_c, \mathbf{t}_c)$ choose $H_\alpha \in [\mathbf{g}_{c\alpha}, \mathbf{g}_{c-\alpha}]$ such that $\alpha(H_\alpha) = 2$. Choose $E_\alpha \in \mathbf{g}_{c\alpha}$ such that $E_{-\alpha} = \sigma E_\alpha$ and $H_\alpha = [E_\alpha, E_{-\alpha}]$ (see [7]). The following can then be proved as in [24, p. 57] or more simply by using the involution η.

Lemma 4.1. *For $\alpha \in \Delta(\mathbf{p}_c, \mathbf{t}_c)$ we can choose E_α such that $\tau E_\alpha = E_{\tau\alpha}$.*

Let $\alpha \in \Delta(\mathbf{g}_c, \mathbf{t}_c)$ such that $\hat{\alpha} \neq 0$. Let $\beta := \hat{\alpha} \in \Delta$ and define

$$\hat{H}_\beta := \begin{cases} H_\alpha & \text{if } -\tau\alpha = \alpha, \\ H_\alpha - \tau H_\alpha & \text{if } -\tau\alpha \neq \alpha. \end{cases}$$

$$\hat{E}_\beta := \begin{cases} E_\alpha & \text{if } -\tau\alpha = \alpha, \\ E_\alpha + \tau E_{-\alpha} & \text{if } -\tau\alpha \neq \alpha. \end{cases}$$

$$\tilde{E}_\beta := \begin{cases} 0 & \text{if } -\tau\alpha = \alpha, \\ E_\alpha - \tau E_{-\alpha} & \text{if } -\tau\alpha \neq \alpha. \end{cases}$$

Lemma 4.2. *Let $\alpha \in \Delta(\mathbf{g}_c, \mathbf{t}_c)$ such that $\hat{\alpha} \neq 0$. Let $\beta := \hat{\alpha} \in \Delta$. Then*

(1) $\hat{H}_\beta \in i\mathbf{a} \cap [\mathbf{g}_{c\beta}, \mathbf{g}_{c-\beta}]$ *and* $\beta(\hat{H}_\beta) = 2$;
(2) $\tilde{E}_\beta = E_\alpha - \eta E_\alpha \in \mathbf{g}_{c\beta}(-)$ *and* $\hat{E}_\beta = E_\alpha + \eta E_\alpha \in \mathbf{g}_{c\beta}(+)$;
(3) $\hat{H}_\beta = [\tilde{E}_\beta, \sigma\tilde{E}_\beta] = [\hat{E}_\beta, \sigma\hat{E}_\beta]$ *and* $[\tilde{E}_\beta, \sigma\hat{E}_\beta] = H_\alpha + \tau H_\alpha$.

Proof. As the claim of the lemma is obvious for $-\tau\alpha = \alpha$ we may assume that this is not the case. As $\tau\sigma = \sigma\tau$, $\tau E_{-\alpha} \in \mathbf{g}_{c(-\tau\alpha)}$, $\alpha \pm \tau\alpha \notin \Delta(\mathbf{g}_c, \mathbf{t}_c)$ and $\sigma(E_\alpha \pm \tau E_{-\alpha}) = E_{-\alpha} \pm \tau E_\alpha$ it follows:

$$[E_\alpha \pm \tau E_{-\alpha}, E_{-\alpha} \pm \tau E_\alpha] = [E_\alpha, E_{-\alpha}] - \tau[E_\alpha, E_{-\alpha}] = H_\alpha - \tau H_\alpha.$$

For $H \in \mathbf{a}_c$ we also have

$$[H, E_\alpha \pm \tau E_{-\alpha}] = [H, E_\alpha] \pm [H, \tau E_{-\alpha}] = \alpha(H)(E_\alpha \pm \tau E_{-\alpha}).$$

By $\alpha \perp -\tau\alpha$ and $H_{-\tau\alpha} = -\tau H_\alpha$ it follows that $\alpha(-\tau H_\alpha) = 0$ and then by direct calculation $\beta(H_\alpha - \tau H_\alpha) = 2$. The rest of the lemma is now obvious. □

Choose $E_j := E_{\gamma_j} \in \mathbf{p}_{c\gamma_j}$ such that $\tau E_j = E_{\tau\gamma_j}$ and define

$$X_j := E_j + \sigma E_j,\ \hat{X}_j := \hat{E}_j + \sigma \hat{E}_j,\ \tilde{X}_j := \tilde{E}_j + \sigma \tilde{E}_j.$$

where $\hat{E}_j = \hat{E}_{\hat{\gamma}_j}$, $\tilde{E}_j = \tilde{E}_{\hat{\gamma}_j}$, $j = 1, \ldots, r$. Then $X_j \in \mathbf{p}$, $\hat{X}_j \in \mathbf{h}_p$, $\tilde{X}_j \in \mathbf{q}_p$ and $2X_j = \hat{X}_j + \tilde{X}_j$. Define

$$\mathbf{b} := \bigoplus_{j=1}^{r} \mathbb{R}X_j.$$

Lemma 4.3. *Let the notation be as above. Let $\pi : \{1, \ldots, s\} \to \mathcal{M} \cup \mathcal{N}$ be the bijection such that $\pi(i) < \pi(j)$ for $i < j$. Then $\mathbf{b}$ is a maximal abelian τ-stable subalgebra of $\mathbf{p}$ such that $\mathbf{b} \cap \mathbf{h}_p = \bigoplus_{j=1}^{s} \mathbb{R}\hat{X}_{\pi(j)}$ is maximal abelian in $\mathbf{h}_p$. Furthermore $\mathbf{b} \cap \mathbf{q}_p = \bigoplus_{j \in \mathcal{N}} \mathbb{R}\tilde{X}_j$.*

The first part is well known, e.g., [7, p. 385]. The second part follows from Lemma 4.2 and [31, Lemma 2.3] (by replacing τ by $\theta\tau$). The last part follows from the fact, that the ortogonal projection of $2X_j$ onto $\mathbf{h}_p$ (resp. $\mathbf{q}_p$) is given by $\hat{X}_j$ (resp. $\tilde{X}_j$). □

Lemma 4.4. *Let $H_j := H_{\gamma_j}$ and $\hat{H}_j := \hat{H}_{\hat{\gamma}_j}$. Define $H_0 := \frac{1}{2} i \sum_{j=1}^{r} H_j$ and $X_0 := \frac{1}{2} \sum_{j=1}^{r} X_j$. Then $H_0 = \frac{1}{2} i \sum_{j=1}^{s} \hat{H}_{\pi(j)} \in \mathbf{a}$ and $X_0 = \frac{1}{2} \sum_{j=1}^{s} \hat{X}_{\pi(j)} \in \mathbf{b} \cap \mathbf{h}_p$.*

Proof. As $H_{-\tau\gamma_j} = -\tau H_j$ it follows that

$$H_0 = \frac{i}{2} \sum_{j=1}^{s} \hat{H}_{\pi(j)} = \frac{i}{2} \sum_{j \in \mathcal{M}} H_j + \frac{i}{2} \sum_{j \in \mathcal{N}} (H_j - \tau H_j) = \frac{i}{2} \sum_{j=1}^{s} \hat{H}_{\pi(j)}.$$

The other part follows in the same way. □

Remark 4.5. In the same way we may construct a maximal abelian subalgebra $\mathbf{b}^q$ in $\mathbf{p}$ such that $\mathbf{b}^q \cap \mathbf{q}_p$ is maximal abelian in $\mathbf{q}_p$. For that we only have to replace τ by $\theta\tau$

everywhere. The corresponding vectors are then:

$$X_j^q = i(E_j - \sigma E_j), \quad \hat{X}_j^q = i(\hat{E}_j - \sigma \hat{E}_j), \quad \tilde{X}_j^q = i(\tilde{E}_j - \sigma \tilde{E}_j),$$

$$X_0^q = \frac{1}{2}\sum_{j=1}^{s} \hat{X}_{\pi(j)}^q \in \mathbf{b}^q \cap \mathbf{h}_p.$$

Notice that for $J = \mathrm{ad}(Z_0)$ as before and $k_0 := \exp(\frac{1}{2}\pi Z_0)$, then $J(\mathbf{b}) = \mathrm{Ad}(k_0)(\mathbf{b}) = \mathbf{b}^q$.

Define $i\mathbf{t}^- := \sum_{j=1}^{r} \mathbb{R}H_{\gamma_j}$ and $i\mathbf{a}^- := i\mathbf{t}^- \cap i\mathbf{a} = \sum_{j=1}^{s} \mathbb{R}\hat{H}_{\hat{\lambda}_{\pi(j)}}$. Let $X_0^q \in \mathbf{b}^q \cap \mathbf{q}_p$ be as in Lemma 4.5 and define

$$\mathsf{c} := \exp \frac{\pi i}{2} X_0^q \quad \text{and} \quad \mathsf{C} = \mathrm{Ad}(\mathsf{c}).$$

By Lemma 4.5, [19] and [40] C is just the usual Cayley transform. As $\mathsf{C} \circ \eta = \eta \circ \mathsf{C}$ we call C the *η-equivariant Cayley transform.* The usual $\mathbf{sl}_2$-reduction gives

$$\mathsf{C}(H_j) = X_j, \quad \mathsf{C}(X_j) = -H_j, \quad \text{and} \quad \mathsf{C}(X_j^q) = X_j^q,$$

as well as

$$\mathsf{c}(i\mathbf{t}^-) = \mathbf{b}, \quad \text{and} \quad \mathsf{c}(i\mathbf{a}^-) = \mathbf{b} \cap \mathbf{h}.$$

By the theorem of Moore [26] (see [39, p. 15]), we have now relatively to $\mathrm{ad}(X_o)$:

$$\mathbf{g} = \mathbf{g}(-1) \oplus \mathbf{g}(-\tfrac{1}{2}) \oplus \mathbf{g}(0) \oplus \mathbf{g}(\tfrac{1}{2}) \oplus \mathbf{g}(1),$$

where $\mathbf{g}(\pm\frac{1}{2})$ may be zero. Let $\varphi := \mathrm{Ad}(\exp(\pi i X_0)) = \mathrm{Ad}(k_0)^{-1} \circ \mathsf{C}^2 \circ \mathrm{Ad}(k_0)$. Then

$$\varphi|_{\mathbf{g}(\pm 1)} = -1, \quad \varphi|_{\mathbf{g}(0)} = 1, \quad \text{and} \quad \varphi|_{\mathbf{g}(\pm\frac{1}{2})} = \pm i.$$

In paricular $\varphi^4 = 1$ and $\varphi^2 = 1 \Leftrightarrow \mathbf{g}(\pm\frac{1}{2}) = 0$. In that case $\varphi = \mathsf{C}^2 \circ \theta$. Let

$$\mathbf{g}_T := \mathbf{g}^{\varphi^2} \quad \text{and} \quad G_T := G^{\varphi^2}.$$

Then G_T is reductive with the Lie algebra $\mathbf{g}_T$ and φ defines by restriction an involution on G_T. We collect now some facts that we need about invariant convex cones (see e.g., [10]). Let L be a Lie group and $\mathbf{V}$ a finite dimensional real Euclidean vector space and a L-module. As we are only interested in closed or open *convex* cones we define $C \subset \mathbf{V}$ to be a *cone* if C is closed or open, convex and $\mathbb{R}^+ C \subset C$ (for C open we replace $\mathbb{R}^+$ by $\mathbb{R}^+ \setminus \{0\}$.) If not otherwise stated we will assume C closed and use the notation Ω for *open* cones. C is an *L-invariant cone* if C is a cone and $LC \subset C$. If C is a cone we define the *dual cone* $C^* \subset \mathbf{V}^*$ by

$$C^* := \{u \in \mathbf{V} \mid \forall v \in C : (u \mid v) \geqslant 0\}.$$

C is *proper* if C and C^* are both non-zero. This is equivalent to one of the following

1. C is pointed, i.e., there exists a $v \in \mathbf{V}$ such that $(v \mid u) > 0$ for all $u \in C \setminus \{0\}$.
2. $C \cap -C = \{0\}$.

We call C *generating* if $C - C := \{u - v \mid u, v \in C\} = \mathbf{V}$ and *regular* if it is proper and generating, or equivalently both C and C^* have non empty interior. Denote by $\mathrm{Con}_L(\mathbf{V})$ the set of L-invariant, regular cones in $\mathbf{V}$. If Ω is an open cone we define *dual cone* by

$$\Omega^* := \{u \in \mathbf{V} \mid \forall v \in \bar{\Omega} \setminus \{0\} : (u \mid v) > 0\}.$$

Definition 4.6. Let Ω be an open and proper convex cone in a real vector space $\mathbf{V}$. Then

$$D(\Omega) := \mathbf{V} + i\Omega \subset \mathbf{V}_c$$

is called a *tube domain over* Ω and also a *Siegel domain of type* I.

Let $\mathbf{W}$ be a complex vector space and Q a Hermitian form on $\mathbf{W}$ with values in $\mathbf{V}_c$ such that

$$Q(u,u) \in \bar{\Omega} \setminus \{0\}, \qquad u \in \mathbf{W} \setminus \{0\}.$$

Then Q is called a Ω*-Hermitian form* and

$$D(\Omega, Q) := \{p = (x + iy, u) \in \mathbf{V}_c \oplus \mathbf{W} \mid y - Q(u,u) \in \Omega\}$$

is called a *Siegel domain of type* II.

Theorem 4.7 (Korányi, Wolf). *Let the notation be as above. Let $D_p = G/K \subset \mathbf{p}_c^+$ be as before. Then the following are equivalent:*

(1) *D_p is of tube type;*

(2) *There exists a 3-dimensional subalgebra of $\mathbf{g}$ containing Z_0 and isomorphic to* $\mathbf{sl}(2,\mathbb{R})$;

(3) $Z_0 = H_0$;

(4) $\varphi(Z_0) = -Z_0$;

(5) $\varphi^2 = 1$;

(6) $\mathbf{g}(\pm\frac{1}{2}) = 0$.

The proof can be found in [19]. For convenience, we extend the definition of 'Hermitian type' as follows. A reductive symmetric pair (G, H) in the Harish-Chandra class is of *Hermitian type* if the center of G is compact and $([\mathbf{g},\mathbf{g}], \tau|_{[\mathbf{g},\mathbf{g}]})$ is of Hermitian type. We will then also call the pair $(\mathbf{g}, \tau)$ of Hermitian type. Let $\mathbf{c}_T$ be the center of $\mathbf{g}_T$.

Lemma 4.8. *Let the notation be as above. Then:*

(1) *$\tau \circ \varphi = \varphi \circ \tau$ and $\theta \circ \varphi = \varphi \circ \theta$. Thus G_T is τ- and θ-stable and G_T/K_T, $K_T := G_T \cap K$, is of tube type with almost complex structure on $\mathbf{p}_T$ given by* $\mathrm{ad}\, H_0|_{\mathbf{p}_T}$, $\mathbf{p}_T := \mathbf{p} \cap \mathbf{g}_T$.

(2) *Let* $\varphi_T := \varphi|_{\mathbf{g}_T}$. *Then* $(\mathbf{g}_T, \varphi_T)$ *is of Hermitian type with* (-1)*-eigenspace* $\mathbf{q}_T = \mathbf{g}(1)\oplus\mathbf{g}(-1)$, $\mathbf{t}_0 \subset \mathbf{g}_T$ *and* $\mathbf{c}_T \subset (\mathbf{t}^-)^\perp$. *Furthermore,* $\mathbf{t}_0^-$ *is maximal abelian subalgebra of* $\mathbf{q}_{Tk}$, $\{\gamma_1, \dots, \gamma_r\}$ *is a maximal set of strongly orthogonal roots and* $-\varphi\gamma_j = \gamma_j$.

(3) $(\mathbf{g}_T, \tau|_{\mathbf{g}_T})$ *is of Hermitian type.*

Proof. The first part of (1) follows from the construction of φ and Theorem 4.7. The last part is Lemma 4.6 in [19, p. 274]. For (2) we first notice that $\mathbf{t} \subset \mathbf{g}_T$. As $\mathbf{t}$ is maximal abelian in $\mathbf{g}$ the center of $\mathbf{g}_T$ has to be contained in $\mathbf{t}$. Furthermore $\mathbf{c}_T$ commutes with X_0 and thus $\mathbf{c}_T \subset \mathbf{g}(0) \cap \mathbf{t} = (\mathbf{t}^-)^\perp$ as we will see in a moment. As $\varphi H_0 = -H_0$ by Theorem 4.7 it follows, that φ_T is of Hermitian type.

$$X = H^- + H^+ + \sum_{\alpha\in\Delta(\mathbf{g}_c,\mathbf{t})} X_\alpha,$$

where $H^- \in \mathbf{t}^-$, $H^+ \in (\mathbf{t}^-)^\perp$, $X_\alpha \in \mathbf{g}_{c\alpha}$. Then

$$[H_j, X] = \sum_\alpha \alpha(H_j)X_\alpha = \sum_\alpha \frac{2(\alpha|\gamma_j)}{(\gamma_j|\gamma_j)} X_\alpha.$$

Thus X commutes with $\mathbf{t}^-$ if and only if all α's are orthogonal to all γ_j's. But then

$$\varphi(X) = -H^- + H^+ + \sum X_\alpha$$

as $\varphi(H_j) = -H_j$. Thus $\mathbf{z}_{\mathbf{g}_T}(\mathbf{t}^-) \cap \mathbf{q}_T = \mathbf{t}^-$ and (2) follows. By Lemma 4.4, $\tau(H_0) = -H_0$ and (3) follows by Lemma 1.4. □

Let $I_C := H_0 - X_0$. By [19] (see [40, p. 16]) there is an open, self-dual $G(0)$-invariant cone

$$\Omega = G(0) \cdot I_C \subset \mathbf{g}(1),$$

where $G(0) = C_{G_T}(X_0) \subset G_T^\varphi$ and $a \cdot X = \mathrm{Ad}(a)X$. Then Ω is a Riemannian symmetric space $\Omega \simeq G(0)/K(0)$ with $K(0) = K \cap G(0)$. Notice that Ω is the unique $G(0)$-invariant cone in $\mathbf{g}(1)$ containing I_C (if Ω_1 contains I_C then $\Omega_1 \cap \Omega \neq \emptyset \Rightarrow \Omega \subset \Omega_1 \Rightarrow \Omega_1^* \subset \Omega^* = \Omega$. As Ω is minimal $\Omega_1^* = \Omega$).

Define a positive system $\Delta^+(\mathbf{g}, \mathbf{b}) := \{\alpha \circ \mathsf{C}^{-1} \mid \alpha \in \Delta^+(\mathbf{g}_c, \mathbf{t}_c),\ \alpha|_{\mathbf{t}^-} \neq 0\}$ and let

$$\mathbf{s} := \mathbf{g}(\Delta^+(\mathbf{g}, \mathbf{b})) = \mathbf{s}_0 \oplus \mathbf{s}_{1/2} \oplus \mathbf{s}_1 \quad \text{and} \quad \mathbf{W} := \mathbf{s}_{1/2c} \cap \mathsf{C}(\mathbf{p}_c^+)$$

where $\mathbf{s}_\lambda := \mathbf{s} \cap \mathbf{g}(\lambda)$. Then there exists a $G(0)$-equivariant bijection $\iota : \mathbf{s}_{1/2} \to \mathbf{s}_{1/2}$ such that $\mathbf{W} = \{X - i\iota X \mid X \in \mathbf{s}_{1/2}\}$. $\mathbf{W}$ is $G(0)$-stable and $\mathbf{s}_{1/2} \ni X \mapsto w(X) := \frac{1}{2}(X - i\iota(X)) \in \mathbf{W}$ is a complex $G(0)$-equivariant isomorphism commuting with η. (The last claim follows by: $\eta(\mathbf{p}^+) = \mathbf{p}^-$ and $\eta(\mathbf{s}_{1/2}) = \mathbf{s}_{1/2}$. As η is conjugate linear, $\eta \circ \iota = -\iota \circ \eta$.) We also let $\mathbf{V} := \mathbf{g}(1)_c$ and define the the Hermitian $\mathbf{V}$-valued form Q on $\mathbf{W}$ by

$$Q(W, W_1) := \frac{i}{2}[W, \sigma(W_1)], \qquad W, W_1 \in \mathbf{W}.$$

We also define $\alpha : \mathsf{c}P^+K_cP^- \to \mathbf{V} \oplus \mathbf{W}$ by

$$\alpha(g) := \mathsf{C}(p_+(\mathsf{c}^{-1}g)).$$

Then the following holds (see [40, p. 17]):

Theorem 4.9 (Korányi, Wolf). *Let the notation be as above. Then*
(1) *Q is a Ω-Hermitian form and for all $a \in G(0)$, $W, W_1 \in \mathbf{W}$*

$$Q(\mathrm{Ad}(a)W, \mathrm{Ad}(a)W_1) = \mathrm{Ad}(a)Q(W, W_1).$$

(2) *α determines a G-invariant biholomorphc isomorphism of G/K onto $D(\Omega, Q)$.*
(3) *Let $a \in G(0)$, $X \in \mathbf{s}_1, Y \in \mathbf{s}_{1/2}$ and let $b = a\exp(X)\exp(Y)$. Then*

$$b \cdot (Z, W) = (\mathrm{Ad}(a)X + \mathrm{Ad}(a)Z + Q(\mathrm{Ad}(a)W, \mathrm{Ad}(a)w(Y)), \mathrm{Ad}(a)W + \mathrm{Ad}(a)w(Y)).$$

We will now describe the H-orbit through iI_C in $D(\Omega, Q)$. For that let $S_\lambda := \exp(\mathbf{s}_\lambda)$ and $B := \exp \mathbf{b}$.

Lemma 4.10. (1) *Let $a \in \mathsf{c}P^+K_cP^-$. Then $\alpha(\eta(a)) = \eta(\alpha(a))$ and in particular $\alpha(H/H \cap K) = D(\Omega, Q)^\eta$.*
(2) *Let $W, W_1 \in \mathbf{W}$. Then $Q(\eta(W), \eta(W_1)) = -\eta(Q(W, W_1))$.*
(3) *Let $a \in G(0)S_1S_{1/2}$ and $(Z, 0)) \in D(\Omega, Q)$ then $\eta(a \cdot (Z, 0) = \eta(a)(\eta Z, 0)$.*

Proof. The first part follows from $\mathsf{C} \circ \eta = \eta \circ \mathsf{C}$ and $\eta(\mathsf{c}) = \mathsf{c}$ whereas the second part follows directly from the definition. The last part follows from part (3) in Theorem 4.9. □

As $\tau|_{G_T}$ and φ_T are commuting involutions of Hermitian type we now that the $G(0)$- and $H \cap G_T$ orbit through $\alpha(0) = iI_C$ are diffeomorphic (see Theorem 1.6). We describe this diffeomorphism now in terms of the data used to define the Siegel domain $D(\Omega, Q)$. We first look at G_T and thus we assume for a moment that $G = G_T$, i.e., G/K of tube type.

Lemma 4.11. *Assume that G/K is of tube type. Then*

$$G(0) = S_0 \cdot B \cdot K(0) = \exp(\mathbf{s}_0 \cap \mathbf{q}) \cdot B \cap \exp(\mathbf{q}) \cdot (G(0) \cap H)K(0)$$

$$H = S_1^\tau \cdot S_0^\tau \cdot B^\tau \cdot K \cap H = S_1^\tau G(0)^\tau K \cap H$$

where the second products are in general not diffeomorphisms. The map

$$S_1^\tau \cdot G(0)^\tau I_C \ni \exp(X) \cdot g \cdot I_C \mapsto \exp([I_C, X]) \cdot g \cdot I_C \in \Omega$$

is a diffeomorphism of $H/H \cap K$ onto Ω. Also $(G^{\varphi\tau}, \tau)$ is of Hermitian type and the Cayley transform induces a diffeomorphism

$$D^{\varphi\tau} \simeq \mathbf{V} \cap \mathbf{q} + i(\Omega \cap \mathbf{q}).$$

In particular $\Omega \cap \mathbf{q}$ is a proper cone in $\mathbf{q}_T \cap \mathbf{q}$.

Proof. The first two decompositions are just the Iwasawa decompositions, $G(0)$ respectively $H \cap G_T$. By [40, Lemma 2.2.4, p. 16], $[I_C, \mathbf{s}_0] = \mathbf{s}_1$. As

$$\mathbf{s}_0 \cap \mathbf{q} \oplus \mathbf{s}_0 \cap \mathbf{h} \ni (X,Y) \mapsto \exp(X)\exp(Y) \in S(0)$$

is a diffeomorphism the first part follows. As $\varphi\tau(Z_0) = -Z_0$ and $\tau Z_0 = Z_0$, $(\mathbf{g}^{\varphi\tau}\tau)$ is Hermitian. As the '$\mathbf{g}(1)$' in this case is $\mathbf{g}(1) \cap \mathbf{q}(1)$ the lemma follows. □

In the general case we know, that

$$H/H \cap K \simeq \exp(\mathbf{s}_{1/2}^{\tau}) \cdot (H \cap G_T)/H \cap K_T.$$

From Lemma 4.10 we now have:

Theorem 4.12. *The H-orbit through iI_C in $D(\Omega, Q)$ is given by*

$$\begin{aligned} D(\Omega,Q)^{\eta} &= \{(X+iZ, W) \in \mathbf{s}_1^{\tau} \oplus i\mathbf{s}_1 \cap \mathbf{q} \times \mathbf{W}^{\eta} \mid Z - Q(W,W) \in \Omega\} \\ &\simeq \{(iZ, W) \in i\mathbf{s}_1 \times \mathbf{W}^{\eta} \mid Z - Q(W,W) \in \Omega\} \end{aligned}$$

where the diffeomorphism is given by

$$\begin{aligned} a\exp(X)\exp(Y)\cdot iI_C &= (\mathrm{Ad}(a)X + \mathrm{Ad}(a)iI_C, \mathrm{Ad}(a)w(Z)) \\ &\mapsto (\exp([I_C, \mathrm{Ad}(a)X])\,\mathrm{Ad}(a)iI_C, \mathrm{Ad}(a)w(Z)) \end{aligned}$$

for $a \in G(0) \cap H$, $X \in \mathbf{s}_1^{\tau}$ and $Y \in \mathbf{W}^{\eta}$.

5. Cayley type involutions

This section is devoted to some special kinds of involutions. The first kind of involutions are those that generalize complex conjugation on $\mathrm{SU}(p,q)$, $\mathrm{SO}^*(2n)$ and $\mathrm{Sp}(n,\mathbb{R})$ (see [29]). Those involutions are charaterized by $\operatorname{rank} M = \operatorname{rank} K/K \cap H = \operatorname{rank} G$, i.e., $\mathbf{a} = \mathbf{t}$. The second type of involutions are those that are inner and then there are the special inner involutions coming from a Cayley transform as the involution φ_T in Lemma 4.8 in the case that G/K is a tube. Except for Lemma 5.2 and 5.3 we assume that $(\mathbf{g}, \tau)$ is of Hermitian type and that $(\mathbf{g}, \mathbf{h})$ is effective without compact ideals.

Definition 5.1. Let $\mathbf{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$. Then $\mathbf{g}$ is *split*, or a *normal real form* of $\mathbf{g}_c$, if there exists a Cartan subalgebra of $\mathbf{g}$ contained in $\mathbf{p}$.

Lemma 5.2. *Assume that $(\mathbf{g}, \tau)$ is regular and $(\mathbf{g}, \mathbf{h}^a)$ effective. Let $\mathbf{b} \subset \mathbf{q}_p$ be maximal abelian. Then $\mathbf{c}_p \subset \mathbf{b}$ and $\mathbf{b}$ is a maximal abelian subalgebra of $\mathbf{p}$.*

Proof. By the same argument as in the proof of Lemma 1.2 it follows, that $\mathbf{z}_{\mathbf{g}}(\mathbf{c}(\mathbf{q}_p)) \supset \mathbf{h}_k \oplus \mathbf{q}_p$. As $\mathbf{a}$ is maximal abelian $\mathbf{c}_p \subset \mathbf{a}$. But by Lemma 2.2 there exists a $Z^o \in \mathbf{c}_p \subset \mathbf{a}$ such that $\mathbf{z}_{\mathbf{g}}(Z^o) = \mathbf{h}^a$. Thus $\mathbf{a} \subset \mathbf{q}_p$. □

Lemma 5.3. *Let* $\mathbf{g}$ *be a semisimple Hermitian Lie algebra without compact ideals. Then there exists an involution* τ *of Hermitian type, unique up to conjugation, such that* $\operatorname{rank} M = \operatorname{rank} K/K \cap H = \operatorname{rank} G$. *This involution is characterized by* $\mathbf{g}^c$ *being split.*

Proof. Let $\mathbf{t}$ be a Cartan subalgebra of $\mathbf{g}$ contained in $\mathbf{k}$. Then multiplication by -1 is an automorphism of $\Delta(\mathbf{g}_c, \mathbf{t}_c)$ in the sense of [7, p. 421]. By Theorem 5.1. (see also [45, p. 289]) there exists an homomorphism τ of $\mathbf{g}_c$ with $\tau|_{\mathbf{t}} = -1$. It can be shown, that τ can be constructed such that $\tau \circ \sigma = \sigma \circ \tau$, i.e., τ leaves $\mathbf{g}$ stable. If τ_1 is another involution of G satisfying $\operatorname{rank} G/H_1 = \operatorname{rank} K/K \cap H_1 = \operatorname{rank} G$, $H_1 = G_o^{\tau_1}$, we can find a Cartan subalgebra $\mathbf{t}_1 \subset \{X \in \mathbf{k} \mid \tau_1(X) = -X\}$. But then we can find a $k \in K$ such that $\operatorname{Ad}(k)\mathbf{t}_1 = \mathbf{t}$, [45, p. 352], and we can assume that $\mathbf{t} = \mathbf{t}_1$. By Theorem 5.9, [7, p. 425], and its proof, there then exists a $X \in \mathbf{t}$ such that $\tau = \operatorname{Ad}(\exp(X)) \circ \tau_1 \circ \operatorname{Ad}(\exp(-X))$. By construction $\mathbf{t} \subset \mathbf{q}_k$ and $i\mathbf{t} \subset \mathbf{q}_p^c$. Thus $\mathbf{g}^c$ is split. Assume now that $\mathbf{g}^c$ is split. Then we can choose a Cartan subalgebra $\mathbf{t}^c$ contained in $\mathbf{p}^c$ and containing $i\mathbf{c}$. By the above lemma $\mathbf{t}^c \subset \mathbf{q}_p^c$. Hence $\mathbf{t} := i\mathbf{t}^c \subset \mathbf{q}_k$ is a Cartan subalgebra of $\mathbf{g}$ contained in $\mathbf{q}_k$. □

Those are $(\mathbf{su}(p,q), \mathbf{so}(p,q))$, $(\mathbf{so}^*(2n), \mathbf{so}(n,\mathbb{C}))$, $(\mathbf{sp}(n,\mathbb{R}), \mathbf{sl}(n,\mathbb{R}) \times \mathbb{R})$, where the involution is given by complex conjugation. Then there is the pair $(\mathbf{so}(2,2p), \mathbf{so}(p,1) \times \mathbf{so}(p,1))$ and $(\mathbf{so}(2,2p+1), \mathbf{so}(p,1) \times \mathbf{so}(p+1,1))$. Here the involution is given as conjugation by $d(1,-1,1,-1,\ldots,1,-1)$ respectively $d(1,-1,1,-1,\ldots,1)$. At last we have the two exceptional cases $(\mathbf{e}_{6(-14)}, \mathbf{sp}(2,2))$ and $(\mathbf{e}_{7(-25)}, \mathbf{su}^*(8))$.

The next type of involutions are those that are inner. If $\mathbf{g} = \mathbf{g}_1 \times \mathbf{g}_1$ and $\tau(X,Y) = (Y,X)$ then every inner automorphism leaves the factors invariant and so $\mathbf{g}$ is never inner. Thus $\mathbf{g}$ is a product of simple factors invariant under τ. Thus we can assume that $\mathbf{g}$ is simple. Notice that by [7, Chapter 9, Theorem 5.7], τ is inner if and only if $\operatorname{rank} \mathbf{g} = \operatorname{rank} \mathbf{h}$ and in fact if $\tau = \exp(\operatorname{ad} X)$ then X may be chosen in $\mathbf{k}^d$. The main idea of the proof is, that if $\operatorname{rank} \mathbf{h} = \operatorname{rank} \mathbf{g}$, then there is a θ-stable Cartan subalgebra $\mathbf{t}$ in $\mathbf{h}$ and $\mathbf{g}$. Then $\mathbf{t}_k \oplus i\mathbf{t}_p$ is a Cartan subalgebra of the compact Lie algebra $\mathbf{k} \oplus i\mathbf{p}$ and $\tau|_{\mathbf{t}_k \oplus i\mathbf{t}_p} = \mathrm{id}$. Hence there exists a $X_k + iX_p \in \mathbf{t}_k \oplus i\mathbf{t}_p$ such that

$$\tau = \operatorname{Ad}(\exp(X_k + iX_p)).$$

As $\mathbf{t}_k \oplus i\mathbf{t}_p$ is σ-stable, $X_k, X_p \in \mathbf{t}$ and in particular $[X_k, X_p] = 0$. Define

$$\tau_k := \operatorname{Ad}(\exp X_k), \quad \text{and} \quad \tau_p := \operatorname{Ad}(\exp iX_p).$$

Before we look at the general case we need the following lemma where $k_o = \exp(\frac{1}{2}\pi i Z_o)$ as before:

Lemma 5.4. *Let* $\tau = \exp(\pi i X)$ *be an involution of Hermitian type with* $X \in \mathbf{h}_p$. *Define* $\xi, \psi : G_c \to G_c$ *by*

$$\xi := \operatorname{Ad}\left(\exp \frac{\pi i}{2} X\right), \quad \textit{and} \quad \psi := \operatorname{Ad}\left(\exp \frac{\pi i}{2} \operatorname{Ad}(k_o) X\right).$$

Then

(1) $\xi : (\mathbf{g}, \tau) \to (\mathbf{g}^c, \tau)$ *is an isomorphism.* $\xi \circ \theta = \theta \circ \tau \circ \xi$ *and* ξ *define isomorphisms*

$$G \to G^c, \quad K \to \tilde{K} \quad \text{and} \quad G/K \to G^c/\tilde{K},$$

where $\tilde{K}$ *is the maximal compact subgroup of* G^c *corresponding to the Cartan involution* τ^a.

(2) $\psi : (\mathbf{g}, \tau) \to (\mathbf{g}^r, \theta)$ *is an isomorphism.* $\psi \circ \theta = \tau \circ \psi$ *and* ψ *define isomorphisms*

$$G \to G^r, \quad K \to K^r \quad \text{and} \quad G/K \to G^r/H^r.$$

Proof. As $\mathrm{Ad}(k_o) \circ \tau \circ \mathrm{Ad}(k_o^{-1}) = \mathrm{Ad}(k_o^2) \circ \tau = \tau^a$ it follows that $\psi^2 = \tau^a$ and we only have to prove (2) as (1) follows in the same manner by replacing τ^a by τ. For simplicity we write $Y = \mathrm{Ad}(k_o)X = JX$. Then $\psi \circ \sigma = \sigma \circ \psi^{-1} = \sigma \circ \tau^a \circ \psi$ and $\psi \circ \theta = \theta \circ \psi^{-1} = \tau \circ \psi$ Here σ is the conjugation of $\mathbf{g}_c$ relative to $\mathbf{g}$ as usually. Hence ψ defines a Lie algebra isomorphism over $\mathbb{R}$ $\mathbf{g} \simeq \mathbf{g}_c^{\eta\theta} = \mathbf{g}^r$ and $\mathbf{k}_c \simeq \mathbf{h}_c$. □

Lemma 5.5. τ_k *and* τ_p *are commuting involutions of* $\mathbf{g}$ *such that* $\tau = \tau_k \tau_p$. τ_k *and* τ_p *commutes with* θ *and* τ_p *is non-trivial. Furthermore* τ_p *is regular, parahermitian and of Hermitian type.*

Proof. That $\tau_k \tau_p = \tau = \tau_p \tau_k$ is clear. Now $\theta \circ \tau_k = \tau_k \circ \theta$ and $\theta \circ \tau_p = \tau_p^{-1} \circ \theta$. Thus

$$\theta \circ \tau = \theta \circ \tau_k \circ \tau_p = \tau_k \circ \tau_p^{-1} \circ \theta = \tau \circ \theta = \tau_k \circ \tau_p \circ \theta.$$

Hence $\tau_k \circ \tau_p = \tau_k \circ \tau_p^{-1}$. From this it follows, that $\tau_p^{-1} = \tau_p$ and so τ_p is an involution commuting with θ. As $\tau^2 = 1$ it also follows that τ_k is an involution commuting with θ.

Let $Z_o \in \mathbf{c}$ be as before. Then $[Z_o, X_k] = 0$. As τ is of Hermitian type $-Z_o = \tau Z_o = \tau_p \tau_k Z_o = \tau_p Z_o$. Hence $\tau_p \neq \mathrm{id}$ and of Hermitian type. Denote now by the superscript ^{r} the dual objects build up from $(\mathbf{g}, \tau_p)$. By the above $\mathbf{g}^r$ (with $\mathbf{h}^r$ as a maximal compact subalgebra) is Hermitian. Hence there exists a X^o in the center of $\mathbf{h}^r$ such that $\mathrm{ad}(X^o)$ has the eigenvalues $0, i, -i$ and $\tau_p = \exp(\pi\, \mathrm{ad}(X^o))$, [7, Chapter 8]. Now the center of $\mathbf{h}^r$ is one-dimensional and θ-stable and thus contained in the $i\mathbf{q}_p$ or $\mathbf{h}_k$ space for τ_p. Thus $X^o \in i\{Y \in \mathbf{g} \mid \tau_p(Y) = -Y\}_p$ or $X^o \in \{Y \mid \tau_p(Y) = Y\}_k$. As τ_p is of Hermitian type $X^o \in i\mathbf{p}$ as otherwise $\tau_p(Z_o) = Z_o$ is a contradiction. Now it follows that $(\mathbf{g}, -iX^o)$ is graded and $\{Y \mid \tau_p(Y) = Y\} = \mathbf{z}_{\mathbf{g}}(-iX^o)$. Thus τ_p is parahermitian. Replace τ_p by $\tau_p^a = \exp(\mathrm{Ad}(k_o)(-iX^o))$. As $(\mathbf{g}, \tau_p) \simeq (\mathbf{g}, \tau_p^a)$ the lemma follows from Lemma 2.6. □

We will now give a characterization of the involutions of the form τ_p. We notice that in this case $(\mathbf{g}, \tau) \simeq (\mathbf{g}^r, \tau)$ etc. and hence those involutions can also be described in terms of properties of the dual resp. c-dual pair. This we leave to the reader, see (6), (7) and (10) in the following theorem. Recall that if G/K is of tube type, then a Cayley transform of G/K is a map C of G/K into a tube domain $D(\Omega)$, $\mathsf{C} = \mathrm{Ad}(\exp \frac{1}{2}\pi i X)$ such that $\mathrm{ad}(X)$ has the eigenvalues $0, 1, -1$. In particular C has order 4.

Theorem 5.6. *Let* $\mathbf{g}$ *be simple and* τ *of Hermitian type. Then the following are equivalent:*

(1) G/K *is a tube domain and there exists a Cayley transform* C *such that* τ *is conjugate to* C^2.

(2) $\mathbf{q}$ *is reducible as a* $\mathbf{h}$ *module.*

(3) *If* $\mathbf{c}_h$ *is the center of* $\mathbf{h}$ *then* $\mathbf{c}_h$ *is non zero. In that case* $\dim \mathbf{c}_h = 1$ *and* $\mathbf{c}_h \subset \mathbf{p}$.

(4) τ *is inner and* $\tau = \tau_p$.

(5) $\tau = \mathrm{Ad}(\exp X)$ *is inner and* $\mathbf{h} \subset \mathbf{z_g}(X)$.

(6) *All the spaces in Theorem 2.7 are isomorphic.*

(7) $(\mathbf{g}, \tau)$ *is isomorphic to one of the pairs* $(\mathbf{g}^c, \tau)$, $(\mathbf{g}^c, \theta)$, $(\mathbf{g}^r, \theta)$ *or* $(\mathbf{g}^r, \tau^a)$.

(8) $(\mathbf{g}, \tau)$ *is regular.*

(9) $(\mathbf{g}, \tau)$ *is parahermitian.*

(10) $(\mathbf{g}^r, \theta)$ *is of Hermitian type.*

Proof. If (1) holds then $\tau = \mathrm{Ad}(\exp \pi i X)$ and $\mathbf{g} = \mathbf{g}(0) \oplus \mathbf{g}(1) \oplus \mathbf{g}(-1)$ relatively to X. But then $\mathbf{h} = \mathbf{g}(0)$ and $\mathbf{q} = \mathbf{g}(1) \oplus \mathbf{g}(-1)$. In particular X is central in $\mathbf{h}$ and thus $\mathbf{g}(\pm 1)$ is $\mathrm{ad}(\mathbf{h})$-stable. As $\mathbf{g}(\pm 1) \neq \{0\}$ it follows that $\mathbf{q}$ is reducible as a $\mathbf{h}$-module. If (2) holds then, as $(\mathbf{p}^r)_c = \mathbf{q}_c$ is reducible, it follows that $\mathbf{g}^r$ is Hermitian. Thus the center of $\mathbf{h}$ is one dimensional by [7, Chapter 8], and as above we see that $\mathbf{c}_h \subset \mathbf{h}_p$.

Assume (3), then (4) follows by using the Riemannian dual form again. (5) is now obvious. As X is then central in $\mathbf{h}$ we have $X \in \mathbf{h}_p$, e.g., $\tau = \tau_p$. By Lemma 5.4 all the spaces are isomorphic. Thus (6) holds and then (7) is obvious. Assume that $(\mathbf{g}, \tau)$ is isomorphic to one of the pairs $(\mathbf{g}^c, \tau)$ or $(\mathbf{g}^c, \theta)$, (the other cases follow in the same way by replacing $(\mathbf{g}, \tau)$ by the associated pair). Then $\mathbf{g}^c$ is Hermitian and it follows that $(\mathbf{g}^c, \tau) \simeq (\mathbf{g}^c, \theta)$ as the Cartan involution on $\mathbf{g}^c$ is $\theta\tau$. Hence $(\mathbf{g}, \tau)$ is regular by Theorem 2.7. If $(\mathbf{g}, \tau)$ is regular then $(\mathbf{g}, \tau^a) \simeq (\mathbf{g}, \tau)$ is parahermitian by Lemma 2.6. As $(\mathbf{g}^r, \tau^a)^c = (\mathbf{g}, \tau^a)$ with respect to the Cartan involution τ on $\mathbf{g}^r$ it follows by Theorem 2.7, part (3) that $(\mathbf{g}^r, \tau^a)$ is of Hermitian type. But using the Cartan involution τ on $\mathbf{g}^r$ it follows by Lemma 2.6 that $(\mathbf{g}^r, \tau^a) \simeq (\mathbf{g}^r, \theta)$ and thus (10) follows from (9).

Assume now (10). Then $(\mathbf{g}^r, \theta)$ is of Hermitian type. But then the Cartan involution τ is given by $\tau = \mathrm{Ad}(\exp(\pi X^o))$ with X^o central in $\mathbf{h}^r$ and $\theta(X^o) = -X^o$. Furthermore $\mathrm{ad}(X^o)$ has the eigenvalues $0, i, -i$. From $\theta(X^o) = -X^o$ it follows that $X^o \in i\mathbf{h}_p$. Thus $X := -iX^o \in \mathbf{h}_p$ and $\mathrm{ad}(X)$ has the eigenvalues $0, 1, -1$. Let $\mathbf{b}$ be a maximal abelian subalgebra of $\mathbf{p}$ containing X. As $\mathbf{z}(X) = \mathbf{h}$ it follows, that $\mathbf{b} \subset \mathbf{h}$. Choose a Cayley transform C_1 transforming $i\mathbf{t}^-$ onto $\mathbf{b}$ and choose $\Delta^+(\mathbf{g}, \mathbf{b})$ as in Section 4. If $\gamma_1, \ldots, \gamma_r$ are the strongly orthogonal roots and $\alpha_j = \gamma_j \circ \mathsf{C}_1^{-1}$ we know by the theorem of Moore, that D is a tube domain if and only if $\Delta^+(\mathbf{g}, \mathbf{a}) = \{\alpha_j, \frac{1}{2}(\alpha_i \pm \alpha_k) \mid 1 \leqslant i, j, k \leqslant r, i < k\}$. Otherwise $\Delta^+(\mathbf{g}, \mathbf{a}) = \{\frac{1}{2}\alpha_j, \alpha_j, \frac{1}{2}(\alpha_i \pm \alpha_k) \mid 1 \leqslant i, j, k \leqslant r, i < k\}$. By this we see that D has to be a tube domain as otherwise $\mathrm{ad}(X)$ would have an eigenvalue $1/2$ or 2. Comparing now the eigenvalues of $\mathrm{ad}(X)$ and $\mathrm{ad}(X_o)$ in Section 4 it follows that $X = X_o$. Hence $\tau = \mathrm{Ad}(k_o) \circ \mathsf{C}^2 \mathrm{Ad}(k_o)^{-1}$ where C is the Cayley transform from Section 4. □

Definition 5.7. Let $(\mathbf{g},\tau)$ be a semisimple pair such that τ leaves every simple factor of $\mathbf{g}$ invariant. Then τ is of *Cayley type* if and only if restricted to each irreducible factor τ satisfies (1)–(10) above. In that case we also call M of *Cayley type.*

The above defined spaces have also been introduced in [16, Section 5], as spaces of Silov type. Our argument for calling this type of involutions Cayley type is their relation to the classical Cayley transform.

Corollary 5.8. *Let* $\mathbf{g}$ *be semisimple. Then there exists an inner involution on* $\mathbf{g}$ *of Hermitian type if and only if* $D = G/K$ *is a tube domain. In this case there exists (up to a conjugation) a unique involution* τ_p *of Cayley type. If* τ *is inner then there exists a* $\gamma \in \mathrm{Ad}(G)$ *such that* $\gamma \circ \tau \circ \gamma^{-1} = \tau_k \tau_p$*, where* $\tau_k \in \mathrm{Ad}_G(K)$ *commutes with* τ_p.

Thus if τ is inner then τ is a product of an inner involution $\tau_k \in \mathrm{Ad}_G(K)$ and an involution of Cayley type. The only claim that has not be proved so far is, that the involution of Cayley type is unique. But if we have two such involutions defined by X_1 and X_2, then we may conjugate say X_1 by an element of K such that $\mathrm{Ad}(k)X_1, X_2 \in \mathbf{b}$ where $\mathbf{b}$ is a maximal abelian algebra in $\mathbf{p}$ and in fact we may assume that $\mathrm{Ad}(k)X_1$ and X_2 are in the same Weyl chamber. But then $\mathrm{Ad}(k)X_1 = X_2$ by looking at the eigenvalues.

Now we can read of the inner involutions from our first table by $\operatorname{rank}\mathbf{h} = \operatorname{rank}\mathbf{g}$. We then find the involutions of Cayley type by $\mathbf{c}_h \neq 0$ or $\mathbf{g} \simeq \mathbf{g}^c$.

Inner involutions

$\mathbf{g}$: Hermitian type	$\mathbf{g}^c$: Regular	$\mathbf{h}$	τ_k
$\mathbf{su}(n,n)$	$\mathbf{su}(n,n)$	$\mathbf{sl}(n,\mathbb{C}) \times \mathbb{R}$	$\tau_k = \mathrm{id}$
$\mathbf{so}^*(4n)$	$\mathbf{so}(2n,2n)$	$\mathbf{so}(2n,\mathbb{C})$	$\tau_k \neq \mathrm{id}$
$\mathbf{so}^*(4n)$	$\mathbf{so}^*(4n)$	$\mathbf{su}^*(2n) \times \mathbb{R}$	$\tau_k = \mathrm{id}$
$\mathbf{so}(2,n)$ *)	$\mathbf{so}(k+1,n+1-k)$	$\mathbf{so}(k,1)\times\mathbf{so}(1,n-k)$	$\tau_k \neq \mathrm{id},\ k \neq 1$
			$\tau_k = \mathrm{id},\ k = 1$
$\mathbf{sp}(2n,\mathbb{R})$	$\mathbf{sp}(n,n)$	$\mathbf{sp}(n,\mathbb{C})$	$\tau_k \neq \mathrm{id}$
$\mathbf{sp}(n,\mathbb{R})$	$\mathbf{sp}(n,\mathbb{R})$	$\mathbf{sl}(n,\mathbb{R}) \times \mathbb{R}$	$\tau_k = \mathrm{id}$
$\mathbf{e}_{7(-25)}$	$\mathbf{e}_{7(7)}$	$\mathbf{su}^*(8)$	$\tau_k \neq \mathrm{id}$
$\mathbf{e}_{7(-25)}$	$\mathbf{e}_{7(-25)}$	$\mathbf{e}_{6(-26)} \times \mathbb{R}$	$\tau_k = \mathrm{id}$

*) n and k not both even.

Example 5.9. For $\mathbf{so}(2,n)$ we define an involution τ by conjugating by the element

$$d(\underbrace{1,-1,1,-1,\ldots,1,-1}_{k+1\ times},1,1,\ldots,1) \quad \text{or} \quad d(\underbrace{-1,1,\ldots,-1,1}_{k+1\ times},1,1,\ldots,1).$$

Then the fixpoint algebra is isomorphic to $\mathbf{so}(1,n-k) \times \mathbf{so}(1,k)$. If n and k are not both even, this involution is inner and of Cayley type if $k = 1$.

Example 5.10. (SU(n,n)). Let $G = \mathrm{SU}(n,n)$. Let τ be the involution

$$\tau\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) := \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = J_n \begin{pmatrix} A & B \\ C & D \end{pmatrix} J_n^{-1}$$

where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then

$$\mathbf{h} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \operatorname{Tr} A = 0,\ A^* = -A,\ B^* = -B \right\} \simeq \mathbf{sl}(n,\mathbb{C}) \times \mathbb{R}$$

where the isomorphism is given by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB,$$

and

$$\mathbf{q} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \middle| A^* = -A,\ B^* = B \right\}.$$

The elements Z_o, X_o^q and X_o are now given by

$$Z_o = \frac{i}{2}\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad X_o^q = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

and

$$X_o = \frac{1}{2}\begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix}.$$

Furthermore

$$\mathbf{g}(1) = \left\{ \begin{pmatrix} iA & A \\ A & -iA \end{pmatrix} \middle| A^* = A \right\} \simeq \mathrm{H}(n,\mathbb{C}),$$

$$\mathbf{g}(-1) = \left\{ \begin{pmatrix} iA & -A \\ -A & -iA \end{pmatrix} \middle| A^* = A \right\} \simeq \mathrm{H}(n,\mathbb{C}),$$

where $\mathrm{H}(n,\mathbb{C}) = \{A \in M_{n,n}(\mathbb{C}) \mid A^* = A\}$. For finding the corresponding operation of H on $\mathrm{H}(n,\mathbb{C})$ we see by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} = \begin{pmatrix} A^* & B^* \\ -B^* & A^* \end{pmatrix} \qquad \text{for} \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ -\mathrm{B} & \mathrm{A} \end{pmatrix} \in \mathrm{H}$$

that the above identification transforms the adjoint action of H on $\mathbf{g}(1)$ into the operation $(a, Z) \mapsto aZa^*$ of $\mathrm{SL}(n,\mathbb{C}) + \mathbb{R} \simeq H$ on $\mathrm{H}(n,\mathbb{C})$. In this case ψ is given by conjugation by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}.$$

If we realize G/K as $D_{n,n} = \{Z \in M_{n,n}(\mathbb{C}) \mid I_n - Z^*Z > 0\}$ and identify $\mathbf{g}(1)$ with $\mathrm{H}(n, \mathbb{C})$ then we get the usual Cayley transform

$$D_p \ni Z \mapsto (Z + iI_n)(iZ + iI_n)^{-1} \in \mathrm{H}(n, \mathbb{C}) + i\mathrm{H}^+(n, \mathbb{C}) \simeq \mathbf{q}^+ \oplus i\Omega.$$

Here the H-orbit $D^\tau = \{Z \in D_p \mid Z^* = -Z\}$ maps onto the cone $H^+(n, \mathbb{C}) \simeq \Omega = H \cdot I_{2n}$.

6. H-invariant cones in q

In this section we will characterize the symmetric pairs of Hermitian type using an *infinitesimal causal orientation* on M in the sense of [42, p. 22]. As $TM \simeq G \times_H \mathbf{q}$ this in turn amounts to give a characterization in terms of H-invariant cones in $\mathbf{q}$. Some of the results in this section may also be found in the papers [35, 36] of Ol'shanskii but without proofs. The main motivation for Ol'shanskii's studying regular real forms is the fact that for $\mathbf{g}$ simple, $C \in \mathrm{Con}_G(i\mathbf{g})$ and $\mathbf{l} = \mathbf{h} \oplus i\mathbf{q}$ a regular real form he proves in [35, p. 281], that $0 \neq C \cap i\mathbf{q} \in \mathrm{Con}_H(i\mathbf{q})$. In particular it now follows from Theorem 2.7, that for $\mathbf{g}$ simple $\mathrm{Con}_H(i\mathbf{q})$ is non empty. We will always assume that $(\mathbf{g}, \tau)$ is effective and without compact ideals. We will also assume that H is connected as otherwise there can arise some problems as explained in [30]. We will also explain that shortly at the end. Recall that we are always assuming $G \subset G_c$ where G_c is a simply connected Lie group with the Lie algebra $\mathbf{g}_c$.

Theorem 6.1. *Let $(\mathbf{g}, \tau)$ be an effective symmetric pair. Let $H := G_o^\tau$. Then $(\mathbf{g}, \tau)$ is of Hermitian type (regular) if and only if there exists a $C \in \mathrm{Con}_H(\mathbf{q})$ with $C^o \cap \mathbf{q}_k \neq \emptyset$ ($C^o \cap \mathbf{p} \neq \emptyset$), where the superscript ^{o} denotes the interior.*

Corollary 6.2. *Let $(\mathbf{g}, \tau)$ and H be as above. Then $(\mathbf{g}, \tau)$ is of Hermitian type if and only if there exists a $C \in \mathrm{Con}_H(\mathbf{g})$ and a $X \in C^o$ such that the geodesic $\mathbb{R} \ni t \mapsto \gamma_X(t) := \exp(tX) \cdot x_0 \in G/H$ is closed.*

Our main tools for proving this and other results about cones are the following theorems of Kostant, Paneitz and Vinberg (see [42, 38, 46]).

Theorem 6.3 (Kostant, Vinberg). *Let L be a connected semisimple Lie group acting on the real vector space $\mathbf{V}$ by a representation π. Let $K \subset L$ be a subgroup of L such that $\pi(K)$ is a maximal compact subgroup of $\pi(L)$ and let $P \subset L$ be a minimal parabolic subgroup of L.*

(1) *There exists a proper L-invariant cone in $\mathbf{V}$ if and only if the space of K-fixed vectors*

$$\mathbf{V}^K := \{v \in \mathbf{V} \mid \forall\, k \in K \;:\; \pi(k)v = v\}$$

is non-zero.

(2) *If π is irreducible, then $\mathrm{Con}_L(\mathbf{V}) \neq \emptyset$ if and only if any of the following equivalent conditions is satisfied:*

(a) $\mathbf{V}^K \neq 0$.
(b) *There exists a ray through* 0 *which is invariant with respect to* P.

Theorem 6.4 (Paneitz,Vinberg). *Let the notation be as in the theorem of Kostant-Vinberg and assume that* $\mathbf{V}$ *is irreducible. Let* $\mathrm{Con}_L(\mathbf{V}) \neq \emptyset$. *Then there exists a unique (up to the multiplication by* (-1)*) minimal invariant cone* $C_{\min} \in \mathrm{Con}_L(\mathbf{V})$ *given by*

$$C_{\min} = \mathrm{con}(L \cdot v_P) = \overline{\mathrm{con}\, L \cdot v_K},$$

where $\mathrm{con}(U) := \{\sum_{j\in I} c_j v_j \mid c_j \in \mathbb{R}^+, v_j \in U, I \subset \mathrm{N}$ *finite* $\}$ *denotes the convex hull of a subset* U *of* $\mathbf{V}$, v_P *is an eigenvector for* P *contained in the ray in* (2) *and* v_K *is a non-zero* K*-invariant vector unique up to a scalar. Furthermore* $v_K \in C^o_{\min}$. *The unique (up to a scalar) maximal cone is then given by* $C_{\max} = C^*_{\min}$.

We point out one idea of the proof as we will use it later on. Let L be a Lie group acting on $\mathbf{V}$ by π. As we will only deal with $\pi(L)$ we can assume that $L = \pi(L)$. Let $K \subset L$ be a compact subgroup of L. If $C \subset \mathbf{V}$ is a proper cone we choose $v \in C^*$ with $(u|v) > 0$ for all $u \in C \setminus 0$. Let $u \in C \setminus 0$. Then $(k \cdot u|v) > 0$ for all $k \in K$. It follows, that

$$u_K := \int_K k \cdot u dk$$

is K-invariant and $(u_K, v) = \int_K (k \cdot u, v) dk > 0$. Thus $u_K \neq 0$. As K is compact it follows that $K \cdot u$ is also compact and thus $\mathrm{con}\, K \cdot u = \overline{\mathrm{con}\, K \cdot u}$ is compact, too. If C is generating we can start with $u \in C^o$. Then for all $c_1, \ldots, c_n > 0$, $\sum_j c_j = 1$ and all $k_1, \ldots k_n \in K$ it holds $\sum_j c_j k_j \cdot u \in C^o$. It follows:

$$u_K \in \overline{\mathrm{con}\, K \cdot u} = \mathrm{con}\, K \cdot u \subset C^o.$$

We now prove Theorem 6.1. We only prove the claim for cones with $C^o \cap \mathbf{k} \neq \emptyset$ resp. of Hermitian type as the other will follow by same method or by using the c-dual construction. We can also assume that $(\mathbf{g}, \tau)$ is irreducible by projecting onto each irreducible factor resp. by constructing cones by $C_1 \oplus \ldots \oplus C_r := \{(X_1, \ldots, X_r) \mid X_j \in C_j\}$ for C_j invariant cone in the irreducible factor $(\mathbf{g}_j, \tau|_{\mathbf{g}_j})$, $\mathbf{g} = \oplus_{j=1}^r \mathbf{g}_j$.

Assume first that there is a cone $C \in \mathrm{Con}_H(\mathbf{q})$ such that $C^o \cap \mathbf{k} \neq \emptyset$. Then we can find by the above a $Z \in C^o \cap \mathbf{k}$. Then $Z \neq 0$ and $[\mathbf{h}_k, Z] = 0$. Let $X \in \mathbf{q}_k$. Then $([Z,X]|[Z,X])_\theta = -(X|[Z,[Z,X]])_\theta = 0$, as $[Z,X] \in \mathbf{h}_k$. Hence $Z \in \mathbf{c}_\mathbf{k}$. And then $\mathbf{z}_\mathbf{g}(Z) = \mathbf{k}$, $\mathbf{z}_\mathbf{q}(\mathbf{c}) = \mathbf{q}_k$. Hence $(\mathbf{g}, \tau)$ is of Hermitian type.

If $\mathbf{q}$ is irreducible as a H-modul it follows by the Paneitz-Vinberg theorem that we can find a cone $C \in \mathrm{Con}_H(\mathbf{q})$ containing a $Z \in \mathbf{c}$ as an inner point, i.e., $C^o \cap \mathbf{k} \neq \emptyset$. If $\mathbf{q}$ is reducible it follows that $(\mathbf{g}, \tau)$ is of Cayley type and we can write $\mathbf{g} = \mathbf{h} \oplus \mathbf{q}^+ \oplus \mathbf{q}^-$ with $\mathbf{q}^\pm = \mathbf{g}(\pm 1)$ and $\mathbf{q}^\pm$ abelian, $\theta \mathbf{q}^+ = \mathbf{q}^-$. Furthermore it is easy to see that $\mathbf{q}^\pm$ is irreducible (otherwise take $0 \neq \mathbf{q}_1^+ \subset \mathbf{q}^+$, $\mathbf{q}_1^+ \neq \mathbf{q}^+$, an H-invariant submodule. Then $\mathbf{q}_1^+ \oplus \theta(\mathbf{q}_1^+) \oplus [\mathbf{q}_1^+, \theta(\mathbf{q}_1^+)]$ is an ideal). Let $Z = X_+ + X_- \in \mathbf{c}$, where $X_+ \in \mathbf{q}^+$ and $X_- \in \mathbf{q}^-$. Then $\theta X_+ = X_-$, and X_+ and X_- are $H \cap K$-invariant as $\mathbf{q}^\pm$ are H-stable.

Thus we can find H-invariant cones $C_\pm \in \mathrm{Con}_H(\mathbf{q}^\pm)$ such that C_+ contains X_+ as an inner point (relatively to $\mathbf{q}^+$), and similarly $X_- \in C_-^o$. The theorem follows now with $C := C_+ \oplus C_-$. □

We notice now that by our previous remarks on Cayley transforms in Section 4 it follows, that the cones $C_\pm$ are self dual and so unique up to a sign. In particular $\theta(C_+) = C_-$.

Theorem 6.5. *If $(\mathbf{g}, \tau)$ is irreducible then* $\dim \mathbf{q}^{H\cap K} \leqslant 2$. *If $(\mathbf{g}, \mathbf{h})$ is irreducible then* $\dim \mathbf{q}^{H\cap K} = 1$ *if and only if $\mathbf{q}$ is irreducible as an H-module. This holds if and only if every proper H-invariant cone is generating. In this case exactly one of the following two cases holds:*

(1) $\mathbf{q}^{H\cap K} \subset \mathbf{q}_k$ *and* $C^o \cap \mathbf{k} \neq \emptyset$ *but* $C \cap \mathbf{p} = \{0\}$ *for every* $C \in \mathrm{Con}_H(\mathbf{q})$. $(\mathbf{g}, \tau)$ *is of Hermitian type but not Cayley type.*

(2) $\mathbf{q}^{H\cap K} \subset \mathbf{q}_p$ *and* $C^o \cap \mathbf{p} \neq \emptyset$ *but* $C \cap \mathbf{k} = \{0\}$ *for every* $C \in \mathrm{Con}_H(\mathbf{q})$. *In this case* $(\mathbf{g}, \tau)$ *is regular.*

Theorem 6.6. *Let $(\mathbf{g}, \mathbf{h})$ be an irreducible symmetric pair. Then* $\dim \mathbf{q}^{H\cap K} = 2$ *if and only if $\mathbf{q}$ is reducible as an H-module. This holds if and only if τ is a Cayley type involution. In this case $\mathbf{q}$ decomposes into $\mathbf{q} = \mathbf{q}^+ \oplus \mathbf{q}^-$ with $\mathbf{q}^\pm$ irreducible.* $\dim(\mathbf{q}^+)^{K\cap H} = \dim(\mathbf{q}^-)^{K\cap H} = \dim \mathbf{q}_k^{H\cap K} = \dim \mathbf{q}_p^{K\cap H} = 1$. *There exist cones* $C_k, C_p \in \mathrm{Con}_H(\mathbf{q})$ *such that*

$$\begin{aligned} C_k^o \cap \mathbf{k} &\neq \emptyset, \quad C_k \cap \mathbf{p} = \{0\}, \\ C_p^o \cap \mathbf{p} &\neq \emptyset, \quad C_p \cap \mathbf{k} = \{0\}, \\ \mathrm{Con}_H(\mathbf{q}) &= \{C_k, -C_k, C_p, -C_p\}. \end{aligned}$$

Proof. First of all the dimension of $\mathbf{q}^{H\cap K}$ is less than or equal to the number of irreducible components of $\mathbf{q}$ by the Paneitz-Vinberg theorem. As we have seen above, this number is $\leqslant 2$ and equals 1 if and only if $\mathbf{q}$ is irreducible. Now any cone $C \in \mathrm{Con}_H(\mathbf{q})$ satisfies $C^{oH\cap K} \subset \mathbf{q}^{H\cap K}$ and the first theorem follows easily by the above arguments using the theorems of Konstant, Paneitz and Vinberg, Theorem 5.6 and noticing that $\mathbf{q}^{H\cap K}$ is θ-stable and thus $\mathbf{q}^{H\cap K} = \mathbf{q}_k^{H\cap K} \oplus \mathbf{q}_p^{H\cap K}$. For the second theorem we only have to chose X_+ as above. Then $X_k + \theta(X_+) \in \mathbf{q}_k^{H\cap K} \setminus 0$ and $X_+ - \theta(X_+) \in \mathbf{q}_p^{H\cap K} \setminus 0$. The theorem now follows by the above arguments and the fact that $C_\pm$ are the (up to a sign) unique invariant cones in $\mathbf{q}^\pm$. □

From now on we assume that $(\mathbf{g}, \tau)$ is of Hermitian type and irreducible. We choose $Z_o \in \mathbf{c}$ defining the almost complex structure on $\mathbf{p}$ as usually. If $\mathbf{q}$ is irreducible then we know, that the (up to a sign unique) minimal H-invariant cone in $\mathbf{q}$ is given by $C_{\min} := \overline{\mathrm{con}(\mathrm{Ad}(H)Z_o)}$ and the maximal cone is $C_{\max} := C_{\min}^*$. If $\mathbf{q}$ is reducible we need the following:

Lemma 6.7. *Assume that $(\mathbf{g}, \tau)$ is irreducible and of Cayley type. Choose X_+, X_- as above such that $Z_o = X_+ + X_-$.*

(1) *If* $C \in \mathrm{Con}_H(\mathbf{q})$ *then*
(a) *If* $C \cap \mathbf{k} \neq \{0\}$ *then* $X_+, X_- \in C$ *or* $\in -C$;
(b) *If* $C^o \cap \mathbf{p} \neq \{0\}$ *then* $X_+, -X_- \in C$ *or* $\in -C$.
(2) *Let* $C \subset \mathbf{q}$ *be an* H*-invariant closed proper cone. Then* $C \cap \mathbf{k} \neq \{0\}$ *and* $C \cap \mathbf{p} \neq \{0\}$ *is impossible.*

Proof. (1) Assume that $C \cap \mathbf{k} \neq \{0\}$. Let $Z \in C^{H\cap K} \cap \mathbf{k}$. Then there is a $t \in \mathbb{R} \setminus 0$ such that $Z = t(X_+ + X_-)$. Thus we may assume that $Z_o \in C$. We choose furthermore $X \in \mathbf{c}_h$ such that $\mathbf{q}^+ = \{Y \in \mathbf{q} \mid [X,Y] = Y\}$. Then $\mathbf{q}^- = \{Y \in \mathbf{q} \mid [X,Y] = -Y\}$. With $Y := X_+ - X_-$ we then have

$$e^{t\,\mathrm{ad}(X)} Z_o = \cosh(t) Z_o + \sinh(t) Y = \cosh(t)(Z_o + \tanh(t) Y).$$

Divide $\cosh(t)$ out and let $t \to \pm\infty$; it follows that $Z_o \pm Y, \in C$. As $2X_+ = Z_o + Y$ and $2X_- = Z_o - Y$ we have the first claim. The first part follows now by similar arguments by starting with Y instead of Z_o in the case where $C^o \cap \mathbf{p} \neq \emptyset$.

If $C \cap \mathbf{p}$ and $C \cap \mathbf{k}$ are both non trivial we can choose $X_\pm$ as above and such that $\pm X_+ \in C$ or $\pm X_- \in C$, thus C would contain a line. □

This implies that the unique (up to a sign) invariant proper cone in $\mathbf{q}$ containing an almost complex structure of $\mathbf{p}$ commuting with K is also given by $\overline{\mathrm{con}(\mathrm{Ad}(H) Z_o)}$ in the case that $\mathbf{q}$ is reducible.

Example 6.8. Let $G = \mathrm{SL}(2,\mathbb{R})$ and define τ by

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Let

$$H := G^\tau = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{R} \setminus 0 \right\}.$$

Then

$$G/H \simeq \mathrm{Ad}(G)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \subset \mathbf{sl}(2,\mathbb{R})$$

is the one-sheeted hyperboloid

$$G/H = \left\{ x\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \middle| \, x^2 + y^2 - z^2 = 1 \right\}.$$

In the above notation we have:

$$Z_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Thus the cones $C_\pm$ are the lines

$$C_+ = \mathbb{R}^+\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_- = \mathbb{R}^+\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The H-orbit through Z_0 is the hyperbola $H \cdot Z_0 = \{xZ_0 + y(X_+ + X_-) \mid x^2 - y^2 = 1\}$ and the cones C_k, C_p are given by

$$C_k = \left\{ \begin{pmatrix} 0 & x \\ -y & 0 \end{pmatrix} \middle| x, y \geqslant 0 \right\}, \quad C_p = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \middle| x, y \geqslant 0 \right\}.$$

Remark 6.9. Replace in the above example the group $\mathrm{SL}(2, \mathbb{R})$ by the locally isomorphic group $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Ad}(\mathrm{SL}(2, \mathbb{R}))$ and define τ on $\mathrm{PSL}(2, \mathbb{R})$ by $\tau(\mathrm{Ad}(g)) := \mathrm{Ad}(\tau(g))$. Then $\theta \in \mathrm{PSL}(2, \mathbb{R})^\tau$ and hence the cone C_p is not $\mathrm{PSL}(2, \mathbb{R})^\tau$-invariant. And in fact $\mathbf{q}$ is now irreducible as a $\mathrm{PSL}(2, \mathbb{R})^\tau$-module as $\theta(\mathbf{q}^+) = \mathbf{q}^-$. In the general case the same problem arises if $(\mathbf{g}, \tau)$ is of Cayley type or regular. Then there may be a H_o-invariant cone but no H-invariant one with $C^o \cap \mathbf{p} \neq \emptyset$. In particular this happens if $\theta \in \mathrm{Ad}(G)^\tau$ as above, see also [30].

7. Remarks on the classification

In this last section we give a short overwiev - partly without proofs - of the classification of invariant cones in $\mathbf{q}$, [30]. For simplicity we assume that $(\mathbf{g}, \tau)$ is irreducible and H connected. Since multiplication by i induces a bijection $\mathrm{Con}_H(\mathbf{q}) \simeq \mathrm{Con}_H(i\mathbf{q})$ we may also assume that τ is of Hermitian type. If τ is of Cayley type then we know that the cones are unique up to a sign by the results of [19], as we have pointed out before, but to make the classification uniform and independent of [19] we notice, that we can find a $X_o \in c_h$ such that $\mathrm{ad}(X_o)$ defines a paracomplex structure on $\mathbf{q}$. It follows that $[\mathrm{ad}(X_o)C]^o \cap \mathbf{p} \neq \emptyset$ if and only if $C^o \cap \mathbf{k} \neq \emptyset$. Hence we only have to describe the set $\mathrm{Con} = \{C \in \mathrm{Con}_H(\mathbf{q}) \mid C^o \cap \mathbf{k} \neq \emptyset\}$.

Choose now $\mathbf{a} \subset \mathbf{q}_k$ maximal abelian as before. Let $W_H = N_H(\mathbf{a})/C_H(\mathbf{a})$ be the Weyl group of Δ in H. Then $W_H = W_k$, the Weyl group of Δ_k. Define $P(C) := \mathrm{pr}(C)$ and $I(C) := C \cap \mathbf{a}$, $C \in \mathrm{Con}$, where pr is the orthogonal projection. Then $P(C)$ and $I(C)$ are W_H invariant cones in $\mathbf{a}$. We also define $F(c) := \overline{\mathrm{con}(\mathrm{Ad}(H)c)}$ for c a cone in $\mathbf{a}$. Then $F(c)$ is an H-invariant cone in $\mathbf{q}$. Let

$$-ic_{\min} := \oplus_{\alpha \in \Delta_p^+} \mathbb{R}^+ \; \hat{H}_\alpha$$

and

$$c_{\max} := c_{\min}^* = -i\{X \in \mathbf{a} \mid \forall \alpha \in \Delta_p^+ \; : \; \alpha(X) \geqslant 0\}.$$

Let A be the analytic subgroup of G corresponding to $\mathbf{a}$. Since $A = \{a \in T \mid \tau(a) = a^{-1}\}_o$, where T is the maximal torus corresponding to $\mathbf{t}$, A is closed. Normalize the Haar measure on A to have total measure 1. Define for $b \in G$ the linear map $\Psi_b : \mathbf{g}_c \to \mathbf{g}_c$ by

$$\Psi_b(X) := \int_A \mathrm{Ad}(a)\,\mathrm{Ad}(b)\,\mathrm{Ad}(a)^{-1} X \, da,$$

and let $\mathcal{H} := \{\Psi_h \mid h \in H\}$.

Lemma 7.1. *The orthogonal projection onto* $\mathbf{a}$ *is given by* $\mathrm{pr}(X) = \int_A \mathrm{Ad}(a)X\, da$. *In particular* $\mathrm{pr} \circ \mathrm{Ad}(h)|_{\mathbf{a}} = \Psi_h|_{\mathbf{a}}$ *for all* $h \in H$.

Proof. First we notice that A acts on each root space $\mathbf{g}_{c\alpha}$ by the non trivial character $a \mapsto a^\alpha$, where a^α is defined by $\exp(Y)^\alpha = e^{\alpha(Y)}$. As this character is unitary it follows that $\int_A a^\alpha da = 0$ for all α. For $X = Y + \sum_\alpha (X_\alpha - \tau(X_\alpha))$, $Y \in \mathbf{a}$ we get

$$\int_A \mathrm{Ad}(a)X\, da = Y$$

and the claim follows. □

Now the first results in the direction of classification are:

Lemma 7.2. *Let* $C \in \mathrm{Con}$. *Then* $P(C)^* = I(C^*)$.

Theorem 7.3. (1) *Let* $C \in \mathrm{Con}$. *Then* $I(C)$ *is a* W_H*-invariant regular cone in* $\mathbf{a}$ *and*

$$c_{\mathrm{min}} \subset I(C_{\mathrm{min}}) \subset I(C) \subset I(C_{\mathrm{max}}) \subset P(C_{\mathrm{max}}) \subset c_{\mathrm{max}}$$

for a suitable chosen c_{min} *and* c_{max}.

(2) $C_{\mathrm{min}} = F(c_{\mathrm{min}})$.

(3) *Let* c *be a closed regular* $\mathcal{H}$*-invariant cone in* $\mathbf{a}$. *Then* c *is* W_H*-invariant,* c^* *is a* $\mathcal{H}$*-invariant cone in* $\mathbf{a}$ *and* $c_{\mathrm{min}} \subset P(C_{\mathrm{min}}) \subset c \subset P(C_{\mathrm{max}}) \subset c_{\mathrm{max}}$. *Moreover* $F(c) \in \mathrm{Con}$ *and* $c = P(F(c)) = I(F(c))$.

The problem is then to relate P and I as well as W_H- and $\mathcal{H}$-invariant regular cones in $\mathbf{a}$. This is done by the following generalization of the convexity theorem of Paneitz [38] or infinitesimal version of the convexity theorem of van den Ban [1]:

Theorem 7.4. *Let* $X \in I(C_{\mathrm{max}})$ *and* $h \in H$, *then* $\Psi_h(X) \in \mathrm{con}(W_k \cdot X) + c_{\mathrm{min}}$.

From this we get immediately:

Theorem 7.5. $P(C_{\mathrm{min}}) = I(C_{\mathrm{min}}) = c_{\mathrm{min}}$ *and* $P(C_{\mathrm{max}}) = I(C_{\mathrm{max}}) = c_{\mathrm{max}}$.

Proof. By the above convexity theorem it follows that c_{min} is $\mathcal{H}$-stable. By Theorem 7.3. $P(F(c_{\mathrm{min}})) = c_{\mathrm{min}} \subset P(C_{\mathrm{min}}) \subset P(F(c_{\mathrm{min}}))$, as $F(c_{\mathrm{min}})$ is a regular cone. Now $I(C)$ is always a subset of $P(C)$ and we are forced to have $P(C_{\mathrm{min}}) = I(C_{\mathrm{min}})$. By Lemma 7.2

$$I(C_{\mathrm{max}}) = I(C_{\mathrm{min}}^*) = P(C_{\mathrm{min}})^* = c_{\mathrm{min}}^* = c_{\mathrm{max}}.$$

Thus $c_{\mathrm{max}} \subset I(C_{\mathrm{max}}) \subset P(C_{\mathrm{max}}) \subset c_{\mathrm{max}}$ by Theorem 7.3. □

Theorem 7.6. *Let* c *be a closed cone in* $\mathbf{a}$. *Then the following are equivalent*:

(1) c *is* W_k*-invariant and* $c_{\mathrm{min}} \subset c \subset c_{\mathrm{max}}$ *for a suitable chosen minimal cone.*

(2) c *is regular and* $\mathcal{H}$*-invariant.*

(3) *There exists a cone* $C \in \mathrm{Con}$ *such that* $P(C) = c$

(4) *There exists a cone* $C \in \mathrm{Con}$ *such that* $I(C) = c$.

Proof. If (1) holds then by Theorem 7.5 and the convexity theorem $\Psi_h X \in c$ for all $h \in H$ and $X \in c$. Thus (2) follows. (3) follows from (2) by Theorem 7.3. If (3) holds then c is W_H-stable and hence $\mathcal{H}$-stable. As $c^o = P(C^o)$ and by Lemma 7.2 $(c^*)^o = (P(C)^*)^o = (I(C^*))^o \neq \emptyset$ it follows that c is regular. Thus by Theorem 7.3 c^* is regular and $\mathcal{H}$-invariant and $c = P(F(c^*))^* = I(F(c^*)^*)$. That (4) $\Longrightarrow$ (1) is obvious from Theorem 7.3 and thus the theorem follows. □

We can now formulate the main theorem. As it stands the theorem holds for semisimple symmetric pairs of Hermitian type such that $(\mathbf{g}, \mathbf{h})$ and $(\mathbf{g}, \mathbf{k})$ are effective.

Theorem 7.7 (Classification of cones). *Let $(\mathbf{g}, \mathbf{h})$ be a irreducible semisimple symmetric pair of Hermitian type. Let H be connected and let* $\mathrm{Con} = \{C \in \mathrm{Con}_h(\mathbf{q}) \mid C^o \cap \mathbf{k} \neq \emptyset\}$. *Let $c_{\min} \subset c^*_{\min} = c_{\max}$ and $C_{\min} \subset C^*_{\min} = C_{\max}$ be as before. Let $C \in$* Con.

(1) *C is uniquely determined by $I(C)$. $C^o = \mathrm{Ad}(H)I(C)^o$ and $C = \overline{\mathrm{Ad}(H)I(C)}$.*

(2) *$I(C) = P(C)$ and $I(C)^* = I(C^*)$.*

(3) *$c_{\min} = I(C_{\min})$ and $I(C_{\max}) = c_{\max}$.*

(4) *A cone c in $\mathbf{a}$ is of the form $I(C)$ for some $C \in$* Con *if and only if c is regular and $\mathcal{H}$-invariant. This is equivalent to c being W_H-invariant and $c_{\min} \subset c \subset c_{\max}$ for a suitable choice of a minimal cone. In this case $C = F(c) = \mathrm{Ad}(H)c$.*

Here the first main point of the proof is part (1), where we use for the first time that H is sitting in a bigger group G. Assume for a moment that we have proved (1). By Theorem 7.6 $I(C)$ is $\mathcal{H}$-invariant. Thus $P(C) = \mathrm{pr}(\mathrm{Ad}(H)I(C)) \subset I(C)$ and the first part of (2) follows as $I(C) \subset P(C)$. The second part follows from this and Lemma 7.2 and $I(C^*) = P(C)^* = I(C)^*$. As we have already proved (3) and (4) we only have to prove (1) and for that we need some facts about G-invariant cones in $\mathbf{g}$ and extension of cones from $\mathbf{q}$ to $\mathbf{g}$. Define for $C \in$ Con a G-invariant cone in $\mathbf{g}$ by

$$F_G(C) := \overline{\mathrm{con}(\mathrm{Ad}(G)C)}.$$

To avoid confusion we use D for G-invariant cones in $\mathbf{g}$. In particular $D_{\min}$ is the minimal cone and $D_{\max}$ the corresponding maximal cone. Define $I_G(D) := D \cap \mathbf{q}$ and $P_G(D) := \mathrm{pr}_{\mathbf{q}}(D)$, $D \in \mathrm{Con}_G(\mathbf{g})$. Here $\mathrm{pr}_{\mathbf{q}} : \mathbf{g} \to \mathbf{q}$ is the orthogonal projection. $C \in$ Con is called *extendable* if there exists a $D \in \mathrm{Con}_G(\mathbf{g})$ such that $C = I_G(D)$. We have the following theorem:

Theorem 7.8. *Every cone in* Con *is extendable.*

For some of the classical groups this was first proved for invariant cones C with $D_{\min} \cap \mathbf{q} \subset C \subset D_{\max} \cap \mathbf{q}$ by J. Hilgert in his notes [8]. Our proof uses the obvious relations

$$C_{\min} \subset D_{\min} \cap \mathbf{q} \subset D_{\max} \cap \mathbf{q} \subset C_{\max}$$

and then the classification as well as the results from Section 3 on the relation between roots of $\mathbf{t}$ and $\mathbf{a}$ as well as the relation between the corresponding root vectors and co-roots. This gives the first step $D_{\min} = F_G(C_{\min})$ and $F_G(C_{\max}) \subset D_{\max}$. Then by general remarks on the different types of involutions the proof is reduced to the cases $(\mathbf{su}(2p,2q), \mathbf{sp}(p,q))$, $(\mathbf{so}(2,n), \mathbf{so}(1,n-k)\times\mathbf{so}(1,k))$, $4 \leqslant 2k \leqslant n$, $2k \neq q$, and $(\mathbf{e}_{6(-14)}, \mathbf{}_{4(-20)})$, where we show this case by case.

Lemma 7.9. *$d_{\min} \cap \mathbf{a} = \mathrm{pr}_{\mathbf{q}}\, d_{\min} = c_{\min}$ and $d_{\max} \cap \mathbf{a} = \mathrm{pr}_{\mathbf{q}}\, d_{\max} = c_{\max}$, where $d_{\min} = D_{\min} \cap \mathbf{t}$ and $d_{\max} = d^*_{\min}$.*

Proof. As $-\tau\beta \in \Delta^+(\mathbf{p}_c, \mathbf{t}_c)$ for all $\beta \in \Delta^+(\mathbf{p}_c, \mathbf{t}_c)$ it follows that $d_{\min}$ is $-\tau$ stable. As $\mathrm{pr}_{\mathbf{q}} X = \frac{1}{2}(X - \tau X)$ it follows that $\delta_{\min} \cap \mathbf{a} \subset \mathrm{pr}_{\mathbf{q}}\, d_{\min} \subset d_{\min} \cap \mathbf{a}$ or $d_{\min} \cap \mathbf{a} = \mathrm{pr}_{\mathbf{q}}\, d_{\min}$. Let $\beta \in \Delta^+(\mathbf{p}_c, \mathbf{t}_c)$ be such that $\beta|_{\mathbf{a}} = \alpha$. Then $\hat{H}_\alpha = H_\beta$ if $\beta = \alpha$ and otherwise $\hat{H}_\alpha = H_\beta - \tau H_\beta = H_\beta + H_{-\tau\beta}$ according to our results on root vectors in Section 4. Thus $c_{\min} \subset d_{\min} \cap \mathbf{a}$.

Let $X \in -i d_{\min} \cap \mathbf{a}$. Then

$$\begin{aligned} X &= \sum_{\alpha \in \Delta^+(\mathbf{g}_c, \mathbf{t}_c)} \lambda_\alpha H_\alpha, \ \lambda_\alpha \geqslant 0, \\ &= -\tau(X) = \sum_{\alpha \in \Delta^+(\mathbf{p}_c, \mathbf{t}_c)} \lambda_\alpha(-\tau(H_\alpha)) \\ &= \sum_{\alpha \in \Delta^+(\mathbf{p}_c, \mathbf{t}_c)} \lambda_{-\tau(\alpha)} H_\alpha. \end{aligned}$$

Thus by replacing λ_α by $\frac{1}{2}(\lambda_\alpha + \lambda_{-\tau(\alpha)})$ we can assume, that $\lambda_\alpha = \lambda_{-\tau(\alpha)}$. Hence

$$\begin{aligned} X &= \frac{1}{2}(X - \tau(X)) \\ &= \sum_{\alpha \in \Delta^+(\mathbf{p}_c, \mathbf{t}_c)} \frac{\lambda_\alpha}{2}(H_\alpha - \tau(H_\alpha)) \\ &= \sum_{\alpha \in \Delta^+_p} \mu_\alpha \hat{H}_\alpha \in c_{\min}, \end{aligned}$$

with $\mu\alpha \geqslant 0$. The claim for $d_{\max}$ follows now by duality. □

From this we get

Lemma 7.10. (1) *$-\tau(D_{\max}) = D_{\max}$, $P_G(D_{\max}) = I_G(D_{\max}) = C_{\max}$ as well as $-\tau(D_{\min}) = D_{\min}$, $P_G(D_{\min}) = I_G(D_{\min}) = C_{\min}$.*
(2) *$d_{\min} \cap \mathbf{a} = \mathrm{pr}_{\mathbf{q}}\, d_{\min} = c_{\min}$ and $d_{\max} \cap \mathbf{a} = c_{\max}$.*
(3) *$D_{\min} = F_G(C_{\min})$ and $F_G(C_{\max}) \subset D_{\max}$.*

We prove now (1). By continiuty and the obvious fact that $\mathrm{Ad}(H)I(C^o) \subset C^o$, we only have to show that $C^o \subset \mathrm{Ad}(H)I(C^o)$. Let $X \in C^o$. Then by the above $X \in D^o_{\max}$ and thus by Lemma 4.1. in [9, p. 193], X is semisimple and $\mathbf{z}_{\mathbf{g}}(X)$ is a compactly imbedded subalgebra in $\mathbf{g}$. By [37, p. 412], X is contained in some Cartan subspace

$\mathbf{a}_1$ of $\mathbf{q}$. Then also by [37] we can find a $h \in H$ such that $\mathrm{Ad}(h)\mathbf{a}_1$ is θ-stable. But as $\mathbf{z}_\mathbf{g}(\mathrm{Ad}(h)(X)) = \mathrm{Ad}(h)\mathbf{z}_\mathbf{g}(X)$ is compact, it follows that $\mathrm{Ad}(h)\mathbf{a}_1 \subset \mathbf{q}_k$. But then once again by [37, Theorem 3]), there exists an $a \in H$ such that $\mathrm{Ad}(ah)\mathbf{a}_1 \in \mathbf{a}$. Thus $\mathrm{Ad}(ah)X \in \mathrm{Ad}(ah)\mathbf{a}_1 = \mathbf{a}$. Hence $\mathrm{Ad}(ah)X \in I(C^o) \subset I(C)^o$ and the theorem follows. □

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