JOURNAL OF DIFFERENTIAL EQUATIONS **66**, 51–61 (1987)

A Simple Almost-Periodicity Criterion and Applications

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Received June 13, 1985; revised November 14, 1985

We introduce a new almost-periodicity criterion for functions: $R \to X$ with X a complete metric space. This result is used to establish asymptotic almost periodicity of precompact positive trajectories to some differential equations of the form $du/dt + A(t)\,u(t) \ni 0$, where $A(t)$ is periodic with respect to $t \in R$. © 1987 Academic Press, Inc.

INTRODUCTION

The purpose of this paper is to study the asymptotic behavior as $t \to \infty$ of the solutions to some differential equations of the general form

$$\frac{du}{dt} + A(t)\,u(t) \ni 0 \tag{1}$$

with $A(t)$ a (possibly unbounded, non-linear or multi-valued) operator which depends on t in a periodic manner.

The first case of interest is of course that of linear, periodic differential systems

$$u' + A(t)\,u(t) = 0, \tag{2}$$

where $A(t) \in C(\mathbb{R}, \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n))$ is a periodic matrix. It is well-known that any *bounded* solution $u(t)$ of (2) on $\mathbb{R}$ is almost periodic: $\mathbb{R} \to \mathbb{R}^n$, and also that this result is not true in general for almost periodic $A(t)$. However, the proof given in the literature is complicated since it relies on comparison with the case $A(t)$ constant by means of Floquet theory, which of course provides additional information in this case but cannot be extended to an efficient infinite-dimensional theory.

0022-0396/87 $3.00

Another important case is when (1) is autonomous with $A(t) \equiv A$ an m-accretive operator in some infinite-dimensional Banach space V.

This has been studied in 1973 by Dafermos and Slemrod [1], who established that any precompact positive trajectory is asymptotically almost periodic: $\mathbb{R} \to V$ (the result is valid for multi-valued A).

Their result contains as a special (degenerated) case the well-known theorem on almost periodicity of solutions in the energy space for the (linear) wave equation with homogeneous Dirichlet boundary conditions in a bounded domain of $\mathbb{R}^n$.

Finally, in 1983 [5] the case of general contraction (or isometric) processes on a complete metric space was considered.

The results obtained there encompass the result of [1] as well as some special cases of (2).

Here we establish that a refinement of the method of [5] permits us to treat the general case of (2) in the same framework as the results of [5].

1. A Simple Almost-Periodicity Criterion

Let (X, d) be a complete metric space.

For any $u \in C(\mathbb{R}, X)$ and any $\alpha \in \mathbb{R}$ we set $(T_\alpha u)(t) = u(t+\alpha)$, $\forall t \in \mathbb{R}$.

A function $u \in C_B(\mathbb{R}, X)$ is called almost periodic: $\mathbb{R} \to X$ if we have

$$\bigcup_{\alpha \in \mathbb{R}} \{T_\alpha u\} \text{ is precompact in } C_B(\mathbb{R}, X). \tag{3}$$

We recall that a set $E \subset \mathbb{R}$ is called *relatively dense* if there exists $l > 0$ such that

$$\forall a \in \mathbb{R}, \qquad [a, a+l] \cap E \neq \varnothing.$$

For example, a doubly infinite sequence $(\alpha_m)_{m \in \mathbb{Z}}$ such that $\alpha_{m+1} > \alpha_m$ for all m is relatively dense if, and only if, $\alpha_{m+1} - \alpha_m$ is bounded.

The following weakened form of (3) will prove to be quite useful in the sequel of this paper.

THEOREM 1. *Let $u \in C_B(\mathbb{R}, X)$ be such that for some relatively dense set $E \subset \mathbb{R}$, we have*

$$\bigcup_{m \in E} \{T_m u\} \textit{ is precompact in } C_B(\mathbb{R}, X). \tag{4}$$

Then u: $\mathbb{R} \to X$ is almost periodic.

Proof. Hypothesis (4) implies in particular that $\bigcup_{m \in E}\{u(\cdot + m)\}$ is precompact in $C(0, 2l; X)$. Hence we have

$$u\colon \mathbb{R} \to X \text{ is uniformly continuous.} \tag{5}$$

[Since if $x_1 \leqslant x_2 \leqslant x_1 + l$ we have $\{x_1, x_2\} \subset [m, m+2l]$ for some $m \in E$.]

Now let $\{\alpha_k\}_{k \in \mathbb{N}}$ be any sequence of reals and $\alpha_k = m_k + \sigma_k$, $m_k \in E$, $\sigma_k \in [0, l]$. We can replace $\{\alpha_k\}$ by a subsequence such that $\sigma_k \to \sigma$ and $u(m_k + t)$ converges uniformly to $v(t)$ in $C_B(\mathbb{R}, X)$. For any $\varepsilon > 0$, taking account of (5), we have for $k \geqslant k(\varepsilon)$

$$\forall t \in \mathbb{R}, \qquad d(u(t+\alpha_k), v(t+\sigma)) \leqslant \varepsilon + d(u(t+m_k+\sigma), v(t+\sigma)).$$

Hence $u(\alpha_k + t)$ converges to $v(t+\sigma)$ in $C_B(\mathbb{R}, X)$, and Theorem 1 is proved. ∎

COROLLARY 2. *Let $u \in C_B(\mathbb{R}, X)$ be such that for some relatively dense set $E \subset \mathbb{R}$, there exists a map $w\colon E \to X$ and φ continuous: $\mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that we have*

$$w(E) \textit{ is precompact in } X. \tag{6}$$

$$\forall (m, n) \in E \times E, \qquad \operatorname*{Sup}_{t \in \mathbb{R}} d(u(t+m), u(t+n) \leqslant \varphi(d(w(m), w(n)). \tag{7}$$

Then $u\colon \mathbb{R} \to X$ is almost periodic.

Proof. Clearly (6) and (7) imply that (4) is satisfied. Then we apply Theorem 1. ∎

An important special case is the following

COROLLARY 3. *Let E, φ be as in the statement of Corollary 2, and assume that $u \in C_B(\mathbb{R}, X)$ satisfies*

$$u(E) \textit{ is precompact in } X, \tag{8}$$

$$\forall (m, n) \in E \times E, \qquad \operatorname*{Sup}_{t \in \mathbb{R}} d(u(t+m), u(t+n)) \leqslant \varphi(d(u((m)), d(u(n)). \tag{9}$$

Then $u\colon \mathbb{R} \to X$ is almost periodic.

Proof. In the above Corollary 2 we take $w = u|_E$. ∎

Remark 4. In the applications we shall have $E = \mathbb{Z}T$ for some $T > 0$ and $\varphi(r) = Kr$, $K \in [1, +\infty[$.

2. Application to Time-Dependent Periodic ODE

In this section, we consider Eq. (2), where $A(t) \in C(\mathbb{R}, \mathscr{L}(\mathbb{R}^n)$ satisfies

$$A(t+T) = A(t), \qquad \forall t \in \mathbb{R}, \tag{10}$$

with T some positive real number.

We give a new and simple proof of the following result.

Corollary 5. *Let $u \in C^1(\mathbb{R}, \mathbb{R}^n)$ be a solution of* (2) *with $A(t)$ as above. Assume that we have*

$$\operatorname*{Sup}_{t \in \mathbb{R}} \|u(t)\| < +\infty. \tag{11}$$

Then u: $\mathbb{R} \to \mathbb{R}^n$ is almost periodic.

Proof. Let V be the linear span of $u(\mathbb{Z}T)$, and $U(t)$ the fundamental matrix associated to (2). Also, let $B = \{b_1, ..., b_r\}$ be an orthonormal basis for V. Because $A(t)$ is T-periodic, it is clear that we have

$$\forall b \in B, \qquad \operatorname*{Sup}_{t \in \mathbb{R}} \|U(t)\, b\| < +\infty, \tag{12}$$

and since V is finite dimensional we infer

$$\forall v \in V, \forall t \in \mathbb{R}, \qquad \|U(t)\, v\| \leqslant C\|v\| \tag{13}$$

for some constant $C \geqslant 1$.

Since $u(t+mT)$ is a solution of (2) with initial value in V for all $m \in \mathbb{Z}$, we deduce from (11) that we have

$$\forall (m, n) \in \mathbb{Z} \times \mathbb{Z}, \quad \forall t \in \mathbb{R}, \qquad \|u(t+mT) - u(t+nT)\| \leqslant C\|u(mT) - u(nT)\|. \tag{14}$$

Therefore Corollary 5 is now an immediate consequence of Theorem 1. ∎

Remark 6. (a) It is well-known (cf., for example [5, Remark 1.3, p. 477]) that the conclusion of Corollary 5 is not satisfied for general almost periodic $A(t)$, even if $n=2$ and $(A(t))^* = -A(t)$ for all t. For a counterexample to a slightly different property in the nonlinear framework, cf. [8].

(b) In the *autonomous* case $A(t) \equiv A$, $\forall t \in \mathbb{R}$, the above method provides an extremely simple proof of the almost periodicity of bounded solutions of (2). (The usual proof is by computation.)

(c) In the same vein as above, we have the following result.

COROLLARY 7. *Let* $u \in C^1(\mathbb{R}^+, \mathbb{R}^n)$ *be a solution of* (2) *with* $A(t)$ *as above. Assume that we have*

$$\operatorname*{Sup}_{t \geqslant 0} \|u(t)\| < +\infty. \tag{15}$$

Then there exists an almost-periodic solution v *of* (2) *such that*

$$\lim_{t \to +\infty} \|u(t) - v(t)\| = 0. \tag{16}$$

Proof. We introduce as previously the fundamental matrix $U(t)$ associated to (2) and we call V_1 the linear span of $u(\mathbb{N}T)$.

As before we obtain

$$\forall w \in V_1, \forall t \geqslant 0, \|U(t)\, w\| \leqslant C_1 \|w\| \qquad \text{for some constant } C_1 \geqslant 1. \tag{17}$$

Now let $z_r(t) = u(t + rT)$, $\forall r \in \mathbb{N}$.

As a consequence of (15), there exists a sequence $r_k \to +\infty$ such that $z_{r_k} \to z$ in $C^1(a, b; \mathbb{R}^n)$ for all a, b such that $-\infty < a < b < +\infty$, and z is a *bounded solution* of (2) on $\mathbb{R}$.

By Corollary 5, z: $\mathbb{R} \to \mathbb{R}^n$ is almost periodic. To summarize, we have

$$\forall t \in \mathbb{R}, \qquad \lim_{k \to \infty} u(t + r_k T) = z(t) \tag{18}$$

with z an almost-periodic solution of (2).

Now we may assume (by refining the sequence r_k if necessary) that we have

$$\lim_{k \to \infty} \operatorname*{Sup}_{t \in \mathbb{R}} \|z(t - r_k T) - v(t)\| = 0 \tag{19}$$

because z is almost periodic: $\mathbb{R} \to \mathbb{R}^n$.

Obviously, v is again a solution of (2) on $\mathbb{R}$. Finally, we have

$$\lim_{k \to \infty} \|u(r_k T) - v(r_k T)\| = 0, \tag{20}$$

$$v(\mathbb{Z}T) \subset V_1 \tag{21}$$

[since $v(\mathbb{Z}T) \subset \overline{z(\mathbb{Z}T)} \subset \overline{u(\mathbb{N}T)}$].

From (17) and (21) we deduce

$$\forall k \in \mathbb{N}, \forall t \geqslant r_k T, \qquad \|u(t) - v(t)\| \leqslant C_1 \|u(r_k T) - v(r_k T)\|. \tag{22}$$

Finally, (20) and (22) clearly imply (16).

3. Application to General "Quasi-Contractive" Periodic Processes

In this section, we give a new proof of the main results of [5] in a slightly generalized form that could be convenient for some applications. First we recall some definitions

— A *process* on a complete metric space (X, d) is by definition (cf. [2]) a two-parameter family $U(t, \tau)$ of maps: $X \to X$ defined for $(t, \tau) \in \mathbb{R} \times \mathbb{R}^+$ and such that

$$\forall t \in \mathbb{R}, \forall x \in X, \qquad U(t, 0)\, x = x, \tag{23}$$

$$\forall (t, \sigma, \tau) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \forall x \in X, \qquad U(t, \sigma + \tau)\, x = U(t + \sigma, \tau)\, U(t, \sigma)\, x, \tag{24}$$

$$\forall x \in X, \forall t \in \mathbb{R}, \qquad U(t, \tau)\, x \in C(\mathbb{R}^+, X). \tag{25}$$

— For any $x \in X$, the function $t \to U(0, t)\, x$ defined for $t \geqslant 0$ is called the *positive trajectory* starting from x.

— A *complete trajectory* is by definition any function $u(t)$: $\mathbb{R} \to X$ such that $\forall (t, \tau) \in \mathbb{R} \times \mathbb{R}^+$, $u(t + \tau) = U(t, \tau)\, u(t)$. As a consequence of (25), any positive trajectory (resp. complete trajectory) is continuous.

— A process U on (X, d) is called T-periodic if we have

$$\forall t \in \mathbb{R}, \forall \tau \geqslant 0, \qquad U(t + T, \tau) = U(t, \tau).$$

Theorem 8. *Let U be a T-periodic process on X such that we have for some $M \geqslant 1$*

$$\forall (t, \tau) \in \mathbb{R} \times \mathbb{R}^+, \forall (x, y) \in X \times X, \qquad d(U(t, \tau)\, x, U(t, \tau)\, y) \leqslant M d(x, y). \tag{27}$$

Then we have the following conclusions

(a) Any complete trajectory $u(t)$ such that $u(\mathbb{R}^-)$ be precompact is almost periodic: $\mathbb{R} \to X$.

(b) If u is a positive trajectory such that $u(\mathbb{R}^+)$ be precompact, there exists an almost periodic, complete trajectory $v(t)$ such that

$$\lim_{t \to +\infty} d(u(t), v(t)) = 0. \tag{28}$$

Proof. We set $E = \mathbb{Z}T$ and $E^+ = \mathbb{N}T$.

(a) It follows immediately from (27) and the above definitions that we have

$$d(u(t+m), u(t+n)) \leqslant Md(u(m-p), u(n-p)),$$
$$\forall (m, n) \in E, \forall p \in E^+, \forall t \geqslant -p. \tag{29}$$

By the diagonal procedure, we can find an increasing sequence $\{p_k\}$ in E^+ such that

$$\forall m \in E, \qquad \lim_{k \to \infty} d(u(m-p_k), w(m)) = 0, \qquad \text{where } w: E \to \overline{u(R^-)}. \tag{30}$$

By letting $p = p_k$ and $k \to \infty$ in (29) we obtain

$$d(u(t+m), u(t+n)) \leqslant Md(w(m), w(n)) \qquad \forall (m, n) \in E, \forall t \in \mathbb{R}. \tag{31}$$

The result then is an immediate consequence of Corollary 2.

(b) The proof is analogous to the deduction of Corollary 7 from Corollary 5. Indeed, by using precompactness of $u(\mathbb{R}^+)$ and the "stability" hypothesis (27), we construct a complete trajectory $w(t)$ such that

$$w(t) = \lim_{k \to +\infty} u(n_k + t), \qquad \forall t \in \mathbb{R}$$

and uniformly on compact intervals of $\mathbb{R}$, where $\{n_k\}$ is an increasing sequence in E^+. By (a), w is almost periodic: $\mathbb{R} \to X$. After refining $\{n_k\}$ if necessary we can assume that $v_k(t) = w(t-n_k)$ converges in $C_B(\mathbb{R}, X)$ to some function v as $k \to +\infty$. By using (27) we easily conclude that (28) is satisfied, and of course v is an almost-periodic trajectory of U.

Remark 9. (a) Theorem 8 contains as a special case Theorem 1.1 of [5]. Our present result is slightly more general, with a new proof, probably more elegant and certainly easier to understand than the argument given in [5].

(b) Theorem 8 allows us to treat some multi-valued non-autonomous problems, such as the evolution equation $du/dt + Au(t) \ni f(t)$, where f is periodic and A is any maximal monotone operator in a Hilbert space H. It implies that any precompact trajectory of this equation is asymptotically almost periodic as $t \to +\infty$. However, for simplicity in Section 4 below we will restrict our attention to examples in the framework of single-valued operators.

4. Examples

In this section, we illustrate the general results obtained in the three sections above by giving a list of simple examples.

EXAMPLE 1. Consider the first order ODE

$$u' + a(t)\,u(t) = 0, \tag{32}$$

where $A = \mathbb{R} \to \mathbb{R}$ is T-periodic. The solution of (32) with initial value u_0 is given by the formula

$$u(t) = u_0 \exp\left(\int_0^t a(s)\,ds\right).$$

In this case, either all solutions with $u_0 \neq 0$ are unbounded on $[0, +\infty[$, or $\int_0^T a(s)\,ds \leqslant 0$. Then we have two cases

— If $\int_0^T a(s)\,ds < 0$, all solutions tend to 0 exponentially as $t \to +\infty$.

— If $\int_0^T a(s)\,ds = 0$, $\exp(\int_0^t a(s)\,ds)$ is T-periodic and so are all the solutions.

EXAMPLE 2. Consider the second order ODE

$$u'' + a(t)\,u' + b(t)\,u = 0, \tag{33}$$

where a and b: $\mathbb{R} \to \mathbb{R}$ are both T-periodic (and continuous). Let u be a solution of (33) which is bounded for $t \geqslant 0$. Then we have the following alternative

— Either u is asymptotic as $t \to +\infty$ to a $2T$-periodic solution (which may be either a constant, or a non-constant T-periodic solution, or a non-trivial T-anti-periodic solution).

— Or all solutions of (33) (and especially the given $u(t)$) are almost periodic.

Proof. It follows from "Esclangon's lemma" (cf., e.g. [3, Theorem 5.4, p. 82]) that $u'(t)$ is also bounded for $t \geqslant 0$.

As a consequence of Corollary 5, $U(t) = (u(t), u'(t))$ is asymptotic as $t \to +\infty$ to $\Omega(t) = (\omega(t), \omega'(t))$ with $\omega(t)$ an almost-periodic solution of (33). If $\Omega(0)$ and $\Omega(T)$ are linearly independent, the fundamental matrix is almost periodic. If $\Omega(T) = \lambda\Omega(0)$ with $\lambda \in \mathbb{R}$, then $\Omega(nT) = \lambda^n\Omega(0)$ and the almost periodicity of $\Omega(t)$ obviously implies $\lambda = \pm 1$ or $\Omega(0) = 0$. The conclusion follows easily.

EXAMPLE 3. Consider the ODE

$$u'' + b(t)\,u = 0, \tag{34}$$

where b: $\mathbb{R} \to \mathbb{R}$ is T-periodic. [This equation is called Hill's equation and the study of stability in (34) is already a delicate problem; cf. [4].]

Let $u = \mathbb{R}^+ \to \mathbb{R}$ be a bounded solution of (34). Then

— Either $u(t) \to 0$ and $u'(t) \to 0$ as $t \to +\infty$.

— Or $u(t)$ is almost periodic. In addition if u is not $2T$-periodic, then all solutions of (34) are almost periodic.

Proof. Let $\Omega(t) = (\omega(t), \omega'(t))$ be as above. Then $w(t) = (u'\omega - u\omega')(t)$ is constant and $w(t) \to 0$ as $t \to +\infty$. Hence we have

$$\forall t \in \mathbb{R}, \qquad (u'\omega - u\omega')(t) = 0. \tag{35}$$

From (35) it follows that $u(t) \equiv \lambda\omega(t)$ or $\omega(t) \equiv 0$. Of course in the first case with $\omega \neq 0$ we must have $\lambda = 1$. Hence if $\omega \neq 0$, $u(t)$ is almost periodic. The last assertion is clearly contained in the previous results concerning (33).

EXAMPLE 4. Let $A(t) \in C(\mathbb{R}, \mathscr{L}(\mathbb{R}^n))$ be T-periodic and such that

$$A(t) \geqslant 0, \qquad \forall t \in \mathbb{R}. \tag{36}$$

Let u be a solution of (2) on $\mathbb{R}$. Then

(a) There exists an almost-periodic solution v of (2) on $\mathbb{R}$ such that

$$\lim_{t \to +\infty} \|u(t) - v(t)\| = 0.$$

(b) If u is bounded on $]-\infty, 0]$, then in fact u is almost periodic: $\mathbb{R} \to \mathbb{R}^n$ and such that $\|u(t)\|$ is constant on $\mathbb{R}$.

EXAMPLE 5. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}^*$). We consider the semi-linear partial differential equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - e^{2\alpha(t)}\Delta u - \alpha'(t)\frac{\partial u}{\partial t} + g\left(t, \frac{\partial u}{\partial t}\right) = 0, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{37}$$

where $\alpha(t) \in C^1(\mathbb{R})$ is T-periodic,
$g(t, v) \in C^1(\mathbb{R} \times \mathbb{R})$ is T-periodic in t and such that

$$g(t, 0) = 0, \qquad \forall t \in \mathbb{R},$$

$$\frac{\partial}{\partial v}(g(t, v)) \geqslant 0, \qquad \forall (t, v) \in \mathbb{R}^2.$$

This equation can be written as a system in $X = H_0^1(\Omega) \times L^2(\Omega)$ by setting $v = e^{-\alpha(t)}(\partial u/\partial t)$: we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} &= e^{\alpha(t)} v, \\ \frac{\partial v}{\partial t} &= e^{\alpha(t)} \Delta u - e^{-\alpha(t)} g(t, e^{\alpha(t)} v). \end{aligned} \tag{38}$$

The operator $A(t)$ defined by

$$D(A(t)) = H^2 \cap H_0^1(\Omega) \times L^2(\Omega),$$
$$A(t)(u, v) = (-e^{-\alpha(t)} v,\ -e^{\alpha(t)} \Delta u + e^{\alpha(t)} g(t, e^{\alpha(t)} v))$$

is monotone in X endowed with the usual inner product, and the *closure* of $A(t)$ in $X \times X$ is maximal monotone for all $t \in \mathbb{R}$.

It is not difficult to check that (38) generates a *contractive periodic process* $U(t, \tau)$ on X. Moreover the subspace $W = H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$ is such that $U(t, \tau)\, W \subset W$ for all $(t, \tau) \in \mathbb{R} \times \mathbb{R}^+$ and we have $p(U(t, \tau)\, x) \leqslant p(x)$, $\forall x \in W$, where $p = W \to \mathbb{R}$ denotes the semi-norm $p(u, v) = \int_\Omega \{|\Delta u|^2 + |\nabla v|^2\}\, dx$. It follows by density that *all positive trajectories* of $U(t, \tau)$ are precompact in X. By Theorem 8, we conclude that any solution u of (37) [in the class $C(\mathbb{R}^+, H_0^1(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega))$] is asymptotic in the sense of X, as $t \to +\infty$, to a solution $(\omega(t, x), (\partial\omega/\partial t)(t, x))$ of

$$\begin{aligned} &\frac{\partial^2 \omega}{\partial t^2} - e^{2\alpha(t)} \Delta\omega - \alpha'(t) \frac{\partial \omega}{\partial t} = 0, \\ &\omega|_{\partial\Omega} = 0. \end{aligned} \tag{39}$$

In addition, any solution of (39) is almost periodic: $\mathbb{R} \to X$.

5. Concluding Remarks

Remark 10. In the finite dimensional setting, Corollary 7 and Theorem 8 are quite powerful. In infinite dimensional cases, a major difficulty will be the proof of precompactness of the orbits, even in the linear case.

Remark 11. Theorem 8 admits a generalization when the fixed space X is replaced by a closed subset $(X_t)_{t \in \mathbb{R}}$ which depends on t in a T-periodic manner and such that $U(t, \tau)\, X_t \subset X_{t+\tau}$, $\forall (t, \tau) \in \mathbb{R} \times \mathbb{R}^+$. This framework, rather complicated and more delicate to handle, is useful to study the evolution equation (1) when $\overline{D(A(t))} = X_t$ depends on t. For some results in this direction when $A(t)$ is a subdifferential, we refer the reader to [6] and [7].

Acknowledgment

The author is grateful to the referee for some useful bibliographical information.

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