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Limiting Behavior of the Eigenvalues of a Multivariate F Matrix

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The spectral distribution of a central multivariate F matrix is shown to tend to a limit distribution in probability under certain conditions as the number of variables and the degrees of freedom tend to infinity.

1. INTRODUCTION

Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ denote the eigenvalues of a random matrix $Z = (z_{ij}) : n \times n$. Then, the distribution function $C(x)$ defined by

$$C(x) = \frac{1}{n} \# \{i : \lambda_i \leqslant x\}$$

is known to be the spectral distribution of the matrix Z where $\#\{\ \}$ denotes the number of elements of the set $\{\ \}$. Jonsson [2, 3] showed that the spectral distribution of the Wishart matrix (divided by its degrees of freedom) has a

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limit distribution as the number of variables and the degrees of freedom both tend to infinity such that their ratio tends to a limit. Wachter [4] proved a similar but more general result. Yin and Krishnaiah [5] showed that the spectral distribution of $Z = WT$ tends to a limit distribution under certain conditions when p and m tend to infinity such that their ratio tends to a limit. Here $W : p \times p$ is a Wishart matrix (divided by its degrees of freedom m) and $T : p \times p$ is a random matrix distributed independent of W and has a limit spectral distribution.

In this paper, we first prove bounds on the distribution functions of the largest and smallest eigenvalues of the Wishart matrix. Then, we prove that the spectral distribution of $nS_1 S_2^{-1}$ tends to a limit distribution in probability as p, m and n tend to infinity such that $(p/m) \to y$ and $(p/n) \to y' < \frac{1}{2}$; here $S_1 : p \times p$ and $S_2 : p \times p$ are distributed independently as central Wishart matrices with m and n degrees of freedom respectively. This is the main result of the paper and it is proved by applying a result of Yin and Krishnaiah [5].

2. Preliminaries

We need the following results in the sequel.

Lemma 2.1. *If $0 < r < \frac{1}{2}$, then the unit ball in R^p can be covered by balls of radius r in such a way that the number of these smaller balls does not exceed $C \exp\{(p/2) \log(2\pi e/r^2)\}$ where C is a positive constant.*

Proof. First, we cover the unit ball by nonoverlapping p-cubes of side length $(2r/\sqrt{p})$. All these cubes can be included in a bigger ball with radius $1 + 2r$. So, the sum of these volumes cannot exceed the volume of the latter ball. Thus the number of these balls cannot exceed V_0/V_1 where V_0 is the volume of p-ball of radius $(1 + 2r)$ and V_1 is the volume of p-cube with side $(r/\sqrt{p})$. But,

$$V_0 = 2\pi^{p/2}(1 + 2r)^p/p\Gamma\left(\frac{p}{2}\right), \qquad V_1 = \left(\frac{r}{\sqrt{p}}\right)^p.$$

Now, applying Stirling's formula, we obtain

$$\frac{V_0}{V_1} \leqslant C \exp\left\{\left(\frac{p}{2}\right) \log\left(\frac{2\pi e}{r^2}\right)\right\}. \tag{2.1}$$

We need the following lemma in the sequel:

LEMMA 2.2. *For $M > 0$ and $\varepsilon > 0$*

$$P[\chi_n^2 \geqslant nM] \leqslant \exp\{-n(M - \log 4)/4\} \tag{2.2}$$

$$P[\chi_n^2 \leqslant n\varepsilon] \leqslant C \exp\left\{-\frac{n}{2}\log(1/e\varepsilon)\right\} \tag{2.3}$$

where χ^2 is distributed as chi-square with n degrees of freedom.

LEMMA 2.3. *If $A \geqslant 0$ is a $p \times p$ matrix, and $\bar{\lambda}$ is the largest eigenvalue of A, then*

$$|x'Ax - y'Ay| \leqslant \bar{\lambda}\,|x-y|\,|x+y| \tag{2.4}$$

for any vectors x and y of order $p \times 1$ where $|x|$ denotes Eucledian norm.

Throughout this paper, the transpose of a matrix B is denoted by B'.

3. BOUNDS ON THE EXTREME EIGENVALUES OF THE WISHART MATRIX

Let $\{Y_{ij};\ i, j = 1, 2, ...\}$ be a double sequence of independent random variables which are distributed normally with mean zero and variance one. Also, let Y_n denote the $p \times n$ matrix $(Y_{ij}$ $(i = 1, 2, ..., p, j = 1, 2, ..., n)$. Here $p = p(n)$ and the ratio p/n has a finite limit $y > 0$ as n tends to infinity. Next, let $\bar{\lambda}_n$ and $\underline{\lambda}_n$ respectively denote the largest and smallest eigenvalues of $A_n = Y_n Y_n'/n$. We now establish the following bound on the distribution function of $\bar{\lambda}_n$.

THEOREM 3.1. *There are positive constants C, D and M_0 such that*

$$P\{\bar{\lambda}_n \geqslant M\} \leqslant C \exp\{-DMn\} \tag{3.1}$$

when $M > M_0$ and $n = 1, 2, ...$

Proof. Let $M_1 > 0$ be such that $\sqrt{2\pi e}\exp(-M/64) < 1/12$ when $M \geqslant M_1$. Also, let $M_0 = \max(M_1, 64 \log 2)$. Now, let $M > M_0$ and $r = \sqrt{2\pi e}\exp(-M/64)$. Then $0 < r < 1/2$ and $r(r+2) < 1/4$. We cover the unit ball $B_p(0, 1)$ in R^p with the origin as center by balls of radius r. Suppose, these smaller balls are $B_p(x_1, r), ..., B_p(x_q, r)$ with centers $x_1, ..., x_q$, respectively. By Lemma 2.1, we can choose the covering in such a way that

$$q \leqslant C \exp\left\{\frac{p}{2}\log(2\pi e/r^2)\right\}.$$

Also,

$$P[\bar{\lambda}_n \geqslant M] = P[\max_{z \in B_p(0,1)} z'A_n z \geqslant M]$$

$$\leqslant \sum_{k=1}^{m} P\{\max_{z \in B_p(x_k, r) \cap B_p(0,1)} z'A_n z \geqslant M, \bar{\lambda}_n \leqslant 2M\} + P\{\bar{\lambda}_n \geqslant 2M\}.$$

But, if $z \in B_p(x_k, r) \cap B_p(0, 1)$, $z'A_n z \geqslant M$, $\bar{\lambda}_n \leqslant 2M$, then

$$\begin{aligned} x_k' A_n x_k &\geqslant z'A_n z - |z'A_n z - x_k' A_n x_k| \\ &\geqslant M - 2M|z - x_k|\,|z + x_k| \\ &\geqslant M - 2Mr(1 + 1 + r) \geqslant \frac{M}{2} \end{aligned}$$

by using (2.3). So,

$$P(\bar{\lambda}_n \geqslant M) \leqslant \sum_{k=1}^{q} P\left\{x_k' A_n x_k \geqslant \frac{M}{2}\right\} + P(\bar{\lambda}_n \geqslant 2M). \tag{3.2}$$

By a well-known property of the Wishart matrix, we observe that

$$P(x_k' A_n x_k \geqslant M/2) = P\left\{\chi_n^2 \geqslant \frac{nM}{2x_k' x_k}\right\} \leqslant P\left\{\chi_n^2 \geqslant \frac{nM}{2(1+r)^2}\right\}. \tag{3.3}$$

Applying Lemma 2.2, we have

$$P\left(x_k' A_n x_k \geqslant \frac{M}{2}\right) \leqslant \exp\left(-\frac{n}{4}\left(\frac{M}{8} - \log 4\right)\right). \tag{3.4}$$

Thus,

$$\begin{aligned} P(\bar{\lambda}_n \geqslant M) - P(\bar{\lambda}_n \geqslant 2M) &\leqslant q \exp\left\{-\frac{n}{4}\left(\frac{M}{8} - \log 4\right)\right\} \\ &\leqslant C \exp\left\{\frac{p}{2} \log \frac{2\pi e}{r^2} - \frac{n}{4}\left(\frac{M}{8} - \log 4\right)\right\} \\ &\leqslant C \exp\left\{\frac{n}{2}\left[y \log \frac{2\pi e}{r^2} - \frac{M}{16} + \log 2\right]\right\} \\ &\leqslant C \exp\left\{\frac{n}{2}\left[\frac{My}{32} - \frac{M}{16} + \log 2\right]\right\} \\ &\leqslant C \exp\left\{\frac{n}{2}\left[-\frac{M}{32} + \log 2\right]\right\} \\ &\leqslant C \exp\left\{\frac{n}{2}\left\{-\frac{M}{64}\right\}\right\}, \end{aligned} \tag{3.5}$$

by using (3.2), (3.4) and Lemma 2.1.

Replace M by $2^k M$, and add these inequalities together to get the result required. We now prove a bound on the distribution function of $\underline{\lambda}_n$.

THEOREM 3.2. *Let $y < \frac{1}{2}$. Then*

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant CD^n \varepsilon^{\alpha n}, \qquad 0 < \varepsilon \leqslant \varepsilon_0, \tag{3.6}$$

where C, D and α are positive constants.

Proof. Let $1 < \beta < 1/2y$, $\alpha = 1/2y - \beta$, $\gamma = \varepsilon^\beta/3$ and $K = \varepsilon^{1-\beta}$. Also, let $B_p(x_1, r),..., B_p(x_q, r)$ be p-balls with radius r and centers $x_1,..., x_q$ respectively, which cover the unit p-sphere $S_p(0, 1)$. In addition, let q satisfy the inequality

$$q \leqslant C \exp\left\{\frac{p}{2} \log \frac{2\pi e}{r^2}\right\}. \tag{3.7}$$

Such balls exist by Lemma 2.1. We have

$$P\{\underline{\lambda}_n \leqslant \varepsilon\} \leqslant \sum_{k=1}^{q} P\{\min_{z \in B_p(x_k, r) \cap S_p(0,1)} z' A_n z \leqslant \varepsilon, \bar{\lambda}_n < K\} + P\{\bar{\lambda}_n \geqslant K\}.$$

If $z \in B_p(x_k, r) \cap S_p(0, 1)$, $z' A_n z \leqslant \varepsilon$, $\bar{\lambda}_n < K$, then

$$\begin{aligned} x_k' A_n x_k &\leqslant z' A_n z + |x_k' A_n x_k - z' A_n z| \\ &\leqslant \varepsilon + \bar{\lambda}_n |x_k - z|\,|x_k + z| \\ &\leqslant \varepsilon + K \cdot r(2 + r) \leqslant 2\varepsilon. \end{aligned}$$

Thus

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant \sum_{k=1}^{q} P\{x_k' A_n x_k \leqslant 2\varepsilon\} + P\{\bar{\lambda}_n \geqslant K\}.$$

But,

$$\begin{aligned} P\{x_k' A_n x_k \leqslant 2\varepsilon\} &= P\left\{\chi_n^2 \leqslant \frac{2n\varepsilon}{x_k' x_k}\right\} \leqslant P\left\{\chi_n^2 \leqslant \frac{2n\varepsilon}{(1-r)^2}\right\} \\ &\leqslant C \exp\left\{-\frac{n}{2} \log \frac{(1-r)^2}{2e\varepsilon}\right\}, \end{aligned}$$

by Lemma 2.2. Therefore

$$\begin{aligned} P(\underline{\lambda}_n \leqslant \varepsilon) &\leqslant C \exp\left\{\frac{p}{2}\log\frac{2\pi e}{r^2}\right\} \exp\left\{-\frac{n}{2}\log\frac{(1-r)^2}{2e\varepsilon}\right\} + P\{\bar{\lambda}_n \geqslant K\} \\ &\leqslant C\left(\frac{2\pi e}{r^2}\right)^{ny/2} \cdot \frac{\varepsilon^{n/2}(2e)^{n/2}}{(1-r)^n} + Ce^{-D_1 Kn} \\ &\leqslant CD_2^n \varepsilon^{-\beta yn + n/2} + Ce^{-n\varepsilon^{1-\beta}D_1} \\ &\leqslant CD_2^n \varepsilon^{\alpha n} + C\exp\{-n\varepsilon^{-\delta}D_1\}, \qquad \text{if} \quad 0 < \varepsilon \leqslant \varepsilon_0. \end{aligned}$$

Here $\alpha = \frac{1}{2} - \beta y > 0$, $\delta = \beta - 1 > 0$. But,

$$D_2^n \varepsilon^{\alpha n} \exp\{n\varepsilon^{-\delta} D_1\} \geqslant 1$$

as ε is small. So

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant CD^n \varepsilon^{\alpha n}$$

for $0 < \varepsilon \leqslant \varepsilon_0$, and for some constant ε_0.

Geman [1] showed that $\bar{\lambda}_n$ tends to $(1 + y^{1/2})^2$ a.s. where $\lim_{p,n \to \infty}(p/n) = y$ and $0 < y < \infty$.

4. Proof of the Main Theorem

We now apply the following theorem of Yin and Krishnaiah [5] to prove our main theorem.

Theorem 4.1 (Yin–Krishnaiah [5]). *Let $\{X_{ij};\, i = 1, 2,..., j = 1, 2,...\}$ and $X_m : p \times m$ be as defined in Section 3. Also, let $W_p = (1/m) X_m X_m^T$. In addition, let T_p be a symmetrie $p \times p$ matrix of random variables with spectral distribution $G_p(x)$. We assume that the following conditions are satisfied:*

(1) *$\{X_{ij}\}$ and T_p are independent for each p,*

(2) *$\lim(p/m) = y$ exists and finite,*

(3) *$\int x^k \, dG_p(x) \to H_k$ exists in $L^2(P)$, for $k = 1, 2,...,$ and $\sum H_{2k}^{-1/2k} = +\infty$,*

Then the spectral distribution of $W_p T_p$, $F_p(x)$, tends to a limit $F(x)$ (nonrandom) in probability for each x.

We will verify that the conditions of the above theorem in Yin and Krishnaiah [5] are satisfied for our case.

In this section, we prove the following main theorem of our paper:

THEOREM 4.2. *Suppose* $\{X_{ij}, Y_{kl}, i, j, k, l = 1, 2, \ldots\}$ *are* iid, $X_{11} \sim N(0, 1)$, $X_m = (X_{ij}, i = 1, \ldots, p; j = 1, \ldots, m)$, $Y_n = (Y_{kl}, k = 1, \ldots, p; l = 1, \ldots, n)$. *Then, the spectral distribution of* $((1/m) X_m X_m')((1/n) Y_n Y_n')^{-1}$ *has a nonrandom limit distribution (in probability) as* $p \to \infty$, *if* $p/m \to y'$, $p/n \to y < \frac{1}{2}$ *exist.*

THEOREM 4.3. $(1/p) \operatorname{tr} T_p^k \to H_k$ *in* $L^2(P)$, *as* $p \to \infty$, *and*

$$\sum H_{2k}^{-1/2k} = \infty.$$

Proof. Let $F_n(x)$ be the spectral distribution of the matrix $(1/n) Y_n Y_n' = T_p^{-1}$. By Jonsson's theorem [2, Theorem 2.1], $F_n(x) \to F_y(x)$ where $F_y(x)$ is a distribution function with density function

$$f_y(x) = \frac{\sqrt{(x-a)(b-x)}}{2\pi x y}, \qquad a < x < b,$$
$$= 0, \qquad \text{otherwise},$$

with $a = (1 - \sqrt{y})^2$, $b = (1 + \sqrt{y})^2$. Since

$$\frac{1}{p} \operatorname{tr} T_p^k = \int_0^\infty x^{-k} \, dF_n(x),$$

it is sufficient to prove

$$\int_0^\infty x^{-k} \, dF_n(x) \to \int_0^\infty x^{-k} \, dF_y(x) \qquad \text{in} \quad L^2(P),$$

and

$$\sum_k \left\{ \int_0^\infty x^{-2k} \, dF_y(x) \right\}^{-1/2k} = +\infty.$$

The latter requirement is easy to see. We have

$$\int_0^\infty x^{-2k} \, dF_y(x) = \int_a^b x^{-2k} \, dF_y(x) \leqslant a^{-2k}.$$

Therefore

$$\sum \left\{ \int_0^\infty x^{-2k} \, dF_y(x) \right\}^{-1/2k} \geqslant \sum_{k=1}^\infty a = +\infty.$$

For the first requirement, by Minkowski's inequality, we have

$$E^{1/2}\left|\int_0^\infty x^{-k}\,dF_n(x)-\int_0^\infty x^{-k}\,dF_y(x)\right|^2$$
$$\leqslant E^{1/2}\left|\int_0^\varepsilon x^{-k}\,dF_n(x)\right|^2+E^{1/2}\left|\int_\varepsilon^K x^{-k}\,d[F_n(x)-F_y(x)]\right|^2$$
$$+E^{1/2}\left|\int_K^\infty x^{-k}\,dF_n(x)\right|^2=I_1^{1/2}+I_2^{1/2}+I_3^{1/2},$$

if $0<\varepsilon<a<b<K$. Now we consider I_1:

$$I_1=E\left[\left|\int_0^\varepsilon x^{-k}\,dF_n(x)\right|^2;\underline{\lambda}_n\leqslant\varepsilon\right]\leqslant E[\underline{\lambda}_n^{-2k};\underline{\lambda}_n\leqslant\varepsilon]$$
$$=\int_0^\varepsilon x^{-2k}\,dP(\underline{\lambda}_n\leqslant x).$$

Integrating by parts, and using Theorem 2.2, we see that

$$I_1\leqslant\int_0^\varepsilon x^{-2k}\,dP(\underline{\lambda}_n\leqslant x)\leqslant\varepsilon^{-2k}P(\underline{\lambda}_n\leqslant\varepsilon)+2kCD^n\frac{\varepsilon^{n\alpha-2k}}{n\alpha-2k}$$
$$\leqslant CD^n\varepsilon^{n\alpha-2k}+kCD^n\frac{\varepsilon^{n\alpha-2k}}{n\alpha-2k}\to 0$$

as $n\to\infty$, if we choose $\varepsilon^\alpha<1/D$.

For I_2, we have

$$I_2^{1/2}\leqslant E^{1/2}\,|K^{-k}(F_n(K)-F_y(K))|^2$$
$$+E^{1/2}\,|\varepsilon^{-k}(F_n(\varepsilon)-F_y(\varepsilon))|^2$$
$$+E^{1/2}\left|k\int_\varepsilon^K x^{-k-1}(F_n(x)-F_y(x))\,dx\right|^2$$
$$=J_1+J_2+J_3.$$

By Jonsson's theorem, $F_n(c)-F_y(c)\to 0$ in prob. for any c, and thus J_1, J_2 both tend to 0. Also,

$$J_3^2\leqslant k^2\int_\varepsilon^K x^{-2k-2}\,dxE\int_\varepsilon^K (F_n(x)-F_y(x))^2\,dx$$
$$=k\varepsilon^{-2k-2}(K-\varepsilon)\cdot\int_0^K E(F_n(x)-F_y(x))^2\,dx$$
$$\to 0\qquad\text{as}\quad n\to\infty.$$

Thus, Theorem 4.3 is proved. Therefore the main theorem is proved.

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