A simple proof for the exponential upper bound for some tenacious patterns

Miklós Bóna

Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, USA

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Abstract

We present a method that provides very simple proofs for some instances of the Stanley–Wilf conjecture. Some of these instances had been proved before by much more complicated arguments, and some others are new.

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1. Introduction

Let \( S_n(q) \) be the number of permutations of length \( n \) (or, in what follows, \( n \)-permutations) that avoid the pattern \( q \). The long-standing Stanley–Wilf conjecture claims that for any given pattern \( q \), there exist an absolute constant \( c_q \) so that \( S_n(q) < c_q^n \) for all \( n \). See [2] or [4] for the relevant definitions.

The Stanley–Wilf conjecture has been open for more than 20 years now. In 2000, Alon and Friedgut [1] proved a slightly weaker upper bound for \( S_n(q) \). As far as proving the exponential upper bound itself, one of the most general results is [3], when the existence of an exponential upper bound is proved for all layered patterns. A pattern is called layered if it is the union of decreasing subsequences so that the entries increase among the subsequences, such as in 1 5432 76 8. The Stanley–Wilf conjecture is also known to be true for monotonic patterns [6], patterns of length four [4,5], unimodal patterns [1], and some other sporadic cases [5].

E-mail address: bona@math.ufl.edu.

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In this paper we present a method that makes it very easy to handle some patterns that were difficult to handle so far. The first one is the pattern 1324, the pattern that proved to be the most difficult pattern of length four. The second is the pattern $1k-1k-2\ldots2k$, that is, a layered pattern of three layers, of lengths $1, k-2$, and 1, respectively. This pattern is important because the fact that the Stanley–Wilf conjecture is true for this pattern is at the heart of the proof that the Stanley–Wilf conjecture is true for all layered patterns [3]. The proof used in [3] is very complicated, so the argument of the present paper simplifies the proof of the Stanley–Wilf conjecture for all layered patterns as well.

Finally, we give an even larger class of patterns for which our method provides an exponential upper bound.

It is not our goal to provide small constants $c_q$ in this paper; our goal is to show simple proofs for the fact that some constants $c_q$ satisfying $S_n(q) < c_q^n$ for all $n$ exist.

2. The pattern 1324

Our crucial definition is the following.

**Definition 2.1.** We will say that an $n$-permutation $p = p_1p_2\ldots p_n$ is *orderly* if $p_1 < p_n$. We will say that $p$ is *dual orderly* if the entry 1 of $p$ precedes the maximal entry $n$ of $p$.

It is clear that $p$ is orderly if and only if $p^{-1}$ is dual orderly.

The importance of these permutations for us is explained by the following lemma.

**Lemma 2.2.** The number of orderly (respectively dual orderly) 1324-avoiding $n$-permutations is less than $8^n/4(n+1)$.

**Proof.** It suffices to prove the statement for orderly permutations as we can take inverses after that to get the other statement.

The crucial idea is this. Each entry $p_i$ of $p$ has at least one of the following two properties:

(a) $p_i \geq p_1$;
(b) $p_i \leq p_n$.

In words, everything is either larger than the first entry, or smaller than the last, possibly both. This would not be the case had we not required that $p$ be orderly.

Define $S = \{ i \mid p_i \geq p_1 \}$ and $T = \{ i \mid p_i < p_1 \}$. Then $S$ and $T$ are disjoint, $S \cup T = [n]$, and crucially, if $i \in T$, then, in particular, $p_i < p_n$. Recall that for any pattern $q$ of length three, we have $S_n(q) = C_n = \binom{2n}{n}/(n+1)$, and that the numbers $C_n$ are the well known Catalan numbers [2]. Let $|S| = s$ and $|T| = t$. Then we have $C_{s-1}$ possibilities for the substring $p_S$ of entries belonging to indices in $S$, and $C_t = C_{n-s}$ possibilities for the substring $p_T$ of entries belonging to indices in $S$. Indeed, $p_S$ starts with its smallest entry, and then the rest of it must avoid 213 (otherwise, together with $p_1$, a 1324-pattern is formed) and $p_T$ must avoid 132 (otherwise, together with $p_n$, a 1324-pattern is formed).
Finally, we have \( \binom{n-2}{s-2} \) choices for the set of indices that we denoted by \( S \). Once \( s \) is known, we have no liberty in choosing the entries \( p_i \ (i \in S) \) as they must simply be the \( s \) largest entries.

Therefore, the total number of possibilities is

\[
\sum_{s=2}^{n} \binom{n-2}{s-2} C_{s-1} C_{n-s} < 2^{n-2} \sum_{s=2}^{n} C_{s-1} C_{n-s} < 2^{n-2} C_n < \frac{8^n}{4(n+1)}.
\]

We have seen that it helps in our efforts to limit the number of 1324-avoiding permutations if a large element is preceded by a small one. To make good use of this observation, look at all non-inversions of a generic permutation \( p = p_1 p_2 \cdots p_n \); that is, pairs \((i, j)\) so that \( i < j \) and \( p_i < p_j \). Find the non-inversion \((i, j)\) for which

\[
\max_{(i, j)} (j - i, p_j - p_i)
\]

is maximal. If there are several such pairs, take one of them, say the one that is lexicographically first. Call this pair \((i, j)\) the critical pair of \( p \).

Recall that an entry of a permutation is called a left-to-right minimum if it is smaller than all entries on its left. Similarly, an entry is a right-to-left maximum if it is larger than all entries on its right.

The following proposition is obvious, but it will be important in what follows, so we explicitly state it.

**Proposition 2.3.** For any permutation \( p_1 p_2 \cdots p_n \), the critical pair \((i, j)\) is always a pair in which \( p_i \) is a left-to-right minimum, and \( p_j \) is a right-to-left maximum.

The following definition proved to be useful for treating 1324-avoiding permutations in the past.

**Definition 2.4.** We say that two permutations are in the same class if they have the same left-to-right minima, and the same right-to-left maxima, and they are in the same positions.

**Example 2.5.** The permutations 3612745 and 3416725 are in the same class.

**Proposition 2.6.** The number of nonempty classes of \( n \)-permutations is less than \( 9^n \).

**Proof.** Each such class contains exactly one 1234-avoiding permutation, namely the one in which all entries that are not left-to-right minima or right-to-left maxima are written in decreasing order. As it is well known that \( S_n(1234) < 9^n \), the statement is proved. \( \square \)

To achieve our goal, it suffices to prove that each class contains at most an exponential number of 1324-avoiding \( n \)-permutations.

Choose a class \( A \). By Proposition 2.3, we see that the critical pair of any permutation \( p \in A \) is the same as it depends only on the left-to-right minima and the right-to-left maxima, and those are the same for all permutations in \( A \).
We will now find an upper bound for the number of $1324$-avoiding $n$-permutations in $A$.

For symmetry reasons, we can assume that in the critical pair of $p \in A$, we have $j - i \geq p_j - p_i$; in other words, the maximum (1) is attained by $j - i$.

We will now reconstruct $p$ from its critical pair. First, all entries that precede $p_i$ must be larger than $p_j$. Indeed, if there existed $k < i$ so that $p_k < p_j$, then the pair $(j, k)$ would be a “longer” non-inversion than the pair $(i, j)$, contradicting the critical property of $(i, j)$. Similarly, all entries that are on the right of $p_j$ must be smaller than $p_i$.

This shows that all entries $p_i$ for which $p_i < p_j < p_j$ must be positioned between $p_i$ and $p_j$, that is, $i < t < j$ must hold for them. However, if $j - i = p_j - p_i + b$, where $b$ is a positive integer, then we can select $b$ additional entries that will be located between $p_i$ and $p_j$. We will call them excess entries; that is, an excess entry is an entry $p_u$ that is located between $p_i$ and $p_j$, but does not satisfy $p_i < p_u < p_j$.

The good news is that we do not have too many choices for the excess entries. No excess entry can be smaller than $p_i - b$. Indeed, if we had $p_u < p_i - b$ for an excess entry, then for the pair $(u, j)$ the value defined by (1) would be larger than for the pair $(i, j)$, contradicting the critical property of $(i, j)$. By the analogous argument, no excess entry can be larger than $p_j + b$. Therefore, the set of $b$ excess entries must be a subset of the at-most-$(2b)$-element set $\{(p_i - b, p_i - b + 1, \ldots, p_i - 1) \cup (p_j + 1, p_j + 2, \ldots, p_j + b)\} \cap [n]$. Therefore, we have at most $\binom{2b}{b}$ choices for the set of excess entries, and consequently, we have $\binom{2b}{b}$ choices for the set of $j - i + b$ elements that are located between $p_i$ and $p_j$. As $p_i < p_j$, the partial permutation $p_i p_{i+1} \cdots p_j$ is orderly, and certainly $1324$-avoiding. Therefore, by Lemma 2.2, we have less than $8^{j-i+1}/4(j - i + 1)$ choices for it once the set of entries has been chosen.

This proves that altogether, we have less than

$$4^b \cdot \frac{8^{j-i+1}}{4(j - i + 1)} < 32^{j-i}$$

possibilities for the string $p_i p_{i+1} \cdots p_j$. We used the fact that $b \leq j - i - 1$ as $b$ counts the excess entries between $i$ and $j$. Note that we have some room to spare here, so we can say that the above upper bound remains valid even if we include the permutations in which the maximum was attained by $(p_i, p_j)$, and not by $(i, j)$.

We can now remove the entries $p_{i+1} \cdots p_{j-1}$ from our permutations. This will split our permutations into two parts, $p_L$ on the left, and $p_R$ on the right. It is possible that one of them is empty. We know exactly what entries belong to $p_L$ and what entries belong to $p_R$; indeed each entry of $p_L$ is larger than each entry of $p_R$. Therefore, we do not lose any information if we relabel the entries in each of $p_L$ and $p_R$ so that they both start at 1 (we call this the standardization of the strings). This will not change the location and relative value of the left-to-right minima and right-to-left maxima either. The string $p_{i+1} \cdots p_{j-1}$ should not be standardized, however, as that would result in loss of information.

See Fig. 1 for the diagram of a generic permutation, its critical pair, and the strings $p_L$ and $p_R$.

Then we iterate our procedure. That is, we find the critical pairs of $p_L$ and $p_R$, denote them by $(i_L, j_L)$ and $(i_R, j_R)$, and prove, just as above, that there are at most $32^{j_L-i_L}$ possibilities for the string between $i_L$ and $j_L$, and there are at most $32^{j_R-i_R}$ possibilities for
the string between \( i_R \) and \( j_R \). Then we remove these strings again, cutting our permutations into more parts, and so on, building a binary tree-like structure of strings. Note that the leaves of this tree will be orderly or dual orderly permutations.

Note that this procedure of decomposing of our permutations is injective. Indeed, given the standardized string \( p_L \), the partial permutation \( p_i \cdots p_j \), and the standardized string \( p_R \), we can easily recover \( p \).

Iterating this algorithm until all entries of \( p \) that are not left-to-right minima or right-to-left maxima are removed, we prove the following.

**Lemma 2.7.** The number of 1324-avoiding \( n \)-permutations in any given class \( A \) is at most \( 32^n \).

**Proof.** The above description of the removal of entries by our method shows that the total number of 1324-avoiding permutations in \( A \) is less than

\[
32 \sum_{k} j_k - i_k
\]

where the summation ranges through all intervals \((i_k, j_k)\) whose endpoints were critical pairs at some point. As these interiors of these intervals are all disjoint, \( \sum_{k} j_k - i_k = n - 1 \), and our claim is proved. \( \Box \)

Now proving the upper bound for \( S_n(1324) \) is a breeze.

**Theorem 2.8.** There exists an absolute constant so that for all \( n \), we have \( S_n(1324) < c^n \).

**Proof.** As there are less than \( 9^n \) classes and less than \( 32^n \) \( n \)-permutations in each class that avoid 1324, \( c = 9 \cdot 32 = 288 \) will do. \( \Box \)

Note that an alternative way of proving our theorem would have been by induction on \( n \).

We could have used the induction hypothesis for the class \( A' \) that is obtained from \( A \) by making \( p_i \) and \( p_j \) consecutive entries by omitting all positions between them, and setting their values so that each entry on the left of \( p_i \) is larger than each entry after \( p_j \).
3. The pattern $1k−1k−2⋯2k$

As a generalization, we look at patterns like $14325$, $154326$, and so on, that is, patterns that start with 1, end with their maximal entry $k$, and consist of a decreasing sequence all the way between.

The fact that the Stanley–Wilf conjecture is true for these patterns is the quintessential part of the proof of the claim that the Stanley–Wilf conjecture is true for all layered patterns [3]. That proof was very complicated. So our much simpler proof also simplifies the proof of the conjecture for all layered patterns quite significantly.

**Theorem 3.1.** Let $k \geq 4$, and let $q_k = 1k−1k−2⋯2k$. Then there exists a positive constant $c_k$ so that

$$S_n(q_k) < c^n_k.$$

**Proof.** We again look at orderly permutations first. If $p$ is orderly and avoids $q_k$, then define $S_p$, $p_S$ and $T_p$, $p_T$ just as in proof of Lemma 2.2. Then $p_S$ starts with its smallest entry, and the rest must avoid $q'_k = k−2⋯21k−1$, whereas $p_T$ must avoid $q''_k = 1k−1k−2⋯2$. It is known that $S_n(q'_k) = S_n(q''_k) = S_n(12⋯(k − 1)) < (k − 2)^{2n}$, so it follows, just as in Lemma 2.2 that the number of orderly (respectively dual orderly) $n$-permutations that avoid $q_k$ is less than $(2(k − 2))^{2n}$.

The transition from orderly permutations to generic permutations is identical to what we described in the case of $q_4 = 1324$.

4. A further generalization

We can find a somewhat more general application of our methodology. For a pattern $q$, let $1q$ denote the pattern obtained from $q$ by adding one to each of the entries and then writing 1 to the front, and let $qm$ denote the pattern that we obtain from $q$ by simply affixing a new maximal element to the end of $q$. Finally, let $1qm$ denote the pattern $(1qm) = (1q)m = (qm)$. So for example, if $q = 2413$, then $1q = 1324$, and $qm = 24135$, while $1qm = 135246$.

**Theorem 4.1.** Let $q$ be a pattern so that there exist constants $c_1$ and $c_2$ satisfying $S_n(1q) < c_1^n$ and $S_n(qm) < c_2^n$ for all $n$. Then there exists a constant $c$ so that

$$S_n(1qm) < c^n.$$

**Proof.** Analogous to the proof of Theorem 3.1.

This theorem allows us to prove the Stanley–Wilf conjecture for certain patterns for which it has not been proved yet.

**Example 4.2.** Let $q = 231$. Then it is known [4] that $S_n(1q) = S_n(1324) < 8^n$. On the other hand, $qm = 2314$, so $S_n(qm) = S_n(2314) = S_n(1423) = S_n(1324) < 8^n$, taking reverse
complements, then inverses, and finally applying the result of [4] again. Therefore, there exists a constant $c$ so that $S_n(1qm) = S_n(13425) < c^n$.

Note added in proof

While this paper was in print, the Stanley–Wilf conjecture was solved for all patterns by G. Tardos and A. Marcus. However, our theorems provide sharper bounds in the specific cases that we treated in this paper.

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References