WORD PROBLEMS AND A HOMOLOGICAL FINITENESS CONDITION FOR MONOIDS

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Communicated by F.W. Lawvere
Received 7 April 1986
Revised 28 April 1986

Introduction

Our purpose is to prove that a monoid which has a 'nice' solution to its word problem satisfies a certain homological finiteness condition. More precisely, we prove: if a monoid $S$ has a finite terminating Church–Rosser presentation, then $S$ is $(FP)_3$; this is Theorem 4.1 below. (See Section 2 for the definition of "terminating" and "Church–Rosser".) Examples of groups that are not $(FP)_3$ are known; see Section 4 for a brief description of several of these. For completeness, we provide an example of a monoid that is not $(FP)_3$. In each case, the monoid (or group) is finitely-presented and has a solvable word problem. These examples answer (in the negative) the following question of Jantzen [15]: does a finitely-presented monoid with a solvable word problem have a finite terminating Church–Rosser presentation?

The Church–Rosser property was discovered by Church and Rosser [9] during the course of research on the $\lambda$-calculus. Properties of terminating relations were investigated by Newman [16]. For a systematic treatment of both topics together with further references, see [14]. Monoids with terminating Church–Rosser presentations have been studied by Nivat [17] and others. See [5] for a recent survey.

We conclude this introduction with a brief outline of what follows and some further discussion.

Section 1 contains basic results on Noetherian relations. In particular, we develop some tools for dealing with free abelian groups which have a basis ordered by a Noetherian relation.

Section 2 introduces terminating and Church–Rosser presentations. (Because of difficulties in verifying that the relation $\rightarrow$ defined in Section 2 is Noetherian, it is common to assume that the rewriting rules $R$ are length-reducing: if $(r, s) \in R$, then $|r| > |s|$. We specifically do not make this assumption, so that our terminology differs, for example, from that of [5].) Variations of Theorem 2.1, which gives
equivalent conditions for a terminating presentation to be Church–Rosser, are well known; see [14]. Condition 2.1(c) gives a simple proof of Theorem 2.4, which shows that a terminating Church–Rosser presentation can be assumed to have a particular form. A version of this theorem was communicated to the author by Friederich Otto. Theorem 2.4 plays an important role in Section 3.

Section 3 contains our main results. After reviewing how a presentation of a monoid $S$ yields a resolution through dimension 2 of $\mathbb{Z}$ as a trivial left $\mathbb{Z}S$-module, we show (Theorem 3.1) how to extend this resolution through dimension 3, in the situation when $S$ has a terminating Church–Rosser presentation. We also give a criterion (Theorem 3.2) for such an $S$ to be 3-dimensional.

In Section 4, after defining the $(FP)_k$-condition, we reinterpret Theorem 3.1 in the situation when $S$ has a finite terminating Church–Rosser presentation and conclude with some examples.

We have made an effort to make this paper self-contained. Nonetheless, Sections 3 and 4 will be difficult for the reader with no background in homological algebra. We suggest [12] as a good introductory text; [8] also contains some material of relevance in homological monoid theory. Much of the material in Section 3 leading up to Theorem 3.1 is well known in group theory: see [6, p. 45, exercise 3 or p. 90, exercise 4].

Notation: we use $\lambda$ to denote the empty word in a free monoid.

1. Noetherian induction

Let $X$ be a set and let $\rightarrow$ be a relation on $X$. The relation $\rightarrow$ is called Noetherian provided there does not exist an infinite sequence $\{x_n \mid n \geq 0\}$ of elements of $X$ such that each $x_n \rightarrow x_{n+1}$. We shall need

**Proposition 1.1** (Principle of Noetherian induction). Let $X$ be a set, let $\rightarrow$ be a Noetherian relation on $X$ and let $P$ be a predicate on $X$. Suppose that whenever $x \in X$ has the property that every $y \in X$ with $x \rightarrow y$ satisfies $P$, it follows that $x$ satisfies $P$. Then every $x \in X$ satisfies $P$. □

For a proof, see [10], and, for applications, see [14].

An element $z$ of $X$ is called $\rightarrow$-irreducible provided for every $x \in X$, $z \rightarrow x$ is false. We remark that the hypothesis of Noetherian induction will often have to be verified separately for irreducibles. (Also, $P(x)$ for a reducible $x$ will often follow from $P(y)$ for a single $y$ satisfying $x \rightarrow y$.)

We let $\Rightarrow$ denote the reflexive transitive closure of $\rightarrow$ and let $\Rightarrow^+$ denote the transitive closure of $\rightarrow$. Note that if $\rightarrow$ is Noetherian, then $\Rightarrow$ is a partial order: if $x \Rightarrow y$ and $y \Rightarrow x$, then $x = y$. Thus, if $\rightarrow$ is Noetherian, then every finite subset $A$ of $X$ has an $\Rightarrow$-maximal element: there exists $y \in A$ such that if $x \in A$ satisfies
x \rightarrow y$, then $x = y$. If $x \in X$, we let $\Delta^+(x) = \{ y \in X | x \rightarrow y \}$. Note that $x$ is $\rightarrow$-irreducible if and only if $\Delta^+(x) = \emptyset$.

Let $F(X)$ denote the collection of finite subsets of $X$. We use the relation $\rightarrow$ on $X$ to define a similar relation (also denoted) $\rightarrow$ on $F(X)$.

**Definition 1.2.** Let $A, B$ be finite subsets of $X$. Then $A \rightarrow B$ means: there exists $x \in A$ and a finite subset $D$ of $\Delta^+(x)$ such that $B = (A \cup D) - \{x\}$.

Using the $\rightarrow$ and $\leftarrow$ notation as above, note that if $A \rightarrow B$, then for each $y \in B$, there exists $x \in A$ such that $x \leftarrow y$. Also note that the only $A \in F(X)$ that is $\rightarrow$-irreducible on $F(X)$ is the empty set.

**Lemma 1.3.** If $\rightarrow$ is Noetherian on $X$, then $\rightarrow$ is Noetherian on $F(X)$.

**Proof.** By way of contradiction, we show that the existence of an infinite $\rightarrow$-chain $A_0 \rightarrow A_1 \rightarrow \cdots$ on $F(X)$ implies the existence of an infinite $\rightarrow$-chain on $X$.

Given $A_0 \rightarrow A_1 \rightarrow \cdots$ as above, define a directed graph $\Gamma$ as follows: a vertex of $\Gamma$ is an ordered pair $(x, n)$ where $n$ is a non-negative integer and $x \in A_n$. There is a directed edge from $(x, n)$ to $(y, m)$ provided $m = n + 1$ and either $y = x$ or $y \notin A_n$ and $y \in \Delta^+(x)$. Clearly, if $(y, m)$ is a vertex of $\Gamma$ with $m > 0$, then there exists a unique vertex $(x, n)$ of $\Gamma$ such that there exists a directed edge from $(x, n)$ to $(y, m)$. (In particular, $n = m - 1$.) It follows that $\Gamma$ is a disjoint union of directed trees, one for each element of $A_0$. Each such tree satisfies the following condition: if there is an edge from $(x, n)$ to $(x, n + 1)$, then there is no edge from $(x, n)$ to any other vertex.

If $\rightarrow$ is Noetherian on $X$, it follows from the principle of Noetherian induction that each such tree has only finitely many edges from a vertex $(x, n)$ to any vertex $(y, m)$ with $y \neq x$. (Note that each vertex is involved in only finitely many edges.) The lemma follows easily. □

Let $G(X)$ denote the free abelian group with basis $X$. If $W \in G(X)$, then the *support* of $W$ consists of those elements of $X$ which have non-zero coefficient in the unique expression for $W$ as a linear combination of elements of $X$. Clearly, each support is a finite subset of $X$.

**Theorem 1.4.** Let $X$ be a set, let $\rightarrow$ be a Noetherian relation on $X$, let $Y$ be a subset of $X$ and let $H$ be a subgroup of $G(X)$. Suppose that for each $y \in Y$, $H$ contains an element of the form $y - W_y$ where the support of $W_y$ is a subset of $\Delta^+(y)$. Then for each $W \in G(X)$, there exists $W' \in H$ such that the support of $W - W'$ is disjoint from $Y$.

**Proof.** By the lemma, $\rightarrow$ is Noetherian on $F(X)$. We prove the following by Noetherian induction on $A \in F(X)$: if the support of $W$ is a subset of $A$, then a
suitable \(W'\) exists. If the support of \(W\) is disjoint from \(Y\), take \(W' = 0\). If \(y \in A \cap Y\), write \(W = ny + W_1\) where \(y\) is not in the support of \(W_1\). By hypothesis, choose \(y - W_y \in H\) where the support of \(W_y\) is a subset of \(\Delta^*(y)\). Let \(D\) denote the support of \(W_y\). Clearly, the support of \(W - n(y - W_y)\) is a subset of \(B = (A \cup D) - \{y\}\). Since \(A \rightarrow B\), by the inductive hypothesis, there exists \(W'' \in H\) such that the support of \(W - n(y - W_y) - W''\) is disjoint from \(Y\). Clearly \(W' = n(y - W_y) + W'' \in H\) and the support of \(W - W'\) is disjoint from \(Y\).

We note a simple consequence of Theorem 1.4. If \(K\) is an abelian group and \(f : G(X) \rightarrow K\) is a homomorphism such that \(H \subseteq \ker f\) and the restriction of \(f\) to the subgroup of \(G(X)\) generated by \(X - Y\) is injective, then \(H = \ker f\). (If \(f(W) = 0\), then choosing \(W'\) as in Theorem 1.4, \(f(W - W') = 0\). Since \(W - W'\) belongs to the subgroup of \(G(X)\) generated by \(X - Y\), \(W = W' \in H\).)

2. Presentations and the Church-Rosser property

Let \(\Sigma\) be a set. We let \(\Sigma^* \) denote the free monoid on \(\Sigma\); elements of \(\Sigma^*\) are finite sequences (called words) of elements of \(\Sigma\). The empty word will be denoted \(\lambda\). If \(w \in \Sigma^*\), the length of \(w\) will be denoted \(|w|\).

Let \(R \subseteq \Sigma^* \times \Sigma^*\). We write \(x \rightarrow y\) for \(x, y \in \Sigma^*\) to mean that there exist \(u, v \in \Sigma^*\) and \((r, s) \in R\) such that \(x = urv\) and \(y = usv\). We use the notation \(\Rightarrow\) and \(\rightarrow\) as in Section 1. In addition, we let \(\sim\) denote the equivalence relation generated by \(\rightarrow\); in other words, \(\sim\) is the reflexive symmetric transitive closure of \(\rightarrow\). It follows that \(\sim\) is a congruence on \(\Sigma^*\): if \(x, y \in \Sigma^*\) satisfy \(x \sim y\) and \(u, v \in \Sigma^*\), then \(uxv \sim uyv\). Therefore, the set of equivalence classes in \(\Sigma^*\) under \(\sim\) forms a monoid \(S\); the pair \((\Sigma, R)\) is called a presentation of \(S\). (When several subsets \(R \subseteq \Sigma^* \times \Sigma^*\) are under consideration, we will use notation such as "\(x \rightarrow y\) modulo \(R\)" to distinguish them.)

We call \(R\) terminating provided the relation \(\rightarrow\) on \(\Sigma^*\) is Noetherian. We use the term "irreducible" (relative to the relation \(\rightarrow\) on \(\Sigma^*\)) as in Section 1. Note that if \(R\) is terminating, then for each \(x \in \Sigma^*\) there exists an irreducible \(z \in \Sigma^*\) such that \(x \Rightarrow z\). (The proof is an easy application of Noetherian induction.)

We call \(R\) Church-Rosser provided whenever \(x, y \in \Sigma^*\) satisfy \(x \sim y\), it follows that there exists \(z \in \Sigma^*\) such that \(x \Rightarrow z\) and \(y \Rightarrow z\).

**Theorem 2.1.** Let \(R \subseteq \Sigma^* \times \Sigma^*\) be terminating. Then the following are equivalent:

(a) \(R\) is Church-Rosser;
(b) Let \((r_1, s_1), (r_2, s_2) \in R\). If \(r_1 = uv\) and \(r_2 = vw\) with \(v \neq \lambda\), then there exists \(z \in \Sigma^*\) such that \(s_1w \Rightarrow z\) and \(us_2w \Rightarrow z\). If \(r_1 = ur_2w\), then there exists \(z \in \Sigma^*\) such that \(s_1 \Rightarrow z\) and \(us_2w \Rightarrow z\);
(c) For each \(x \in \Sigma^*\) there exists a unique irreducible \(z \in \Sigma^*\) such that \(x \Rightarrow z\).
Proof. (a) implies (b). This is automatic, since \( s_1w \sim uS_2 \).

(b) implies (c). We use Noetherian induction. Existence of \( z \) has already been noted. If \( x \) is irreducible, uniqueness is easy. In general, suppose that \( x \rightarrow z_1 \) and \( x \rightarrow z_2 \) with \( z_1, z_2 \) irreducible. Write \( x \rightarrow z_i \) as \( x \rightarrow y_i \rightarrow z_i \). Either the relation applications involved in \( x \rightarrow y_1 \) and \( x \rightarrow y_2 \) are identical or are disjoint or (b) applies. In any case, there exist \( y \in \Sigma^* \) such that \( y_1 \rightarrow y \) and \( y_2 \rightarrow y \). Choose an irreducible \( z \in \Sigma^* \) such that \( y \rightarrow z \). Thus each \( y_i \rightarrow z \). Applying the inductive hypothesis twice, each \( z_i = z \). Thus \( z_1 = z_2 \) as required.

(c) implies (a). Note first that if (c) holds and \( u \sim v \), then \( u \) and \( v \) have the same irreducible; (a) follows by an easy induction on the length of a relation chain connecting \( x \) and \( y \) in the definition of the Church–Rosser property. \( \square \)

The equivalence of (a) and (b) is essentially [14, Lemma 2.4] and was originally due to Newman [16]. Pairs of elements of \( R \) as in (b) will play an important role in Section 3 below. The equivalence of (c) is also well known (see the discussion in [14]). The particular version of (c) above allows

Corollary 2.2. Let \( R \subseteq \Sigma^* \times \Sigma^* \) be terminating and Church–Rosser. Suppose that \( R' \subseteq \Sigma^* \times \Sigma^* \) is terminating, has the same irreducibles as \( R \) and satisfies: if \( x, z \in \Sigma^* \) with \( z \) irreducible, then \( x \rightarrow z \) modulo \( R' \) if and only if \( x \rightarrow z \) modulo \( R \). Then \( R' \) is Church–Rosser and if \( x, y \in \Sigma^* \), then \( x \sim y \) modulo \( R \) if and only if \( x \sim y \) modulo \( R' \).

Proof. That \( R' \) is Church–Rosser follows from condition (c) of Theorem 2.1. To prove the second conclusion, note the following consequence of Theorem 2.1: \( x \sim y \) if and only if there exists an irreducible \( z \) such that \( x \rightarrow z \) and \( y \rightarrow z \) (all relative to a terminating Church–Rosser subset of \( \Sigma^* \times \Sigma^* \)). \( \square \)

For convenience, we refer to subsets \( R, R' \subseteq \Sigma^* \) which satisfy the second conclusion of Corollary 2.2 as equivalent.

We use Corollary 2.2 to replace an arbitrary terminating Church–Rosser system with one in a particularly simple form. Before turning to this, given \( R \subseteq \Sigma^* \times \Sigma^* \), define \( R_1 \subseteq \Sigma^* \) to consist of all \( r \in \Sigma^* \) such that there exists \( s \in \Sigma^* \) such that \( (r, s) \in R \). Note that \( z \in \Sigma^* \) is \( R \)-irreducible if and only if \( z \not\in \Sigma^* \) \( R_1 \Sigma^* \).

Definition 2.3. \( R \subseteq \Sigma^* \) is reduced provided for each \( (r, s) \in R, R_1 \cap \Sigma^*r \Sigma^* = \{r\} \) and \( s \) is \( R \)-irreducible.

Theorem 2.4. Let \( R \subseteq \Sigma^* \times \Sigma^* \) be terminating and Church–Rosser. Then there exists a reduced \( R' \subseteq \Sigma^* \times \Sigma^* \) that is terminating, Church–Rosser and equivalent to \( R \).

Proof. Let \( R_1 \) consist of all \( r \in R_1 \) such that if \( urv \in R_1 \), then \( u = v = \lambda \). (In other words, \( R_1 \) consists of the minimal elements of \( R \) with respect to a suitable
subword ordering.) Let $R' = \{(r, s) \in R \mid r \in R'_1\}$. We show that $R'$ is Church–Rosser and equivalent to $R$ by showing that the hypotheses of Corollary 2.2 are satisfied.

Since $R' \subseteq R$ and $R$ is terminating, it follows that $R'$ is terminating.

Since $R'_1 \subseteq R_1$, it follows that if $z$ is $R$-irreducible, then $z$ is $R'$-irreducible.

For the converse, note that if $r \in R_1$, then these exist $r' \in R'_1$ and $u, v \in \Sigma^*$ such that $r = ur'v$. It follows that if $z$ is $R$-reducible, then $z$ is $R'$-reducible, as required.

Finally, since $R' \subseteq R$, if $x \rightarrow z$ modulo $R'$, then $x \rightarrow z$ modulo $R$. To complete this part of the proof, we prove: if $x \rightarrow z$ modulo $R$ and $z$ is irreducible, then $x \rightarrow z$ modulo $R'$. We proceed by Noetherian induction on $\rightarrow$ modulo $R$. If $x$ is irreducible, then $x = z$, so $x \rightarrow z$ modulo $R'$. If $x$ is reducible modulo $R$, then, as noted above, $x$ is reducible modulo $R'_1$, so there exists $y \in \Sigma^*$ such that $x \rightarrow y$ modulo both $R$ and $R'$. Since $R$ is terminating and Church–Rosser, $y \rightarrow z$ modulo $R$, by the uniqueness of $z$. By the inductive hypothesis, $y \rightarrow z$ modulo $R'$, so that $x \rightarrow z$ modulo $R'$, as required.

It follows from Corollary 2.2 that $R'$ is Church–Rosser and equivalent to $R$. Clearly, if $(r, s) \in R'_1$, then $R'_1 \cap \Sigma^*r\Sigma^* = \{r\}$, so $R'$ satisfies half of the definition of reduced. We modify $R'$ to obtain $R''$ which is reduced.

Define $R''_1$ to consist of all pairs $(r, s)$ where $(r, s) \in R$, $s \rightarrow z$ modulo $R'$ and $s$ is $R'$-irreducible. Proceeding as above, we show that $R''$ is Church–Rosser and equivalent to $R'$.

Note that if $x \rightarrow y$ modulo $R''$, then $x \rightarrow y$ modulo $R$. Since $R'$ is terminating, we conclude that $R''$ is terminating.

Note that $R''_1 = R'_1$. We conclude that $R''$ and $R'$ have the same irreducibles.

Finally, since $x \rightarrow y$ modulo $R''$ implies $x \rightarrow y$ modulo $R'$, we conclude that if $x \rightarrow z$ modulo $R''$, then $x \rightarrow z$ modulo $R'$. To complete this part of the proof, we prove: if $x \rightarrow z$ modulo $R'$ and $z$ is irreducible, then $x \rightarrow z$ modulo $R''$. We proceed by Noetherian induction on $\rightarrow$ modulo $R''$. If $x$ is irreducible, then $x = z$ as above. Otherwise, there exists $y \in \Sigma^*$ such that $x \rightarrow y$ modulo $R''$. Then $x \rightarrow y$ modulo $R'$. Since $R'$ is terminating and Church–Rosser, we conclude as above that $y \rightarrow z$ modulo $R'$ so, by the inductive hypothesis, $y \rightarrow z$ modulo $R''$ so, in turn, $x \rightarrow z$ modulo $R''$, as required.

Since $R''_1 = R'_1$, we conclude that if $(r, s) \in R''_1$, then $R''_1 \cap \Sigma^*r\Sigma^* = \{r\}$. By definition, if $(r, s) \in R''$, then $s$ is irreducible. We conclude that $R''$ is reduced, terminating, Church–Rosser and equivalent to $R$, as required.

For convenience, we call a reduced terminating Church–Rosser $R \subseteq \Sigma^* \times \Sigma^*$ uniquely terminating. (Thus, our terminology differs from [15].)

Note that if $R \subseteq \Sigma^* \times \Sigma^*$ is uniquely terminating and $(r, s_1), (r, s_2) \in R$, then $s_1 = s_2$. (Clearly, $s_1 \sim s_2$. Since $s_1$ and $s_2$ are irreducible, $s_1 = s_2$.) Also, if $u_1r_1v_1 = u_2r_2v_2$ with $r_1, r_2 \in R_1$ and $|u_1r_1| = |u_2r_2|$, then $u_1 = u_2$, $r_1 = r_2$ and $v_1 = v_2$. (Neither $r_1$ nor $r_2$ can be a proper subword of the other.) Finally, we note the following:
Corollary 2.5. Each finite terminating Church–Rosser presentation is equivalent to a finite uniquely terminating presentation.

Proof. In the proof of Theorem 2.4, assume that \( R \) is finite. Since \( R' \subseteq R \), \( R' \) is finite, so \( R'_1 \) is finite. Clearly, the cardinality of \( R'' \) equals the cardinality of \( R'_1 \), so \( R'' \) is finite. \( \square \)

For various reasons, we will need to consider both finite and infinite presentations below.

3. A partial free resolution

Let \( S \) be a monoid with identity element 1 and (associative) multiplication denoted \((x, y) \mapsto xy\). Let \( \mathbb{Z} \) denote the ring of (ordinary) integers and let \( \mathbb{Z} S \) denote the monoid ring of \( S \) with coefficients in \( \mathbb{Z} \). Modules over \( \mathbb{Z} S \) will be left modules.

View \( \mathbb{Z} \) as a \( \mathbb{Z} S \)-module on which each element of \( S \) acts as the identity: if \( w \in S \) and \( n \in \mathbb{Z} \), then \( wn = n \). Let \( C_0 \) be the free \( \mathbb{Z} S \)-module on a single formal symbol \([0]\). (Essentially, \( C_0 \) is \( \mathbb{Z} S \) viewed as a left module over itself.) Define a \( \mathbb{Z} S \)-module homomorphism \( \varepsilon : C_0 \to \mathbb{Z} \) by \( \varepsilon([0]) = 1 \); \( \varepsilon \) is called the augmentation and the kernel of \( \varepsilon \) is called the augmentation ideal. Clearly, \( \ker \varepsilon \) is a free abelian group with basis \( \{ (w-1)[0] \mid w \in S, \ w \neq 1 \} \).

To describe \( \ker \varepsilon \) as a \( \mathbb{Z} S \)-module, suppose that \( S \) is generated as a monoid by a set \( \Sigma \). Let \( C_1 \) be the free \( \mathbb{Z} S \)-module on the set of formal symbols \([a]\), one for each \( a \in \Sigma \). Define a \( \mathbb{Z} S \)-module homomorphism \( \partial_1 : C_1 \to C_0 \) by \( \partial_1([x]) = (x-1)[0] \). Clearly, \( \text{im} \partial_1 \subseteq \ker \varepsilon \). In fact, \( \text{im} \partial_1 = \ker \varepsilon \), as will become apparent (and be crucial) below.

To describe \( \ker \partial_1 \), we shall need the free differential calculus ([11] or see [6, pp. 45, 90]). Letting \( \Sigma^* \) denote the free monoid on (formal symbols) \( \Sigma \), we define, for each \( a \in \Sigma \), a function \( (\partial/\partial a) : \Sigma^* \to \mathbb{Z} \Sigma^* \) inductively as follows:

\[
\frac{\partial}{\partial a} (1) = 0 ,
\]

and if \( w \in \Sigma^* \) and \( b \in \Sigma \), then

\[
\frac{\partial}{\partial a} (wb) = \begin{cases} 
\frac{\partial}{\partial a} (w) & \text{if } b \neq a , \\
\frac{\partial}{\partial a} (w) + w & \text{if } b = a . 
\end{cases}
\]

It is easy to verify that if \( u, v \in \Sigma^* \), then \( (\partial/\partial a)(uv) = (\partial/\partial a)(u) + u(\partial/\partial a)(v) \).

Moreover, the following ‘fundamental theorem of calculus’ holds: if \( w \in \Sigma^* \), then

\[
w - 1 = \sum_{a \in \Sigma} \frac{\partial}{\partial a} (a - 1) .
\]
In particular, the equality \( \text{im} \partial_1 = \text{ker} \varepsilon \) is now apparent.

In order to describe \( \text{ker} \varepsilon \), we assume that \( R \) is a set of defining relations of \( S \) in terms of the generating set \( \Sigma \) of \( S \). In other words, \( R \subseteq \Sigma^* \times \Sigma^* \) and the congruence generated by \( R \) is the kernel of the natural homomorphism from \( \Sigma^* \) onto \( S \). Let \( C_2 \) be the free \( \mathbb{Z}S \)-module on the set of formal symbols \([r \mapsto s]\), one for each \((r, s) \in R\). To define \( \partial_2 : C_2 \rightarrow C_1 \), it is convenient to introduce the following notation: \( \phi : \Sigma^* \rightarrow S \) denotes the natural homomorphism and if \( W \in \mathbb{Z}\Sigma^* \), then \( W^\phi \) denotes the image in \( \mathbb{Z}S \) of \( W \) under the natural extension of \( \phi \) to \( \mathbb{Z}\Sigma^* \). With this notation, we define a \( \mathbb{Z}S \)-module homomorphism \( \partial_2 : C_2 \rightarrow C_1 \) by the formula

\[
\partial_2([r \mapsto s]) = \sum_{a \in \Sigma} \left( \frac{\partial r}{\partial a} - \frac{\partial s}{\partial a} \right)^\phi [a].
\]

It is an easy consequence of the fundamental theorem of calculus that \( \text{im} \partial_2 \subseteq \text{ker} \partial_1 \). In fact, \( \text{im} \partial_2 = \text{ker} \partial_1 \). We will outline a proof of this equality below.

Our main goal is to define a \( \mathbb{Z}S \)-module \( C_3 \) and a homomorphism \( \partial_3 : C_3 \rightarrow C_2 \) that satisfy \( \text{im} \partial_3 = \text{ker} \partial_2 \) in the situation when \( R \) is uniquely terminating. We will not assume that \( R \) is uniquely terminating until after giving the proof that \( \text{im} \partial_2 = \text{ker} \partial_1 \).

For each \( m \in S \), choose a ‘normal form’ \( w \in \Sigma^* \) so that \( w^\phi = m \). For each \( w \in \Sigma^* \), choose a relation chain from \( w \) to the normal form for \( w^\phi \). Note that if \( u, v \in \Sigma^* \) and \((r, s) \in R\), then

\[
\left( \frac{\partial}{\partial a} (urv) - \frac{\partial}{\partial a} (usv) \right)^\phi = \left( \frac{\partial u}{\partial a} + u \frac{\partial r}{\partial a} + ur \frac{\partial v}{\partial a} - \frac{\partial u}{\partial a} - u \frac{\partial s}{\partial a} - us \frac{\partial v}{\partial a} \right)^\phi = u^\phi \left( \frac{\partial r}{\partial a} - \frac{\partial s}{\partial a} \right)^\phi
\]

since \( r^\phi = s^\phi \).

Let \( x \in \Sigma^* \) and let \( y \) be the normal form of \( x^\phi \). Let \( x = u_1 r_1 v_1, u_is_i v_i = u_2 r_2 v_2, \ldots, u_n s_n v_n = y \) be the chosen relation chain from \( x \) to \( y \), where for each \( i \) either \((r_i, s_i) \in R\) or \((s_i, r_i) \in R\). Applying the note above several times gives

\[
\left( \frac{\partial x}{\partial a} - \frac{\partial y}{\partial a} \right)^\phi = \sum_{i=1}^{n} u_i^\phi \left( \frac{\partial r_i}{\partial a} - \frac{\partial s_i}{\partial a} \right)^\phi.
\]

Define

\[
\Phi(x) = \sum_{i=1}^{n} \epsilon_i u_i^\phi [r_i' \mapsto s_i']^\phi
\]

where if \((r_i, s_i) \in R\), then \( \epsilon_i = 1 \), \( r_i' = r_i \) and \( s_i' = s_i \) and if \((s_i, r_i) \in R\), then \( \epsilon_i = -1 \), \( r_i' = s_i \) and \( s_i' = r_i \). Clearly, \( \Phi(x) \in C_2 \). Note that
Define an abelian group homomorphism $s_1 : C_0 \rightarrow C_1$ by the formula

$$s_1(w^\phi[\emptyset]) = \sum_{a \in \Sigma} \left( \frac{\partial w}{\partial a} \right)^\phi [a]$$

where $w$ is the normal form of $w^\phi$. (It is easy to verify that $\partial_1 s_1 (w^\phi[\emptyset]) = (w - 1)^\phi[\emptyset]$. This yields an alternate proof of the equality $\text{im} \partial_1 = \ker \Sigma.$) It follows easily that

$$s_1 \partial_1 (w^\phi[b]) = \sum_{a \in \Sigma} \left( \frac{\partial z}{\partial a} - \frac{\partial w}{\partial a} \right)^\phi [a]$$

where $w$ is the normal form of $w^\phi$ and $z$ is the normal form of $(wa)^\phi$. Define an abelian group homomorphism $s_2 : C_1 \rightarrow C_2$ by the formula

$$s_2(w^\phi[b]) = \Phi(wb)$$

where $w$ is the normal form of $w^\phi$. Using the formula for $\partial_2 \Phi$ above, it follows that

$$\partial_2 s_2 (w^\phi[b]) = \sum_{a \in \Sigma} \left( \frac{\partial (wb)}{\partial a} - \frac{\partial z}{\partial a} \right)^\phi [a] = w^\phi[b] + \sum_{a \in \Sigma} \left( \frac{\partial w}{\partial a} - \frac{\partial z}{\partial a} \right)^\phi [a]$$

where, as above, $z$ denotes the normal form of $(wb)^\phi$. We conclude that $\partial_2 s_2 + s_1 \partial_1$ is the identity on $C_1$, from which im $\partial_2 = \ker \partial_1$ follows easily.

We are at last ready to define $C_3$ and $\partial_3$ under the assumption (from now on in force) that $R$ is uniquely terminating. Under this assumption, there is a natural choice of normal form for an element of $S$: if $m \in S$, then the normal form of $m$ will be the unique irreducible $w \in \Sigma^*$ which satisfies $w^\phi = m$. We also assume that for each $w \in \Sigma^*$, the chosen relation chain from $w$ to the normal form of $w$ consists of reductions only. (In the relevant notation above, each $(r_i, s_i) \in R$ and not $(s_i, r_i) \in R$. In particular, the definition of $\Phi(z)$ simplifies as follows: each $r'_i = r_i$, $s'_i = s_i$ and $e_i = 1$.)

Let $C_3$ be the free $\mathbb{Z}S$-module on the set of formal symbols $[r_1 r_2 \rightarrow s_{12}, r_2 r_3 \rightarrow s_{23}]$, one for each pair $(r_1 r_2, s_{12}), (r_2 r_3, s_{23}) \in R$ where $r_2 \neq \lambda$. Define a $\mathbb{Z}S$-module homomorphism $\partial_3 : C_3 \rightarrow C_2$ by the formula

$$\partial_3([r_1 r_2 \rightarrow s_{12}, r_2 r_3 \rightarrow s_{23}]) = r_1[r_2 r_3 \rightarrow s_{23}] + \Phi(r_1 s_{23}) - [r_1 r_2 \rightarrow s_{12}] - \Phi(s_{12} r_3).$$

(The reader should be warned about a slight abuse of notation: given words $r_1 r_2$ and $r_2 r_3$, there may be several choices of $r_2$. There will be one generator of $C_3$ for
each choice of \( r_2 \) and a given choice will be used consistently in the definition of \( \partial_3 \). It follows from the definition of \( \partial_2 \), the formula for \( \partial_2 \Phi \) above and the fact that \( r_1 s_{23} \) and \( s_{12} r_3 \) have the same normal form that \( \partial_2 \partial_3 = 0 \), so that \( \text{im} \partial_3 \subseteq \text{ker} \partial_2 \).

**Theorem 3.1.** \( \text{im} \partial_3 = \text{ker} \partial_2 \).

**Proof.** We will use Noetherian induction as developed in Section 1 and the full force of the assumption that \( R \) is uniquely terminating. Let \( X \) denote the set of all \( w^\phi[r \to s] \) where \( w \in \Sigma^* \) is irreducible and \( (r, s) \in R \). Recall that \( C_2 \) is a free abelian group with basis \( X \). We define a relation \( \to \) on \( X \) as follows:

\[ w^\phi_1[r_1 \to s_1] \to w^\phi_2[r_2 \to s_2] \]

provided either \( w_1 r_1 \to w_2 r_2 x \) for some \( x \in \Sigma^* \) or \( w_1 r_1 = w_2 r_2 x \) for some non-empty \( x \in \Sigma^* \). Since \( \to \) and the 'proper prefix' relation are Noetherian and since a reduction of a prefix is a prefix of a reduction, it follows that \( \to \) is Noetherian on \( X \). Extend \( \to \) to finite subsets of \( X \) as in Section 1. We apply Theorem 1.4 to obtain the following: for each \( W_2 \in C_2 \), there exists \( W_3 \in C_3 \) such that if \( w^\phi[r \to s] \) is in the support of \( W_2 - \partial_3 W_3 \), then each proper prefix of \( w^\phi[r \to s] \) is irreducible. In the notation of Theorem 1.4, let \( Y \) consist of all \( w^\phi[r \to s] \) in \( X \) such that some proper prefix of \( w^\phi[r \to s] \) is reducible. Since \( w^\phi[r \to s] \) and \( \text{im} \partial_3 \subseteq \text{ker} \partial_2 \), it follows that \( \text{im} \partial_3 = \text{ker} \partial_2 \).

To prove \( \text{im} \partial_3 = \text{ker} \partial_2 \), we assume that \( W \in C_2 \) has the property that for each \( w^\phi[r \to s] \) in the support of \( W \), each proper prefix of \( w^\phi[r \to s] \) is irreducible and we show that \( \partial_3 W \neq 0 \). To do this, choose \( w^\phi[r \to s] \) maximal in the support of \( W \) with respect to the Noetherian relation \( \to \) on \( X \) and write \( r = xa \) with \( x \in \Sigma^* \) and \( a \in \Sigma \). Thus \( (wx)^\phi[a] \) occurs in the expansion of \( \partial_3(w^\phi[r \to s]) \) as the longest term in \( w(\partial r/\partial a) \). We show that \( (wx)^\phi[a] \) can occur nowhere else in the expansion of \( \partial_2 W \). (Thus \( (wx)^\phi[a] \) is in the support of \( \partial_2 W \), so that \( \partial_3 W = 0 \).

Note first that if \( w^\phi_1[r_1 \to s_1] \) occurs in the support of \( W \) and \( r_1 = uav \), then, by hypothesis, \( w_1 u \) is irreducible and is therefore the normal form of \( (w_1 u)^\phi \). (In particular, \( wx \) in the normal form of \( (wx)^\phi \).

Assume that \( (wx)^\phi[a] \) cancels with a term of some \( (w_1(\partial r_1/\partial a))^\phi[a] \) where \( w^\phi_1[r_1 \to s_1] \) is in the support of \( W \). Writing \( r_1 = uav \) as above gives \( wx = w_1 u \), so that \( w = wxa \) is a prefix of \( w_1 r_1 = w_1 uav \). If \( v \neq \lambda \), this contradicts the maximality of \( w^\phi[r \to s] \). If \( v = \lambda \), the fact that \( R \) is uniquely terminating leads to the conclusion that \( w = w_1 \) and \( r = r_1 \), so that \( w^\phi[r \to s] = w^\phi_1[r_1 \to s_1] \). This contradicts the assumption that \( (wx)^\phi[a] \) cancels 'somewhere else'.

Assume that \( (wx)^\phi[a] \) cancels with a term of some \( w_1(\partial s_1/\partial a)^\phi[a] \) where \( w^\phi_1[r_1 \to s_1] \) is in the support of \( W \). Then \( wx \) is a (not necessarily proper) reduction
of some \( w_1u \) where \( w_1u \) is a prefix of \( w_1s_1 \). But then \( wr = wxa \) is a prefix of a proper reduction of \( w_1r_1 \) which contradicts the maximality of \( w^\phi [r \rightarrow s] \).

To finish the proof that \( \text{im} \partial_3 = \ker \partial_2 \), see the discussion following Theorem 1.4. \( \Box \)

We use the ideas developed in the proof of Theorem 3.1 to give a sufficient condition for \( \ker \partial_3 = 0 \).

**Theorem 3.2.** Suppose that \( R \) is uniquely terminating and satisfies the following two conditions:

(a) If \( (r_1r_2, s_{12}), (r_2r_3, s_{23}) \in R \), then either \( r_2 = \lambda \) or \( r_3 = \lambda \);

(b) If \( (r_1r_2r_3, s_{123}), (r_2r_3r_4, s_{234}) \in R \), then either \( r_2 = \lambda \) or \( r_4 = \lambda \).

Then \( \ker \partial_3 = 0 \).

**Proof.** We begin by noting a consequence of (a): if \( (r_1r_2, s_{12}), (r_2r_3, s_{23}) \in R \) with \( r_2 \neq \lambda \) and \( w \in \Sigma^* \) is irreducible, then \( wr_1 \) is irreducible. (The only alternative is that there exists \( (r, s) \in R \) such that some proper suffix of \( r \) is a proper prefix of \( r_1 \), a contradiction.) It follows that if \( w^\phi [r_1r_2 \rightarrow s_{12}, r_2r_3 \rightarrow s_{23}] \) is a basis element for \( C_3 \) (as an abelian group), then, in the formula for \( \partial_3 (w^\phi [r_1r_2 \rightarrow s_{12}, r_2r_3 \rightarrow s_{23}]) \), \( w^\phi [r_1r_2 \rightarrow s_{12}] \) and each term of \( w^\phi \Phi (r_1, s_{23}) \) and \( w^\phi \Phi (s_{12}, r_3) \) is a proper reduction of \( (wr_1)^\phi [r_2r_3 \rightarrow s_{23}] \), using the \( \rightarrow \) on \( C_2 \) defined in the proof of Theorem 3.1. (The fact that \( wr_1 \) is irreducible is crucial here.)

Assume that \( W \in C_3 \) satisfies \( \partial_3 W = 0 \). If \( W \neq 0 \), choose a term \( w^\phi [r_1r_2 \rightarrow s_{12}, r_2r_3 \rightarrow s_{23}] \) in the support of \( W \) so that \( (wr_1)^\phi [r_2r_3 \rightarrow s_{23}] \) is maximal among all terms that occur in \( \partial_3 W \) (before cancellation). Arguing as in the proof of Theorem 3.1, \( (wr_1)^\phi [r_2r_3 \rightarrow s_{23}] \) can only cancel with another maximal element. This requires the existence of \( w' \in \Sigma^* \), \( (r'_1r'_2, s_{12}') \in R \) and an equality \( r'_2r'_3 = r_2r_3 \) so that \( wr_1 = w'r_1' \) (again using the irreducibility of \( wr_1 \) and \( w'r_1' \)). Unless \( w = w' \) and \( r_1 = r_1' \) (in which case \( (wr_1)^\phi [r_2r_3 \rightarrow s_{23}] \) and \( (w'r_1')^\phi [r_2r_3 \rightarrow s_{23}] \) arise from the same term of \( W \), we obtain a violation of condition (b). \( \Box \)

### 4. Homological finiteness and examples

We begin with an important consequence of Theorem 3.1. We call a monoid \((\text{FP})_k\) provided there is a sequence

\[
\xrightarrow{\partial_k} C_k \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z}
\]

of left \( Z \)-modules and \( Z \)-module homomorphisms (as indicated) which satisfy:

- each \( C_i \) is a finitely-generated free left \( Z \)-module;
- for each \( i > 0 \), \( \text{im} \partial_{i+1} = \ker \partial_i \);
im \delta = \ker \varepsilon \text{ and } \varepsilon \text{ is surjective. (The } \mathbb{Z}S\text{-module structure on } \mathbb{Z} \text{ is the 'trivial' one: if } x \in S \text{ and } n \in \mathbb{Z}, \text{ then } xn = n\text{.) The notation of an } (FP)_k\text{-group is well known (see, for example, [3, Chapter I or 6, Chapter VIII]). Technically, the concept we just defined should be called "}(FP)_k\text{ on the left"}, \text{ a distinction that need not be made in group theory.}

We record the following consequence of Theorem 3.1:

**Theorem 4.1.** If a monoid \(S\) has a finite terminating Church–Rosser presentation, then \(S\) is \((FP)_3\).

**Proof.** By Theorem 2.4, we may assume that \(S\) has a finite uniquely terminating presentation. In this situation, the modules \(C_0, C_1, C_2\) and \(C_3\) defined in Section 3 are all finitely-generated free \(\mathbb{Z}S\)-modules; Theorem 3.1 and the other results summarized in Section 3 show that \(S\) is \((FP)_3\). \(\square\)

We conclude with some examples of finitely-presented monoids that have solvable word problems but are not \((FP)_3\) and therefore do not have a finite uniquely terminating presentation. This answers a question raised in [15]. The first few examples are groups that arose in various contexts; we do little more than refer the reader to the relevant literature. For completeness (and to illustrate how the homological algebra developed in Section 3 can be carried out in practice), we give our own (monoid) example.

**Example 4.2.** The first example of a finitely-presented group that is not \((FP)_3\) was given by Stallings [18]. For a description of this group that makes it clear that it has a solvable word problem, see [3, p. 37].

**Example 4.3.** In [1], Abels gave examples of groups of \(4 \times 4\) matrices which are (somewhat surprisingly) finitely-presented. The definition of these groups is sufficiently explicit to make it clear that they have solvable word problems. Bieri [4] showed that these groups are not \((FP)_3\). For further discussion, see [2].

**Example 4.4.** In [13], Houghton defined a group generated by two permutations of a countably infinite set. Again, the definition is sufficiently explicit to solve the word problem. Burns and Solitar (unpublished) showed that this group is finitely-presented. Recently, Brown [7] showed that this group is not \((FP)_3\). Here is our example:

**Example 4.5.** For each non-negative integer \(k\), let \(S_k\) denote the monoid defined by the following presentation:
Word problems and a homological finiteness condition

generators: \( a, b, t, x_1, \ldots, x_k, y_1, \ldots, y_k \);

relations: \( at^nb \rightarrow \lambda, \) \((P_n)\)
\( x_ia \rightarrow ax_i, \) \((A_i)\)
\( x_it \rightarrow tx_i, \) \((T_i)\)
\( x_ib \rightarrow bx_i, \) \((B_i)\)
\( x_iy_i \rightarrow \lambda. \) \((Q_i)\)

(For convenience, we have used \( \rightarrow \) notations in the relations and have given each relation a name.) In \( P_n, n \) ranges over all non-negative integers; in \( A_i, T_i, B_i \) and \( Q_i, i \) ranges from 1 to \( k. \) For each \( k, \Sigma_k \) denotes the indicated set of generators of \( S_k \) and \( R_k \) denotes the indicated set of (one-way) relations. Clearly, each \( S_k \) is finitely-generated.

Claim. If \( k \geq 1, \) then \( S_k \) is finitely-related.

Proof. We show that if \( k \geq 1, \) then for each \( n \geq 1, P_n \) follows from \( P_0 \) and the other relations, by induction on \( n: \)

\[
\begin{align*}
at^{n+1}b & \rightarrow at^{n+1}bx_iy_i \quad (Q_i) \\
& \sim at^{n+1}x_iby_i \quad (B_i) \\
& \sim at^nby_i \quad (T_i, n \text{ times}) \\
& \sim x_iat^nby_i \quad (A_i) \\
& \sim x_iy_i \quad (P_n) \\
& \sim \lambda. \quad (Q_i)
\end{align*}
\]

Thus, if \( k \geq 1, \) then \( P_{n+1} \) follows from \( P_n \) and any choice of \( Q_i, B_i, A_i \) and \( T_i.\)

(We will eventually show that \( S_0 \) is not finitely-related.) \( \square \)

Claim. For each \( k \geq 0, R_k \) is terminating.

Proof. We define a function \( f \) from \( \Sigma_k^* \) to a well-ordered set such that if \( w, w' \in \Sigma_k^* \) satisfy \( w \rightarrow w' \) (modulo \( R_k \)), then \( f(w) > f(w') \). The fact that \( R_k \) is terminating follows easily. First, if \( w \rightarrow w' \) arises via \( P_n \) or \( Q_i \), then \( n_1(w) > n_1(w') \), where, for \( w \in \Sigma_k^*, n_1(w) \) is defined by

\[ n_1(w) = \text{the total number of } a\text{'s, } b\text{'s, } x_i\text{'s or } y_i\text{'s that occur in } w. \]

Second, if \( w \rightarrow w' \) arises via \( A_i \), then \( n_1(w) = n_1(w') \) and \( n_2(w) > n_2(w') \), where, for \( w \in \Sigma_k^*, n_2(w) \) is defined by

\[ n_2(w) = \text{the total number of factorizations } w = u_1x_1u_2au_3 \text{ of } w \text{ with } u_1, u_2, u_3 \in \Sigma_k^*. \]
Third, if $w \rightarrow w'$ arises via $T_i$ or $B_i$, then $n_1(w) = n_1(w')$, $n_2(w) = n_2(w')$ and $n_3(w) > n_3(w')$ where, for $w \in \Sigma_k^*$, $n_3(w)$ is defined by

$$n_3(w) = \text{the total number of factorizations } w = u_1u_2tu_3 \text{ or } w = u_1x_iu_2tu_3 \text{ of } w \text{ with } u_1, u_2, u_3 \in \Sigma_k^*.$$

The function $f(w) = (n_1(w), n_2(w), n_3(w))$ from $\Sigma_k^*$ to the set of ordered triples of non-negative integers ordered lexicographically satisfies: if $w \rightarrow w'$, then $f(w) > f(w')$, as required. □

**Claim.** For each $k \geq 0$, $R_k$ satisfies the Church–Rosser property.

**Proof.** Since each $R_k$ is terminating, it suffices to verify that $R_k$ satisfies Theorem 2.1(b). In the notation of Theorem 2.1(b), the critical $r_1 = uw$ and $r_2 = vw$ are given by $u = x_i, v = a$ and $w = t''b$. (In particular, $R_0$ satisfies Theorem 2.1(b) vacuously.) For convenience in computing $\partial_3$, while finding the common reduction of $s_1w$ and $s_2w$, we will record the relevant relation applications and their locations. Reducing $x_i at''b$, starting with $at''b \rightarrow \lambda$, we have

$$x_i at''b \rightarrow x_i \quad \text{via } x_i[R_n].$$

Reducing $x_i at''b$, starting with $x_i a \rightarrow atx_i$, we have

$$x_i at''b \rightarrow atx_i t''b \quad \text{via } [A_i],$$

$$\rightarrow at_i^{n+1}x_ib \quad \text{via } a\frac{t^n-1}{t-1}[T_i],$$

$$\rightarrow at_i^{n+1}bx_i \quad \text{via } at_i^{n+1}[B_i],$$

$$\rightarrow x_i \quad \text{via } [R_{n+1}].$$

By Theorem 2.1(b), $R_k$ has the Church–Rosser property. □

It follows easily that each $S_k$ has a solvable word problem. (The defining relations of each $S_k$ are simple enough to allow any $w \in \Sigma_k^*$ to be reduced to an irreducible $z$ as in Theorem 2.1(c); by Theorem 2.1(a), two irreducibles $z_1, z_2$ satisfy $z_1 \sim z_2$ if and only if $z_1 = z_2$ in $\Sigma_k^*$.)

It is also easy to check that each $R_k$ is reduced (Definition 2.3), so that each $R_k$ is uniquely terminating (Theorem 2.4). Thus all of Section 3 applies. For convenience, we record the formulae for $\partial_2$ and $\partial_3$.

$$\partial_2([P_n]) = [a] + a\frac{t^n-1}{t-1}[t] + at^n[b],$$

$$\partial_2([A_i]) = (1 - at)[x_i] + (x_i - 1)[a] - a[t],$$

$$\partial_2([T_i]) = (1 - t)[x_i] + (x_i - 1)[t],$$

$$\partial_2([B_i]) = (1 - b)[x_i] + (x_i - 1)[b],$$

$$\partial_2([Q_i]) = [x_i] + x_i[y_i].$$
(We have omitted the homomorphism symbol $\phi$ and written $1$ for $\lambda$ in $ZS_k$.) We write $[R_\eta, A_\eta]$ for the generator of $C_3$ which corresponds, in the notation of Section 3, to $r_1 = x_1$, $r_2 = a$ and $r_3 = t'b$.

$$\partial_3([R_\eta, A_\eta]) = x_1[R_\eta] - \left([A_\eta] + a \frac{t-1}{t} [T_1] + at^{n+1}[B_1] + [R_{n+1}]\right).$$

We will show that if $k \geq 2$, then $S_k$ is not $(FP)_3$. For completeness, we begin with a discussion of $S_0$ and $S_1$.

**Claim.** $S_0$ is not finitely-related.

**Proof.** When $k = 0$, $C_3 = 0$, so $\ker \partial_2 = \{0\}$. We show that

$$H_2(S_0, Z) = \ker \left\langle Z \otimes C_2 \xrightarrow{1 \otimes \partial_2} Z \otimes C_1 \right\rangle$$

is not a finitely-generated abelian group. Here, $Z$ is viewed as a trivial right $ZS_0$-module, so that $Z \otimes C_2$ is a free abelian group on $\{[P_n] \mid n \geq 0\}$, $Z \otimes C_1$ is a free abelian group on $\{[a], [t], [b]\}$ and so that $1 \otimes \partial_2$ is given by

$$(1 \otimes \partial_2)[P_n] = [a] + n[t] + [b].$$

It follows that $H_2(S_0, Z)$ can be viewed as the free abelian group on $\{[P_n] - [P_0] - n([P_1] - [P_0]) \mid n \geq 2\}$ and is therefore not finitely-generated. Thus $S_0$ is not finitely-related. (A direct proof of this fact is also possible.)

**Claim.** $S_1$ is $(FP)_k$ for all $k$.

**Proof.** We will show that $S_1$ is 2-dimensional. Since $S_1$ is finitely-presented, the claim follows.

Let $C'_2$ denote the $ZS_1$-submodule of $C_2$ generated by $[P_0]$, $[A_0]$, $[T_0]$, $[B_0]$ and $[Q_0]$. Clearly, $C'_2$ is a free left $ZS_1$-module on these generators. Let $\partial'_2$ denote the restriction of $\partial_2$ to $C_2$. The general discussion of Section 3 applies, so that $\im \partial'_2 = \ker \partial_1$. We show $\ker \partial'_2 = 0$.

Define a module homomorphism $\pi : C_2 \to C'_2$ by letting $\pi$ be the identity on $C'_2 \subseteq C_2$ and inductively defining

$$\pi([R_{n+1}]) = x_1 \pi([R_n]) = \left([A_\eta] + a \frac{t^n-1}{t-1} [T_1] + at^{n+1}[B_1]\right)$$

for $n \geq 0$. An easy consequence of this definition is: $\pi \partial_3 = 0$. If $Z \in \ker \partial_2$, then by Theorem 3.1, there exists $Y \in C_3$ such that $\partial_3 Y = Z$. If, in addition, $Z \in C'_2$, then $Z = \pi Z = \pi \partial_3 Y = 0$, so $\ker \partial'_2 = \{0\}$. It follows that $S_1$ is $(FP)_k$ for every $k$; take $C'_k = \{0\}$ for $k \geq 3$. (Since $S_1$ is $(FP)_3$, Theorem 4.1 does not apply; the
author does not know whether or not $S_i$ has a finite uniquely terminating presentation.) □

Claim. *If $k \geq 2$, then $S_k$ is not (FP)$_3$.*

Proof. We proceed as in the proof that $S_0$ is not finitely-related, except now in dimension 3. It is easy to verify that each $R_k$ satisfies the hypotheses of Theorem 3.2, so that $\ker \partial_3 = \{0\}$. It follows that

$$H_3(S_k, \mathbb{Z}) = \ker \{\mathbb{Z} \otimes C_3 \xrightarrow{1 \otimes \partial_3} \mathbb{Z} \otimes C_2\}.$$

The claim follows from the fact that if $k \geq 2$, then $H_3(S_k, \mathbb{Z})$ is not a finitely-generated abelian group. Clearly, $\mathbb{Z} \otimes C_2$ and $\mathbb{Z} \otimes C_3$ are free abelian groups on the 'same' generators as $C_2$ and $C_3$; $1 \otimes \partial_3$ is given by the formula

$$(1 \otimes \partial_3)[R_n, A_i] = [R_n] - ([A_i] + n[T_i] + [B_i] + [R_{n+1}]).$$

To determine $\ker(1 \otimes \partial_3)$, define $U_i = [R_0, A_i] - [R_0, A_1]$ and $V_i = [R_1, A_i] - [R_1, A_1] - U_i$. It is easy to verify that $\ker(1 \otimes \partial_3)$ is a free abelian group with basis

$$[[R_n, A_i] - [R_n, A_0] + U_i + nV_i | n \geq 2, i \geq 2]$$

and therefore is not finitely-generated when $k \geq 2$. □

Claim. *If $k \geq 2$, then $S_k$ does not have a finite uniquely terminating presentation.*

Proof. Since $S_k$ is not (FP)$_3$ when $k \geq 2$, Theorem 4.1 applies. □

References


