Pointwise Degeneracy of Linear, 
Time-Invariant, Delay-Differential Equations

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1. Introduction

It is an elementary fact that the solutions of the ordinary differential equation $Dx = Ax$ (where, for instance, $x(t) \in \mathbb{R}^n$, $A$ is a constant, $n \times n$-matrix and $Dx$ represents the derivative of $x$) have the property that, for every $x_1 \in \mathbb{R}^n$ and every $t_1 > 0$, there exists an initial condition $x_0$ such that the corresponding solution of the differential equation takes the value $x_1$ at time $t_1$.

L. Weiss [1-3] was the first to investigate whether a similar property is true for the solutions of delay-differential equations. He found that the property is not true in the time-varying case. The question has been harder to answer for the simpler, linear, time-invariant delay differential systems of the form

$$Dx(t) = Ax(t) + Bx(t - h),$$

where $x(t) \in \mathbb{R}^n$, $A \times \times B$ are constant, $n \times n$ matrices and $h > 0$.

After L. Weiss [3], system (1) is called "pointwise degenerate" if there exists a proper subspace of $\mathbb{R}^n$ and a number $t_1 > 0$ such that all the solutions of Eq. (1) (for continuous initial functions) are contained in the mentioned subspace at time $t_1$. Otherwise the system is called "pointwise complete."

L. Weiss proposed the following conjecture to be proved or disproved: "every system of the form (1) is pointwise complete."

This problem has been studied by several authors who obtained a partial confirmation of the above conjecture. J. A. Yorke and J. Kato proved (independently) the conjecture for $n = 2$ (see also [4]). E. B. Lee proved the conjecture if $B$ is nonsingular. R. M. Brooks and K. Schmitt [5] proved the conjecture if $BA = AB$. All the mentioned results have had a limited circulation and the author is indebted to Professor J. A. Yorke for the above information.

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In a note [6] (which also had a limited circulation) the author proved the conjecture if rank $B = 1$, but also proved, by the counterexample reproduced below, that the conjecture is not true in general.\footnote{Professor A. Halanay has kindly informed the author that another counterexample was given by A. M. Zverkin in his communication "Pointwise Completeness of Delay-Differential Equations" at the Conference of the University of Friendship of Peoples, Moscow, May 24–27, 1971.} Indeed, consider the system (for $t \geq 0$)

$$
\begin{align*}
D\xi(t) &= 2\eta(t), \\
D\eta(t) &= -\psi(t) + \xi(t - 1), \\
D\psi(t) &= 2\eta(t - 1),
\end{align*}
$$

where $\xi(t)$, $\eta(t)$ and $\psi(t)$ are scalars. Then, for $t \geq 1$, one obtains successively

$$
\begin{align*}
D\xi(t - 1) - D\psi(t) &= 0, \\
\xi(t - 1) - \psi(t) &= \gamma_0, \\
\eta(t) &= \gamma_1 + \gamma_0 t, \\
\xi(t) &= \gamma_2 + 2\gamma_1 t + \gamma_0 t^2,
\end{align*}
$$

where $\gamma_0$, $\gamma_1$ and $\gamma_2$ are arbitrary constants. Hence, for $t \geq 2$, one obtains the relation

$$
\xi(t) - 2\eta(t) - \psi(t) = 0
$$

which shows that, for $t \geq 2$, all the solutions of the considered system are contained in a plane of $R^3$ and therefore the system is pointwise degenerate.

Obviously, the property of pointwise degeneracy is worth a thorough study. The present paper aims to answer some basic questions concerning this problem and to facilitate in this way the development of the potential applications of this property. In Section 2 of the present paper we determine the largest set of points $t$ at which system (1) exhibits the property of pointwise degeneracy. Section 3 contains two different (necessary and sufficient) algebraic criteria of pointwise degeneracy. In Section 4 one identifies a class of pointwise-degenerate systems which have a remarkably simple structure. In general, there exist pointwise-degenerate systems with a more complicated structure; however, as proved in Section 5, every pointwise-degenerate system of the third order has the simple structure mentioned above. Finally, in Section 6, one examines a related control problem.

Before finishing this paper, the author read a first draft of the paper [7] which was being prepared by A. K. Choudhury and was informed that R. Zmood had obtained a necessary and sufficient criterion of pointwise completeness. Although the final forms of these results are not known to the
author, they seem to be strongly different from the results from Section 3 of this paper. This indicates that there exist other useful forms of the results given by Theorem 2 and Corollary 2 of this paper. There is no doubt that many other results can be obtained in this new and broad area of research.

### 2. The Maximal Set of Pointwise Degeneracy

One can easily see that the definition of pointwise degeneracy (see L. Weiss [3]) is equivalent, for the linear system (1), to the following definition:

**Definition 1.** System (1) is called "pointwise degenerate" (p.d.) iff there exists an $n$-vector $q \neq 0$ and a number $t_1 > 0$ such that every continuous function $x: [-h, t_1] \to \mathbb{R}^n$ which satisfies Eq. (1) in the open interval $(0, t_1)$ satisfies also the condition $q^T x(t_1) = 0$ (where the upper index $T$ denotes transpose). When one wants to specify $t_1$, or $q$, or both, one also says that "Eq. (1) is p.d. at $t_1$" or that "Eq. (1) is p.d. for $q$" or that "Eq. (1) is p.d. at $t_1$ for $q$".

If Eq. (1) is p.d., there exist, in general, several vectors $q$ with the property from Definition 1. For every such vector $q$, the theorem below gives the largest set of points $t_1$ with the property from Definition 1. (This theorem improves a previous partial result from [6; see formula (19)].)

**Theorem 1.** Assume that Eq. (1) is p.d. for the vector $q \neq 0$. Then the largest set of points $t_1$ at which Eq. (1) is p.d. for the considered $q$ is the interval $[lh, \infty)$, where $l$ is the smallest integer with the property

$$q^T (S(\sigma))^{l+1} = 0$$

(2) and $S(\sigma)$ is the $n \times n$ polynomial matrix given by the equation

$$(\sigma I - A)S(\sigma) = B \det(\sigma I - A),$$

(3) where $I$ is the $n \times n$ identity matrix. Moreover, $l \geq 2$.

**Proof.** (I) One shows first that the quantities introduced above are well defined. The existence of the matrix $S(\sigma)$ from (3) can be easily derived from [8, Chapter IX, Section 3, Theorem 6]. [One determines first a polynomial matrix $R(\sigma)$ such that

$$(\sigma I - A)R(\sigma) = I \det(\sigma I - A)$$

(4) and then one multiplies (4) on the right with $B$; this gives (3) for $S(\sigma) = R(\sigma)B$.] Observe that $S(\sigma)$ can be determined in this way even if $B$ is
not square. An explicit expression for $S(\sigma)$ will be given later. The uniqueness of the polynomial matrix $S(\sigma)$ satisfying (3) is also very easy to prove.

To prove the existence of the number $l$ from Theorem 1 we start with a lemma. To avoid confusions, since this lemma will be applied for various matrices, we distinguish some of the symbols with the sign $\sim$. Remark that the matrix $B$ in this lemma is not necessarily square. One denotes by $D^i s(t)$ the $i$-th derivative of function $s$ at point $t$ (if this derivative exists) or the $i$-th derivative on the right (if only this derivative can be defined) or the $i$-th derivative on the left (if only this derivative can be defined).

**Lemma 1.** Let $\tilde{A}$ be an $\tilde{n} \times \tilde{n}$ matrix and let $B$ be an $\tilde{n} \times \tilde{m}$ matrix. Let $S(\sigma)$ be the polynomial matrix given by

$$
(aI - \tilde{A})S(\sigma) = B \det(\sigma I - \tilde{A})
$$

and let $\tilde{S}_i$, $i = 0, 1, \ldots, \tilde{n} - 1$ be the coefficients of this polynomial:

$$
\tilde{S}(\sigma) = \tilde{S}_0 + \tilde{S}_1 \sigma + \cdots + \tilde{S}_{\tilde{n} - 1} \sigma^{\tilde{n} - 1}.
$$

Let $\tilde{\alpha}_i$, $i = 0, 1, \ldots, \tilde{n}$, be the coefficients of the characteristic polynomial of the matrix $\tilde{A}$:

$$
\det(\sigma I - \tilde{A}) = \tilde{\alpha}_0 + \tilde{\alpha}_1 \sigma + \cdots + \sigma^{\tilde{n}}, \quad (\tilde{\alpha}_{\tilde{n}} = 1).
$$

Let $\tau_1$ and $\tau_2$ be two real numbers ($\tau_1 < \tau_2$).

Then for every function $s : [\tau_1, \tau_2] \to \mathbb{R}^\tilde{m}$, $\tilde{n}$ times continuously differentiable, the functions $\tilde{x} : [\tau_1, \tau_2] \to \mathbb{R}^\tilde{m}$ and $u : [\tau_1, \tau_2] \to \mathbb{R}^\tilde{m}$, defined by

$$
\tilde{x}(t) = \tilde{S}_0 s(t) + \tilde{S}_1 Ds(t) + \cdots + \tilde{S}_{\tilde{n} - 1} D^{\tilde{n} - 1} s(t),
$$

$$
u(t) = \tilde{\alpha}_0 s(t) + \tilde{\alpha}_1 Ds(t) + \cdots + D^{\tilde{n}} s(t),
$$
satisfy the ordinary differential equation $D\tilde{x} = \tilde{A}\tilde{x} + \tilde{B}u$ in the interval $(\tau_1, \tau_2)$.

Indeed, from (5)–(7) one obtains

$$
\tilde{S}_{\tilde{n} - 1} = B,
$$

$$
\tilde{S}_i = \tilde{A}\tilde{S}_{i+1} + \tilde{B}\tilde{\alpha}_{i+1}, \quad i = n - 2, n - 3, \ldots, 0,
$$
or, equivalently,

$$
\tilde{S}_i = \tilde{B}\tilde{\alpha}_{i+1} + \tilde{A}\tilde{B}\tilde{\alpha}_{i+2} + \cdots + \tilde{A}^{n-1} B, \quad i = 0, 1, \ldots, n - 1.
$$

Hence and from the theorem of Cayley–Hamilton one obtains

$$
\tilde{A}\tilde{S}_0 + \tilde{B}\tilde{\alpha}_0 = 0.
$$
The conclusion of the lemma follows easily from (10) and (12). It is also easy to verify that from (6), (10) and (12) one obtains (5).

The matrices $\bar{A}\bar{B}$, from (11), occur also in the problem of complete controllability (see R. E. Kalman [9]). This is not surprising if one observes that Lemma 1 leads to the solution of a generalized problem of controllability (see Lemma 2 in Section 3). Other variants of this lemma have been introduced by the author in [6] and [10].

In applications, the function $s$ is chosen so as to satisfy an arbitrary (but finite) number of conditions of the form

$$Dy(t_i) = v_{i,j}, \quad i = 0, 1, \ldots, N_1, \quad j = 0, 1, \ldots, N_2,$$

where $v_{i,j}$ are some given vectors and $t_k$ are some given real numbers. This can be done if, for instance, $s$ is determined by the "full Hermite interpolation" [11, Chapter II, p. 28, Ex. 6]. In this case $s(t)$ is obtained as a polynomial in $t$.

(II) One now applies Lemma 1, taking for $\bar{A}$ and $\bar{B}$ some special matrices. For every positive integer $k$, consider the $nk \times nk$ matrix $A_k$ and the $nk \times n$ matrix $B_k$, of the form

$$A_k = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ B & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A & 0 \\ 0 & 0 & \cdots & B & A \end{bmatrix},$$

$$B_k = \begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

where $A$ and $B$ are the square matrices from Eq. (1). Then one can state the following immediate remark:

**Remark 1.** Let $x : [-h, \infty) \rightarrow \mathbb{R}^n$ be a continuous function satisfying Eq. (1) for $t \geq 0$. Let $k$ be a positive integer and let $y : [(k - 2)h, \infty) \rightarrow \mathbb{R}^{nk}$ be the function defined by

$$y(t) = \begin{pmatrix} x(t - (k - 1)h) \\ x(t - (k - 2)h) \\ \vdots \\ x(t) \end{pmatrix}.$$
Then $y$ satisfies the equation

$$Dy(t) = A_k y(t) + B_k \phi(t - kh)$$

in the interval $[(k - 1)h, \infty)$.

A converse result is also true.

**Remark 2.** Let $k$ be a positive integer and let $y : [(k - 1)h, kh] \rightarrow \mathbb{R}^{n_k}$ and $\phi : [-h, 0] \rightarrow \mathbb{R}^{n}$ be two continuous functions satisfying the equation $Dy(t) = A_k y(t) + B_k \phi(t - kh)$ in the interval $((k - 1)h, kh)$. Let $y_i$, $i = 1, 2, \ldots, k$, $(y_i \in \mathbb{R}^{n_i})$, be the "components" of $y$, in the form

$$y = \left( \begin{array}{c} y_1 \\ \vdots \\ y_k \end{array} \right)$$

and suppose that

$$y_1((k - 1)h) = \phi(0),$$

$$y_j((k - 1)h) = y_{j-1}(kh), \quad j = 2, 3, \ldots, k.$$  

Then the function $x : [-h, kh] \rightarrow \mathbb{R}^{n_k}$, defined by

$$x(t) = \begin{cases} 
\phi(t) & \text{for } -h < t < 0, \\
y_1(t + (k - 1)h) & \text{for } 0 \leq t < h, \\
y_2(t + (k - 2)h) & \text{for } h \leq t < 2h, \\
\vdots \\
y_k(t) & \text{for } (k - 1)h \leq t < kh,
\end{cases}$$

is continuous and satisfies Eq. (1) in the interval $(0, kh)$.

If in Lemma 1 one takes $\tilde{A} = A_k$ and $\tilde{B} = B_k$, then $\tilde{n} = nk$ and $\tilde{m} = n$ and Eqs. (5)-(7) and (11) are replaced by

$$S_k(\sigma) = S_{0k} + S_{1k} \sigma + S_{2k} \sigma^2 + \cdots + S_{n_k-1,k} \sigma^{n_k-1},$$

$$\det(\sigma I_k - A_k) = \alpha_{0k} + \alpha_{1k} \sigma + \alpha_{2k} \sigma^2 + \cdots + \alpha_{n_k},$$

$$S_{1k} = B_k \alpha_{t+1,k} + A_k B_k \alpha_{t+2,k} + \cdots + A_k^{n_k-t-1} B_k,$$

$$t = 0, 1, \ldots, nk - 1.$$  

From Lemma 1 it follows that, for every function $s : [(k - 1)h, kh] \rightarrow \mathbb{R}^{n}$, $nk$ continuously differentiable, the functions $y : [(k - 1)h, kh] \rightarrow \mathbb{R}^{n_k}$ and $\phi : [-h, 0] \rightarrow \mathbb{R}^{n}$, defined by

$$y(t) = S_{nk}s(t) + S_{1k} Ds(t) + \cdots + S_{nk-1,k} D^{nk-1}s(t),$$

where $D$ is the derivative.
and
\[
\phi(t - kh) = \alpha_{0k} s(t) + \alpha_{1k} Ds(t) + \cdots + D^{nk}s(t)
\] (25)
satisfy the differential equation
\[
Dy(t) = A_k y(t) + B_k \phi(t - kh)
\] (26) in the interval \(((k - 1)h, kh)\). Observe also that
\[
S_k(\sigma) = \left( \begin{array}{c}
S(\sigma)(\det(\sigma I - A))^{k-1} \\
(S(\sigma))^2(\det(\sigma I - A))^{k-2} \\
\vdots \\
(S(\sigma))^{k-1} \det(\sigma I - A) \\
(S(\sigma))^k
\end{array} \right).
\] (27)
Indeed, from (13) one obtains
\[
\det(\sigma I_k - A_k) = (\det(\sigma I - A))^k
\] and from Eqs. (13), (14) and (3) one easily finds that the right-hand member of Eq. (27) satisfies Eq. (20)—which uniquely determines \(S_k(\sigma)\).

(III) It is easy to see that if Eq. (1) is p.d. (for \(q\)) at \(t_1\) (Definition 1) then Eq. (1) is p.d. (for \(q\)) at every \(t\) from the interval \([t_1, \infty)\). Indeed, consider any number \(t_2 > t_1\). For every continuous function \(x : [t_1, t_2] \rightarrow \mathbb{R}^n\), satisfying Eq. (1) in \((0, t_2)\), the “shifted” function \(\tilde{x} : [-h, t_2] \rightarrow \mathbb{R}^n\), defined by \(\tilde{x}(t) = x(t + t_2 - t_1)\), is continuous and satisfies Eq. (1) in \((0, t_1)\). Since Eq. (1) is p.d. (for \(q\)) at \(t_1\), one has \(q^T\tilde{x}(t_1) = 0\), that is \(q^Tx(t_2) = 0\). Hence the conclusion.

(IV) Now one proves the existence of the number \(l\) from Theorem 1. Suppose that Eq. (1) is p.d., for \(q\), at \(t_1\). Hence, as shown above, Eq. (1) is p.d. in the whole interval \([t_1, \infty)\). Thus there exists an integer \(k > 0\) and a number \(t_2 \in ((k - 1)h, kh)\) such that Eq. (1) is p.d., for \(q\), at \(t_2\). If one introduces the \(nk\)-vector
\[
q_k^T = \begin{pmatrix} 0 & 0 & \cdots & 0 & q^T \end{pmatrix},
\] (28)
one can prove that \(q_k^T S_k(\sigma) = 0\). Indeed, for every integer \(j \in [0, nk - 1]\) one can determine a polynomial function \(s : [(k - 1)h, kh] \rightarrow \mathbb{R}^n\) satisfying the conditions
\[
D^i s((k - 1)h) = D^i s(kh) = 0, \quad i = 0, 1, \ldots, nk,
\] (29)
\[
D^i s(t_2) = \begin{cases} 0 & \text{if } i \neq j \text{ and } i \in [0, nk], \\
\sigma_i^{T_k} q_k & \text{if } i = j.
\end{cases}
\] (30)
Then, as observed before, the corresponding functions \( y \) and \( \phi \), given by Eqs. (24) and (25), satisfy Eq. (26). From Eqs. (30) and (24) one obtains that

\[
q_k^T y(t_2) = q_k^T S_{jk} S_{jk}^T q_k.
\]  

(31)

Since one has \( y((k - 1)h) = y(kh) = 0 \) and \( \phi(0) = 0 \) [see (29), (24) and (25)], Eqs. (18) are satisfied. Therefore (Remark 2) the function \( x \), given by Eq. (19) satisfies Eq. (1) in \((0, kh)\). From Eqs. (28), (17) and (19) one obtains [since \( t_2 \in ((k - 1)h, kh) \)] \( q_k^T y(t_2) = q^T x(t_2) \). The right-hand member is zero since Eq. (1) was supposed to be p.d., for \( q \), at \( t_2 \). Hence \( q_k^T S_{jk} = 0 \) [31]. Since \( j \) is arbitrary, this implies \( q_k^T S_k(\sigma) = 0 \) [21]. This is equivalent to \( q^T (S(\sigma))^k = 0 \) [27, 28]. Since \( q \neq 0 \), it immediately follows that there exists a minimal integer \( l \) with the properties from Theorem 1.

(V) One proves now that the largest set of points at which Eq. (1) is p.d. (for the considered \( q \)) is the interval \([lh, \infty)\). Equation (1) cannot be p.d. (for \( q \)) at a point \( t_1 < lh \) because then Eq. (1) is p.d. (for \( q \)) at a point \( t_1 \in ((l - 1)h, lh) \) and this implies [see Section (IV) of this proof] \( q^T (S(\sigma))^k = 0 \), for \( k = l \), which contradicts the definition of \( l \) (see Theorem 1).

It remains to show that, if Eq. (1) is p.d. (for \( q \)) at \( t_1 > 0 \) then Eq. (1) is p.d. (for \( q \)) at every \( t \) from the interval \([lh, \infty)\). From \( q^T (S(\sigma))^t = 0 \) (see the definition of the number \( l \) in Theorem 1) or \( q_{t+1}^T S_{t+1}(\sigma) = 0 \) [see (27) and (28), for \( k = l + 1 \)] one obtains

\[
q_{t+1}^T A_{t+1}^T B_{t+1} = 0, \quad t = 0, 1, 2, ..., n(l + 1) - 1
\]  

(32)

[see (21) and (23), for \( k = l + 1 \)]. Let \( x \) and \( y \) be defined as in Remark 1 (for \( k = l + 1 \)). Then from (16) and (32) one obtains

\[
D^i[q_{t+1}^T y(t)] - q_{t+1}^T A_{t+1}^T y(t), \quad i = 0, 1, 2, ..., n(l + 1) - 1, \quad t > lh.
\]

Hence, if one defines

\[
\xi(t) = q_{t+1}^T y(t) = q^T x(t), \quad t > lh
\]  

(33)

[see (15) and (28), for \( k = l + 1 \)] and uses the theorem of Cayley–Hamilton, one sees that \( \xi \) satisfies, for every \( t > lh \), the ordinary, homogeneous differential equation [see (22), for \( k = l + 1 \)]

\[
\alpha_{0, t+1} \xi(t) + \alpha_{1, t+1} D \xi(t) + \cdots + D^{n(t+1)-1} \xi(t) = 0.
\]  

(34)

However \( \xi(t) = 0 \) for \( t \geq t_1 \) [since Eq. (1) is p.d., for \( q \), at \( t_1 \) and hence in the whole interval \([t_1, \infty)\)]. This implies \( \xi(t) = 0 \) for every \( t > lh \), and our conclusion follows from (33) and the continuity of \( x \) at \( t = lh \).

(VI) It remains to prove that \( l \geq 2 \). If \( l < 2 \) then Eq. (1) is p.d. at
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every point in \([lh, \infty)\) and, in particular, at \(t = h\). One applies now Lemma 1 for \(A = A\) and \(B = B\) (then \(n = \tilde{m} = n\)). Choose an arbitrary integer \(j \in [0, n - 1]\) and consider a polynomial function \(s : [0, h] \to R^n\) satisfying the conditions

\[
D^i s(0) = 0, \quad i = 0, 1, 2, \ldots, n,
\]

\[
D^j s(h) = \begin{cases} 
0 & \text{if } i \neq j, \ i \in [0, n - 1], \\
S_i^T q & \text{if } i = j, \\
-\tilde{x}_i S_i^T q & \text{if } i = n,
\end{cases}
\]  

(35)

where \(q\) is a vector for which Eq. (1) is p.d. at \(t = h\). Then the functions \(\tilde{x}\) and \(u\), given by (8) and (9) satisfy the equations

\[
\tilde{x}(0) = u(h) = 0.
\]  

(36)

According to Lemma 1 these functions satisfy the equation \(D\tilde{x} = A\tilde{x} + Bu\), in the interval \((0, h)\). In other words, the function \(x : [-h, h] \to R^n\), defined by

\[
x(t) = \begin{cases} 
(u(t + h) & \text{for } -h \leq t < 0, \\
(\tilde{x}(t)) & \text{for } 0 \leq t \leq h,
\end{cases}
\]

satisfies Eq. (1) in the interval \((0, h)\). Moreover, this function is continuous, as a consequence of (36). Since Eq. (1) is p.d., for \(q\), at \(t = h\), one has \(q^T x(h) = 0\) or \(q^T S_t^T S_i^T q = 0\) [see (8) and (35)]. Since \(j\) is arbitrary, this implies \(q^T S(\sigma) = 0\) [see (6)]. Therefore \(l = 0\) (see the definition of \(l\) in Theorem 1 and note that \(q \neq 0\)). Thus Eq. (1) is p.d. at every point in \([lh, \infty) = [0, \infty)\). This is an obvious contradiction [since one can chose \(x(0) = q \neq 0\)]. Theorem 1 is proved.

(VII) In Section 4 one needs the following result (which was introduced and proved before in [6]).

**Corollary 1.** If Eq. (1) is pointwise degenerate, then \(\text{rank } B \geq 2\).

**Proof.** If \(\text{rank } B < 2\), \(B\) can be written as \(B = bc^T\), where \(b\) and \(c\) are \(n\)-vectors. Using the matrix \(R(\sigma)\) from Eq. (4) one obtains \(S(\sigma) = R(\sigma)B = R(\sigma)b c^T = r(\sigma)c^T\), where \(r(\sigma) = R(\sigma)b\). The number \(l\) given by Theorem 1 satisfies, by definition, the conditions \(q^T(S(\sigma))^l \neq 0\) and \(q^T(S(\sigma))^{l+1} = 0\). Or, since \(S(\sigma) = r(\sigma)c^T\), \(q^T r(\sigma)(c^T r(\sigma))^{l-1} c^T \neq 0\) and \(q^T r(\sigma)(c^T r(\sigma))^l c^T = 0\). Since \(l \geq 2\) and \(c^T r(\sigma)\) is a scalar number, the last two conditions are contradictory and the corollary is proved.
3. Algebraic Criteria of Pointwise Degeneracy

**Theorem 2.** Equation (1) is pointwise degenerate, for the vector \( q \neq 0 \), at time \( t_1 > 0 \) (Definition 1) if there exist: an integer \( m > 0 \), \( k \) matrices \( P_j, m \times n \), (where \( k \) is the largest integer such that \( kh \leq t_1 \)), an \( m \times m \) matrix \( V \) and an \( m \)-vector \( v \) such that

\[
P_1 B = 0, \tag{37}
\]

\[
P_j A + P_{j+1} B = VP_j, \quad j = 1, 2, \ldots, k - 1, \tag{38}
\]

\[
P_k A = VP_k, \tag{39}
\]

\[
v^T e^{vh} P_1 = 0, \tag{40}
\]

\[
v^T e^{vh} P_{j+1} - v^T P_j = 0, \quad j = 1, 2, \ldots, k - 1, \tag{41}
\]

\[
v^T P_k = q^T. \tag{42}
\]

Moreover, if the above quantities exist, one can always choose them such that

\[
\text{rank}(P_1, P_2, P_3, \ldots, P_k) = m. \tag{43}
\]

**Proof.** (I) Suppose that Eqs. (37)-(42) are satisfied. Then for every continuous function \( x : [-h, \infty) \to R^n \), satisfying Eq. (1) for \( t > 0 \), the function \( z : [(k - 1)h, \infty) \to R^n \), defined by

\[
z(t) = P_1 x(t - (k - 1)h) + P_2 x(t - (k - 2)h) + \cdots + P_k x(t)
\]

satisfies the equation \( Dz(t) = Vz(t) \), for \( t \geq (k - 1)h \) [see Eqs. (1), (37)-(39)]. Therefore, for \( t = kh \), one has \( z(t) = e^{vh} x(t - h) \). Multiplying this equation, on the left, with \( v \), and using Eqs. (40)-(42), one obtains \( q^T x(t) = 0 \) for \( t \geq kh \). In particular, this shows that Eq. (1) is p.d., for \( q \) at \( t_1 \).

(II) To establish the converse property, one needs the following lemma:

**Lemma 2.** Let \( A_k \) and \( B_k \) be given by Eqs. (13) and (14). If the equations

\[
p^T A_k B_k = 0, \quad i = 0, 1, 2, \ldots, nk - 1, \tag{44}
\]

have a solution \( p \neq 0 \), let \( p_j, j = 1, 2, \ldots, m \) be a maximal set of linearly independent \( nk \)-vectors with the property

\[
p_j^T A_k B_k = 0, \quad i = 0, 1, \ldots, nk - 1, \quad j = 1, 2, \ldots, m. \tag{45}
\]
Let $P$ be the $m \times nk$ matrix

$$P = \begin{pmatrix} p_1^T \\ \vdots \\ p_{nk}^T \end{pmatrix},$$

(46)

and let $P_i$, $i = 1, 2, \ldots, k$, be the $m \times n$ matrices which are obtained when $P$ is partitioned as

$$P = (P_1 \quad P_2 \quad \cdots \quad P_k).$$

(47)

Let $V$ be the $m \times m$ matrix such that

$$PA_i = VP.$$

(48)

Let $x_0, x_1, \ldots, x_k$ be a set of $n$-vectors such that

$$e^{\nu h}P_1x_0 + (e^{\nu h}P_2 - P_1)x_1 + (e^{\nu h}P_3 - P_2)x_2 + \cdots + (e^{\nu h}P_k - P_{k-1})x_{k-1} - P_kx_k = 0,$$

(49)

if $P_i$ are defined as above. [The vectors $x_i$ are completely arbitrary if Eq. (44) is satisfied only by $p = 0$.]

Then there exists a continuous function $x : [-h, kh] \to \mathbb{R}^n$ which satisfies Eq. (1) for $0 < t < kh$ and the conditions

$$x(jh) = x_j, \quad j = 0, 1, \ldots, k.$$

(50)

Before proving this lemma we observe that a converse property is also true, but we do not need it in this paper. Observe that $V$ is well defined by Eq. (48). Indeed, from Eqs. (45) and (46) one obtains $PA_i^TB_k = 0$, $i = 0, 1, \ldots, nk - 1$. Hence $(PA_k)A_k^TB_k = 0$, $i = 0, 1, \ldots, nk - 1$ (one uses the theorem of Cayley–Hamilton). Thus any row of the matrix $PA_k$ satisfies a condition like (44) and therefore is a linear combination of the vectors $P_i^T$ from (46) [since these vectors form a maximal set of linearly independent vectors with the property (45)]. Hence one obtains Eq. (48). The uniqueness of $V$ follows from the fact that rank $P = m$.

One proves now that the conditions concerning $x_0, x_1, \ldots, x_k$, required in Theorem 2, are satisfied iff one can solve the equation

$$S_0x_0 + S_{1h}x_1 + \cdots + S_{nh-1, h}x_{n-1} = x_1 - e^{\nu h}x_0,$$

(51)
where \( s_j \) are \( n \)-vectors, \( z_0 \) and \( z_1 \) are two \( nk \)-vectors given by

\[
\begin{pmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{k-1} \\
x_k
\end{pmatrix}, \quad \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_k
\end{pmatrix}
\]

(52)

and \( S_{jk} \) are obtained from Eqs. (20) and (21). Indeed, Eq. (51) can be solved iff any \( nk \)-vector \( p \) which satisfies the equations \( p^T S_{jk} = 0, j = 0, 1, \ldots, nk - 1 \) is orthogonal to the right-hand member of (51). Using the expression of \( S_{jk} \) [see Eq. (23)] one sees that the conditions \( p^T S_{jk} = 0 \) are equivalent to (44). Thus Eq. (51) can be solved if there exists no vector \( p \neq 0 \) with the property (44). If however such a vector exists, then, using the definition of \( P \) and the above remark, one finds that Eq. (51) can be solved iff

\[
P(z_1 - e^{Ah} z_0) = 0.
\]

(53)

From Eq. (48) one obtains \( PA_k^i = V^i P, i = 0, 1, \ldots \) and hence \( P e^{Ah} = e^{Ah} P \). Therefore Eq. (53) becomes \( Pz_1 - e^{Ah} Pz_0 = 0 \). This condition takes the form (49) if one uses Eqs. (47) and (52).

One shows now that if Eq. (51) can be solved then there exists a continuous function \( x \) satisfying Eq. (1) and conditions (50). Indeed, if \( s_j, j = 0, 1, \ldots, nk - 1 \) satisfy Eq. (51), one can always find a new \( n \)-vector \( s_{nk} \) such that

\[
\alpha_{0k} s_0 + \alpha_{1k} s_1 + \cdots + \alpha_{nk-1, k} s_{nk-1} + s_{nk} = x_0,
\]

(54)

where \( \alpha_{jk} \) are given by Eq. (22). Choose now a polynomial function \( s : [(k - 1)h, kh] \rightarrow \mathbb{R}^n \) such that

\[
D^i s((k - 1)h) = 0, \quad i = 0, 1, 2, \ldots, nk,
\]

(55)

\[
D^i s(kh) = s_i, \quad i = 0, 1, 2, \ldots, nk.
\]

(56)

Then the functions \( y \) and \( \phi \), defined by

\[
y(t) = S_{0k}s(t) + S_{1k}Ds(t) + \cdots + S_{nk-1, k}D^{nk-1}s(t) + e^{Ak(t-(k-1)h)}z_0,
\]

(57)

\[
\phi(t - kh) = \alpha_{0k}s(t) + \alpha_{1k}Ds(t) + \cdots + \alpha_{nk-1, k}D^{nk-1}s(t) + D^{nk}s(t),
\]

(58)

satisfy the differential equation

\[
Dy(t) = A_k y(t) + B_k \phi(t - kh) \quad \text{for} \quad (k - 1)h < t < kh.
\]

Indeed, for \( z_0 = 0 \), Eqs. (57) and (58) become identical to Eqs. (24) and (25).
DEGENERACY OF DELAY EQUATIONS

and it was proved before that in this case Eq. (26) is satisfied. It immediately follows that the same equation is satisfied even if \( z_0 \neq 0 \).

From Eqs. (57) and (55) one obtains \( y((k - 1)h) = z_0 \), from Eqs. (57), (56) and (51) one obtains \( y(kh) = z_1 \), and from Eqs. (58), (56) and (54) one obtains \( \phi(0) = x_0 \). Using Eqs. (52) and (17) one sees that Eqs. (18) are satisfied. Therefore (Remark 2) the corresponding function \( x \), given by Eq. (19), satisfies Eq. (1) in the interval \((0, kh)\). Equations (50) follow from Eqs. (17), (19) and (52), since, as shown above, one has \( y((k - 1)h) = z_0 \) and \( y(kh) = z_1 \).

(III) One uses now Lemma 2 to finish the proof of Theorem 2. Suppose that Eq. (1) is p.d., for \( q \), at \( t_1 \) and that the integer \( k \) is chosen such that \( kh \leq t_1 < (k + 1)h \). Then from Theorem 1 it follows that \( l \leq k \) and therefore Eq. (1) is p.d. (for \( q \)) at \( t = kh \). If there exists no vector \( p \neq 0 \) with the property (44) then (Lemma 2) the vectors \( x_0, x_1, ..., x_k \) from Eq. (50) can be chosen arbitrarily. Choosing \( x_k = q \), one obtains, from Eq. (50), \( q^T x(kh) = q^T q \neq 0 \). This contradicts the fact that Eq. (1) is p.d. (for \( q \)) at \( t = kh \). Therefore there exists \( p \neq 0 \) such that Eqs. (44) are satisfied. Thus the matrices \( P \) and \( V \) from Lemma 2 can be defined. Then from Eqs. (48), (47) and (13) one obtains Eqs. (38)-(39). Equation (37) follows from Eqs. (44)-(45).

Suppose now that there exists no vector \( v \) with the properties (40)-(42). This implies that the system formed by Eq. (49) and

\[
q^T x_k = 1
\]  

(59)

has a solution \( x_0, x_1, ..., x_k \). Therefore (Lemma 2) there exists a solution of Eq. (1) with the property \( q^T x(kh) = 1 \) [which follows from Eqs. (50) and (59)]. This again contradicts the assumption that Eq. (1) is p.d. (for \( q \)) at \( t_1 \) and shows that there exists a vector \( v \) satisfying Eqs. (40)-(42).

Since the vectors \( p_i \), in the expression (46) of the matrix \( P \), are linearly independent, the rank of matrix \( P \) is maximal and from Eq. (47) one obtains Eq. (43). Theorem 2 is proved.

The following result expresses the property of pointwise degeneracy in a more conventional way.

**Corollary 2.** Equation (1) is p.d. at \( t = kh \) (where \( k \) is a positive integer) iff

\[
\text{rank}(B_k A_k B_k \cdots A_k^{nk-1} B_k) < nk
\]

(60)

and

\[
\text{rank}(W_1 W_2 \cdots W_k) < m.
\]

(61)
where $A_\ell$ and $B_\ell$ are given by (13) and (14), $W_\ell$ are the $m \times n$ matrices defined by

$$W_1 = e^{\ell b} P_1,$$

$$W_j = e^{\ell b} P_j - P_{j-1}, \quad j = 2, 3, \ldots, k,$$

and $P_\ell$ and $V$ are defined as in Lemma 2.

Proof. If Eq. (1) is p.d. at $t = kh$ (for some vector $q$) then (Theorem 2) Eqs. (37)-(42) are satisfied and from the proof of Theorem 2 it follows that the matrices $P_\ell$ and $V$ can be determined as in Lemma 2. Since the vectors $p_\ell$ from Eq. (46) satisfy Eq. (45), one obtains condition (60). From Eqs. (40) and (41) one obtains $v^T W_j = 0, j = 1, 2, \ldots, m$ [see Eqs. (62) and (63)] and from Eq. (42), since $q \neq 0$, it follows that $v \neq 0$; hence (61).

Suppose now that conditions (60) and (61) are satisfied. Then [see (60)] one can determine $P_\ell$ and $V$ as in Lemma 2. Moreover there exists a vector $v \neq 0$ such that $v^T W_j = 0, j = 1, 2, \ldots, k$ [see (61)]. This gives Eqs. (37)-(41) [see Eqs. (45)-(48), (13), (14), (62) and (63)]. If one defines $q$ by Eq. (42) then Eqs. (37)-(42) are satisfied and thus (Theorem 2), if $q \neq 0$, Eq. (1) is p.d. at $t = kh$. It remains to prove that $q \neq 0$, that is [Eq. (42)] that $v^T P_k \neq 0$.

If $v^T P_k = 0$, Eqs. (37)-(42) can be written as

$$P_\ell A + P_{\ell+1} B = V P_\ell, \quad j = 0, 1, \ldots, k + 1,$$

$$v^T e^{\ell b} P_{\ell-1} - v^T P_\ell = 0, \quad j = 0, 1, 2, \ldots, k + 1,$$

where, for convenience, we introduced the matrices $P_0 = P_{k+1} = P_{k+2} = 0$. Then from Eqs. (65) one obtains

$$(v^T e^{\ell b} P_{\ell+1} v^T P_\ell) A + (v^T e^{\ell b} P_{\ell+1} v^T P_{\ell+1}) B = 0$$

and thus gives, using Eq. (64),

$$v^T e^{\ell b} V P_{\ell+1} - v^T V P_\ell = 0, \quad j = 0, 1, 2, \ldots, k.$$ 

Hence Eqs. (40)-(42) (with $q = 0$) are also satisfied if $v^T$ is replaced by $v^T V$. Therefore this is also true if $v^T$ is replaced by $v^T V^i$, $i = 0, 1, 2, \ldots$, or by

$$v^T = \beta_0 v^T + \beta_1 v^T + \cdots + \beta_{m-1} v^T V^{m-1}.$$ 

One can choose $\beta_i$ such that $v \neq 0$ and $v^T V = \rho v^T$ (for some scalar $\rho$). Then one has $v^T V^i = \rho^i v^T$ for $i = 0, 1, 2, \ldots$ and further $v^T e^{\ell b} = e^{\rho b} v^T$. Then from Eqs. (40) and (41)—with $v^T$ replaced by $v^T$—one obtains successively $v^T P_\ell = 0, i = 0, 1, \ldots, k$. Since $v \neq 0$, this contradicts (43). Therefore $v^T P_k \neq 0$ and the corollary is proved.
4. A Class of Pointwise-Degenerate Systems

From Theorem 1 it follows that, for every p.d. system of the form (1), one has \( l \geq 2 \). The limit case, \( l = 2 \), is worth a special study. Let us introduce the following class of systems:

**Definition 2.** A pointwise-degenerate system of the form (1) is called "regular" iff the pair \((A, B)\) is completely controllable and there exists an \( n \)-vector \( q \) such that the pair \((q^T, A)\) is completely observable and Eq. (1) is p.d., for \( q \), at every \( t \) in the interval \([2h, \infty)\).

We recall that the pair \((q^T, A)\) is called completely observable iff the vectors \( q^T A^i \), \( i = 0, 1, \ldots, n - 1 \), are linearly independent; the pair \((A, B)\) is called completely controllable iff

\[
\text{rank}(B \begin{bmatrix} AB & \cdots & A^{n-1}B \end{bmatrix}) = n
\]

(see [9] or [2]).

The regular p.d. systems have a simple structure, given by the following theorem:

**Theorem 3.** Suppose that Eq. (1) is pointwise degenerate and regular. Let \( q \) be a vector with the properties from Definition 2. Then there exists an \( n \times n \) matrix \( Z \) such that

\[
ZAZ = Z^2 A,
\]

\[
q^T Z^2 = 0,
\]

\[
q^T Z = q^T e^{A h},
\]

and Eq. (1) has the form

\[
Dx(t) = Ax(t) + (AZ - ZA)x(t - h)
\]

(that is, \( B = AZ - ZA \)). Conversely, every equation of the form (69), in which \( Z \) satisfies Eqs. (66)-(68), is pointwise degenerate, for \( q \), at every point in the interval \([2h, \infty)\).

**Proof.** (I) Since \( l = 2 \), the necessary and sufficient conditions of pointwise degeneracy become (Theorem 2)

\[
P_1 B = 0,
\]

\[
P_1 A + P_2 B = VP_1,
\]

\[
P_2 A = VP_2,
\]

\[
V^T e^{Vh} P_1 = 0,
\]

\[
V^T e^{Vh} P_2 - VP_1 = 0,
\]

\[
V^T P_2 = q^T,
\]
where $P_1$ and $P_2$ are two $m \times n$ matrices ($m > 0$), $V$ is an $m \times m$ matrix and $v$ is an $m$-vector. Moreover

$$\text{rank}(P_1 P_2) = m. \quad (76)$$

(II) One proves now that

$$w^T P_2 = 0 \iff w = 0. \quad (77)$$

Indeed, from $w^T P_2 = 0$ and from Eq. (72) it follows that the vector $w^T V$ has also the property $w^T V P_2 = 0$. Therefore,

$$w^T V^i P_2 = 0, \quad i = 0, 1, \ldots. \quad (78)$$

Observe now that

$$w^T P_1 A^i = w^T V^i P_1, \quad i = 0, 1, \ldots, \quad (79)$$

because this relation is true for $i = 0$ and—assuming that it is true for an arbitrary $i = j \geq 0$—one obtains that the relation is also true for $i = j + 1$. [One uses successively Eq. (79)—for $i = j$—and Eqs. (71) and (78), as follows:

\[
\begin{align*}
\begin{align*}
w^T P_1 A^{i+1} &= (w^T P_1 A^i)A = (w^T V^i P_1)A = w^T V^i (P_1 - P_2 B) \\
&= w^T V^{i+1} P_1 - (w^T V^i P_2) B = w^T V^{i+1} P_1.
\end{align*}
\end{align*}
\]

From Eqs. (79) and (70) one obtains $(w^T P_2) A^i B = 0$ for $i = 0, 1, \ldots$. Since the pair $(A, B)$ is completely controllable, this implies $w^T P_1 = 0$. From (76) it follows that the only vector $w$ with the properties $w^T P_1 = w^T P_2 = 0$ is $w = 0$. This proves (77). We remark that (77) was proved using only Eqs. (70)–(76) and the condition that the pair $(A, B)$ is completely controllable.

(III) One also has

$$P_2 w = 0 \iff w = 0. \quad (80)$$

Indeed, observe that one has the relation

$$q^T A^i = v^T V^i P_2 \quad (81)$$

because for $i = 0$ this relation follows from Eq. (75) and—assuming that the relation is true for an arbitrary $i = j \geq 0$—one finds that the relation is also true for $i = j + 1$. [One uses Eq. (81)—for $i = j$—and Eq. (72), as follows:

\[
\begin{align*}
\begin{align*}
q^T A^{i+1} &= (q^T A^i)A = (v^T V^i P_2) A = v^T V^i (P_2 A) = v^T V^i (V P_2) = v^T V^{i+1} P_2.
\end{align*}
\end{align*}
\]

Therefore from $P_2 w = 0$ one obtains $q^T A^i w = 0, i = 0, 1, \ldots$, and since the pair $(q^T, A)$ is completely observable, this implies $w = 0$—which proves (80).
From (77) and (80) it follows that matrix \( P_2 \) is square \((m = n)\) and non-singular. Therefore Eq. (72) gives

\[
V = P_2 AP_2^{-1}. \tag{82}
\]

Define the \( n \times n \) matrix \( Z \) as \( Z = P_2^{-1}P_1 \). Then Eq. (71) gives \( B = AZ - ZA \) and therefore Eq. (1) takes the form (69). Moreover from Eq. (70) and the above expression of \( B \) one obtains Eq. (66). From Eq. (75) one obtains

\[
v^T = q^T P_2^{-1}
\]

and further

\[
v^T e^h = q^T e^{Ah} P_2^{-1} \tag{83}
\]

[because, using Eq. (82), one has

\[
v^T e^h = q^T P_2^{-1} e^{vh} P_2 P_2^{-1} = q^T [\exp(P_2^{-1} V P_2)] P_2^{-1} = q^T e^{Ah} P_2^{-1}.
\]

Therefore Eq. (74) takes the form (68) and Eq. (73) becomes \( q^T e^{4h} Z = 0 \). Hence, using Eq. (68), proved above, one obtains Eq. (67).

Finally, to prove the last part of the theorem, it only remains to observe that, if \( B = AZ - ZA \), the quantities \( P_1 = Z, P_2 = I, V = A \) and \( v^T = q^T \) satisfy all the relations (70)-(76), as a consequence of Eqs. (66)-(68). [One also uses the relation \( q^T e^{4h} Z = 0 \), which follows from Eqs. (67) and (68).]

Note that the last statement of Theorem 3 holds true even if the pair \((A, B)\) is not completely controllable and the pair \((q^T, A)\) is not completely observable. However, the theorem, as a whole, is not true without these assumptions.

5. Pointwise-Degenerate Systems of the Third Order

**Theorem 4.** Any pointwise-degenerate system of the form (1), with \( n = 3 \), is regular (Definition 2) and the rank of \( B \) is 2.

**Proof.** (I) If \( n = 3 \), one has \( l = 2 \). Indeed, according to the definition of \( l \) (Theorem 1), there exists a number \( \sigma_0 \) such that

\[
q^T (S(\sigma_0))^{i+1} = 0 \quad \text{and} \quad q^T (S(\sigma_0))^i \neq 0. \tag{84}
\]

We introduce the coefficients \( \gamma_j, i = 0, 1, 2, 3 \) and the integer \( j \in [0, 3] \) such that

\[
det(\lambda I - S(\sigma_0)) = \lambda \gamma_j + \cdots + \lambda^3, \quad \gamma_j \neq 0
\]

[thus \( \gamma_j \) is the first nonvanishing coefficient of the characteristic equation of matrix \( S(\sigma_0) \)]. Using the theorem of Cayley–Hamilton one obtains

\[
(S(\sigma_0))^{i+1} + \cdots + (S(\sigma_0))^3 = 0.
\]
If $l > 2$, then $l \geq j$ and the above equation can be multiplied, on the left, with $q^T(S(\sigma_n))^{l-j}$. Using (84), this gives $\gamma_j = 0$, a contradiction which shows that $l \leq 2$. [Observe that this proof can be carried out even for an arbitrary $n$ and gives the general conclusion $l \leq (n - 1)$.] Since $l \geq 2$ (Theorem 1), one obtains $l = 2$. Consequently, Eqs. (70)–(76) are satisfied (Theorem 2).

(II) If $n = 3$, one also has rank $B = 2$. Indeed, since $l = 2$ one has $q^T(S(\sigma))^3 = 0$ (Theorem 1) and since $\lim_{\sigma \to \infty} S(\sigma)/\sigma^2 = B$ [see (5), (6) and (10), for $n = 3$], one obtains

$$q^TB^3 = 0. \tag{85}$$

Since $q \neq 0$, this implies that rank $B < 3$ (which also follows from the result of E. B. Lee, mentioned in the introduction). But rank $B \geq 2$ (Corollary 2); therefore, rank $B = 2$.

Observe now that from (74) and (70) one obtains

$$v^Te^{\theta_h}P_2B = 0. \tag{86}$$

On the other hand, from (72) it follows that

$$P_2e^{\theta_h} = e^{\theta_h}P_2 \tag{87}$$

and using (75) one finds

$$v^Te^{\theta_h}P_2 = q^Te^{\theta_h}. \tag{88}$$

Hence, since $q \neq 0$, one sees that the vector which multiplies $B$ in (86) is different from zero. Since $n = 3$ and rank $B = 2$ one has

$$r^TB = 0 \quad \text{iff} \quad r^T = \rho v^Te^{\theta_h}P_2 \quad \text{(for some $\rho$)}. \tag{89}$$

In particular, since $P_1B = 0$ [Eq. (70)], one has

$$P_1 = \rho v^Te^{\theta_h}P_2, \tag{90}$$

and substituting this expression in Eqs. (74), (73) and in the equation $\nu^Te^{\theta_h}VP_1 = 0$ [which is obtained by multiplying Eq. (71), on the left, with $\nu^Te^{\theta_h}$ and using Eqs. (74), (73) and (70)], one sees that the vector $\rho$ from (90) must satisfy the equations

$$\nu^T\rho = 1, \tag{91}$$

$$\nu^Te^{\theta_h}\rho = 0, \tag{92}$$

$$\nu^Te^{\theta}V\rho = 0. \tag{93}$$
The vectors $v^T$, $v^T e^{Vh}$ and $v^T e^{VhV}$, in the above equations, are linearly independent. To show this, consider the equation

$$\alpha v^T + \beta v^T e^{Vh} + \gamma v^T e^{VhV} = 0 \quad (94)$$

with scalar coefficients. Multiplying this equation, on the right, with $p$, and using Eqs. (91)-(93), gives $\alpha = 0$. If $\gamma = 0$, then $\beta = 0$ [since $v^T e^{Vh} \neq 0$, as a consequence of (91)]. Finally, if $\gamma \neq 0$ and $\alpha = 0$, one divides Eq. (94) by $\gamma$ and one obtains $v^T e^{Vh} = \delta v^T e^{Vh}$, where $\delta = -\beta/\gamma$. Hence, since $e^{Vh}V = Ve^{Vh}$, one further obtains $(v^T V - \delta v^T) e^{Vh} = 0$, or $v^T V = \delta v^T$ and also $v^T e^{Vh} = e^{\delta h} v^T$. Introducing this expression in (92) gives $v^T p = 0$, which contradicts Eq. (91) and shows that Eq. (94) can be satisfied only if $x = \beta = \gamma = 0$. An immediate consequence of the fact that the $m$-vectors $v^T$, $v^T e^{Vh}$ and $v^T e^{VhV}$ are linearly independent is the inequality $m \geq 3$.

(III) One shows now that the pair $(A, B)$ is completely controllable. Consider the equations

$$w^T A^i B = 0, \quad i = 0, 1, 2, \ldots, n - 1. \quad (95)$$

For $i = 0$ one obtains $w^T B = 0$ and therefore [see (88) and (89)] there exists a scalar $\rho$ such that $w^T = \rho q^T e^{A h}$. From the theorem of Cayley–Hamilton it follows that Eq. (95) is satisfied for any positive integer $i$. This implies $w^T e^{-A h} A^i B = 0$ or, using the above expression of $w^T$, $\rho q^T A^i B = 0$, for $i = 0, 1, 2, \ldots$. Hence, from Eqs. (6) and (11) (deleting the sign $\sim$) one obtains $\rho q^T S(\sigma) = 0$. On the other hand, since $l = 2$, one has (Theorem 1) $q^T (S(\sigma))^2 \neq 0$. Thus one must have $\rho = 0$ and therefore the only solution of Eq. (95) is $w = 0$ [that is, the pair $(A, B)$ is completely controllable]. Since the pair $(A, B)$ is completely controllable, one obtains the consequence (77), as in Section (II) of the proof of Theorem 3. Since $P_z$ is an $m \times 3$ matrix and $m \geq 3$ [see Section (II) of this proof] one obtains, from (77), that $m = 3$ and $P_z$ is nonsingular.

(IV) It remains to show that the pair $(q^T, A)$ is completely observable. Consider the equations

$$q^T w = q^T A w = q^T A^2 w = 0, \quad (96)$$

which imply [using Eqs. (72) and (75)] $v^T P_{3w} = v^T V P_{3w} = v^T V^2 P_{3w} = 0$. Since $m = 3$, one obtains, using the theorem of Cayley–Hamilton $v^T V P_{3w} = 0$ for $i = 0, 1, 2, \ldots$, which implies $v^T P_{3w} = v^T e^{Vh} P_{3w} = v^T e^{Vh} V P_{3w} = 0$ and hence $P_{3w} = 0$ [since the vectors $v^T$, $v^T e^{Vh}$ and $v^T e^{VhV}$ are linearly independent—as shown in Section (II) of this proof]. Finally, $P_{3w} = 0$ implies $w = 0$ [since $P_a$ is nonsingular, as shown in Section (III) of
this proof]. Therefore the pair \((q^T, A)\) is completely observable and Theorem 4 is proved.

**COROLLARY 3.** Any pointwise-degenerate system of the form (1) with \(n = 3\) can be written as Eq. (69), where

\[
Z = rq^Te^{4h}
\]  
(97)

and \(q\) and \(r\) are two vectors which satisfy the equations

\[
q^Tr = 1, \tag{98}
\]

\[
q^Te^{4h}r = 0, \tag{99}
\]

\[
q^Te^{4h}Ar = 0. \tag{100}
\]

**Proof.** Since the system is regular (see Theorem 4) one can apply Theorem 3. Therefore Eq. (1) can be written as Eq. (69). In Section (III) of the proof of Theorem 3, \(Z\) was defined as \(Z = g'P\). Therefore Eq. (97) is obtained from Eqs. (90) and (88), taking \(r = P^{-1}g\). Equations (98)–(100) are obtained by substituting (97) in Eqs. (66)–(68).

6. **Final Comments**

As shown in this paper, the property of pointwise degeneracy can be reduced to some precise algebraic conditions which become very simple in the case of the “regular” systems, studied in Sections 4 and 5 of this paper.

Obviously, the study of the problem can be further developed from many different points of view. We confine ourselves to mention the following problem: *Given an \(n \times n\) matrix \(A\) and an \(n\)-vector \(q\), to find an \(n \times n\) matrix \(B\) so as to have \(q^T x(t) = 0\) for every \(t \geq 2h\) and for every continuous function \(x: [-h, \infty) \to \mathbb{R}^n\) which satisfies Eq. (1) for \(t > 0\).* A solution of this problem follows readily from Theorem 3: If there exists a vector \(r\) satisfying Eqs. (98)–(100), then the matrix \(B = AZ - ZA\), where \(Z\) is given by Eq. (97), is such a solution. [One has only to remark that Eqs. (66)–(68) are satisfied and thus the result is obtained from the last part of Theorem 3.] Using Corollary 3 one can see that, if \(n = 3\), the only possible solution of this problem has the above form and the solution is unique. The problem can be solved in this way whenever Eqs. (98)–(100) have a solution \(r\). Obviously, these conditions are not restrictive, even in the case \(n = 3\).

Thus, if one is allowed to modify the matrix \(B\) (and this is possible in some control problems) the property of pointwise degeneracy does not appear as a highly singular property, but rather as a property which can be almost always secured and which can give rise to interesting applications.
REFERENCES