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Stable manifolds under very weak hyperbolicity conditions

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Abstract

We present an argument for proving the existence of local stable and unstable manifolds in a general abstract setting and under very weak hyperbolicity conditions.

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1. Introduction and Results

1.1. Stable sets and stable manifolds

One of the most fundamental concepts in the modern geometric theory of dynamical systems is that of the *stable set* associated to a point: given a map $\varphi : M \to M$ on a

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metric space M, and a point $z \in M$ we define the (global) *stable set* of z as

$$W^s(z) = \{x \in M : d(\varphi^k(x), \varphi^k(z)) \to 0 \text{ as } k \to \infty\}.$$

If φ is invertible, (global) *unstable set* can be defined in the same way by taking $k \to -\infty$. The situation is completely analogous and so we will concentrate here on stable sets.

This definition gives an equivalence relation on M which defines a partition into sets which are invariant under the action of φ and which are formed of orbits which have the same asymptotic behaviour. An understanding of the geometry of the stable and unstable sets of different points, of how they depend on the base point z, and of how they intersect, forms the core of many powerful arguments related to all kinds of properties of dynamical systems, from ergodicity to structural stability to estimates on decay of correlations.

In general $W^s(z)$ can be extremely complicated, both in its intrinsic geometry and/or in the way it is embedded in M. A first step towards understanding this complexity is to focus on the *local stable set*

$$W^s_\varepsilon(z) = \{x \in W^s(z) : d(\varphi^k(x), \varphi^k(z)) \leqslant \varepsilon \ \forall\ k \geqslant 0\}.$$

A key observation is that the local stable set may, under suitable conditions, have a regular geometrical structure. In particular, if M is a smooth Riemannian manifold and φ is a differentiable map, a typical statement of a "Stable Manifold Theorem" is the following:

$$W^s_\varepsilon(x) \quad \text{is a smooth submanifold of } M.$$

This implies in particular that the global stable manifold, which can be written as

$$W^s(x) = \bigcup_{n \geqslant 0} \varphi^{-n}(W^s_\varepsilon(z)),$$

is also a smooth (immersed) submanifold of M; it may however fail to be an *embedded* manifold (i.e. a manifold in the topology induced from M) due to the complicated way in which it may twist back on itself.

1.2. Historical remarks

As befits such a fundamental result, there exists an enormous literature on the subject, tackling the problem under a number of different conditions. A key idea is that of *hyperbolicity*. In the simplest setting we say that a fixed point p is hyperbolic if the derivative $D\varphi_p$ has no eigenvalue on the unit circle. In the analytic, two-dimensional, area preserving case Poincaré proved that the local stable (and unstable) sets are analytic submanifolds [41]. Hadamard and Perron independently developed more geometric methods allowing them to assume only a C^1 smoothness condition [22,23,37,38]; the stable manifold theorem for hyperbolic fixed points is sometimes called the *Hadamard–Perron Theorem*. In [44], Sternberg used a simple geometric argument, related to Hadamard's technique, to obtain existence and regularity results assuming only *partial* hyperbolicity of the fixed point, i.e. assuming only that the two eigenvalues are real and distinct. Other early work on the subject includes [6,7,16,17,24,25,42] in which the techniques were generalized to deal with stable manifolds associated to more general compact sets as opposed to just fixed points.

In the late 1960s and early 1970s the theory of stable manifolds became fundamental to the theory of *Uniformly Hyperbolic* dynamics pioneered by Anosov [2] and Smale [43]: there exists a *continuous* decomposition

$$T\Lambda = E^s \oplus E^u$$

of the tangent bundle over some set Λ into subbundles on which uniform contraction and exponential estimates hold under the action of the derivative. A straightforward generalizations of this set-up is that of *partial* or *normal* uniform hyperbolicity which allows for the possibility of a neutral subbundle

$$T\Lambda = E^s \oplus E^c \oplus E^u.$$

This is a significant weakening of the uniform hyperbolicity assumptions as it allows the dynamics tangent to E^c to be quite general. Such situations have been systematically and thoroughly investigated using variations and generalizations of the basic methods of Hadamard and Perron [18,19,26–28,31,33], see [29] for a comprehensive treatment.

An even more general set-up is based on the *Multiplicative Ergodic Theorem* of Oseledets [36] which says that there always exists a *measurable* decomposition

$$T\Lambda = E^1 \oplus E^2 \oplus \cdots \oplus E^k$$

with respect to *any* invariant probability measure μ, such that the *asymptotic exponential growth rate*

$$\lim_{n\to\infty} \frac{1}{n} \log \|D\varphi_x^n(v)\| = \lambda^i$$

is well defined, and for ergodic μ even independent of x, for all non-zero $v \in E^i$. The condition $\lambda^i \neq 0$ for all $i = 1, \ldots, k$ is a condition of *non-uniform hyperbolicity* (with respect to the measure μ) since it implies that all vectors as asymptotically contracted or expanded at an exponential rate. The non-uniformity comes from the fact that the convergence to the limit is in general highly non-uniform and thus one may have to wait an arbitrarily long time before this exponential behaviour becomes apparent. Pesin [39,40] extended many results of the theory of uniform hyperbolicity concerning stable manifolds to the non-uniform setting.

There have also been some recent papers introducing new approaches and focussing on particular subtleties of interest in various contexts, see [1,3,8–15,20,21].

1.3. Very weak hyperbolicity

The aim of this paper is to develop some techniques suitable for dealing with situations with *very weak* forms of hyperbolicity. From now on, for the rest of the paper, we shall always assume that the ambient manifold M has dimension 2. For $z \in M$ and $k \geqslant 1$ let

$$F_k(z) = \|(D\varphi_z^k)\| = \max_{\|v\|=1} \{\|D\varphi_z^k(v)\|\}$$

and

$$E_k(z) = \|(D\varphi_z^k)^{-1}\|^{-1} = \min_{\|v\|=1} \{\|D\varphi_z^k(v)\|\}.$$

These quantities have a simple geometric interpretation: since $D\varphi_z^k : T_z M \to T_{\varphi(z)} M$ is a linear map, it sends circles to ellipses; then $F_k(z)$ is precisely half the length of major axis of this ellipse and $E_k(z)$ is precisely half the length of the minor axis of this ellipse. Then let

$$H_k(z) = \frac{E_k(z)}{F_k(z)}.$$

Notice that we always have $H_k(z) \leqslant 1$. The weakest possible *hyperbolicity condition* one could assume on the orbit of some point x is perhaps the condition

$$H_k(z) < 1$$

for all $k \geqslant 1$ (or at least all k sufficiently large), equivalent to saying that the image of the unit circle is strictly an ellipse or that $D\varphi_z^k$ is not conformal. At the other extreme, perhaps the strongest hyperbolicity condition is to assume that $H_k(z) \to 0$ exponentially fast in k. This is the case in the classical hyperbolic setting, both uniform and non-uniform. In this paper, we prove a stable manifold theorem essentially under the "summable" hyperbolicity condition

$$\sum H_k(z) < \infty.$$

The precise statement of the results requires some additional technical conditions which will be given precisely in the next section, however the main idea is that the usual exponential decay of H_k is an unnecessarily strong condition. Existing arguments rely on a contraction mapping theorem in some suitable space of "candidate" stable manifolds which yields a fixed point (corresponding to the real stable manifold) by the observation that a certain sequence is Cauchy and thus converges. In our approach we construct an *canonical* sequence of *finite time local stable manifolds* and use the summability condition to show directly that this sequence is Cauchy and thus converges to a real stable manifold. Also, we make no a priori assumptions on the existence of any tangent space decomposition.

1.4. Finite time local stable manifolds

Our method is based on the key notion of *finite time* local stable manifold. Let $k \geqslant 1$ and suppose that $H_k(z) < 1$; then we let $e^{(k)}(z)$ and $f^k(z)$ denote unit vectors in the directions which are *most contracted* and *most expanded*, respectively, by $D\varphi_z^k$. Notice that these directions are solutions to the differential equation $d\|D\varphi_z^k(v)\|/d\theta = 0$ which are given by

$$\tan 2\theta = \frac{2(\partial_x \Phi_1^k \partial_y \Phi_1^k + \partial_x \Phi_2^k \partial_y \Phi_2^k)}{(\partial_x \Phi_1^k)^2 + (\partial_x \Phi_2^k)^2 - (\partial_y \Phi_1^k)^2 - (\partial_y \Phi_2^k)^2}. \tag{1}$$

In particular, $e^{(k)}$ and $f^{(k)}$ are *orthogonal* and, if φ^k is C^2, *continuously differentiable* in some neighbourhood $\mathcal{N}^{(k)}(z)$ in which they are defined. Therefore they determine two orthogonal foliations $\mathcal{E}^{(k)}$ and $\mathcal{F}^{(k)}$ defined by the integral curves of the unit vector fields $e^{(k)}(x)$ and $f^{(k)}(x)$, respectively. We let $\mathcal{E}^{(k)}(z)$ and $\mathcal{F}^{(k)}(z)$ denote the corresponding leaves through the point z. These are the natural finite time versions of the local stable and unstable manifolds of the point z since they are, in some sense, the most contracted and most expanded curves through z in $\mathcal{N}^{(k)}(z)$. Notice that they are uniquely defined locally. We will show that under suitable conditions the finite time local stable manifolds converge to real local stable manifold.

The idea of constructing finite time local stable manifolds is not new. In the context of Dynamical Systems, as far as we know it was first introduced in [4] and developed further in several papers including [5,30,34,35,45] in which systems satisfying some non-uniform hyperbolicity are considered. All these papers deal with families of systems in which, initially, hyperbolicity cannot be guaranteed for all time for all parameters. A delicate parameter-exclusion argument requires information about the geometrical structure of stable and unstable leaves based only on a finite number of iterations and thus the notion of finite time manifolds as given above is very natural. We emphasise however that in these papers the construction is heavily embedded in the global argument and no particular emphasis is placed on this method as an algorithm for the construction of real local stable manifolds per se. Moreover the decay rate of H_k there is exponential and the specific properties of the systems (such as the small determinant and various other hyperbolicity and distortion conditions) are heavily used, obscuring the precise conditions required for the argument to work.

One aim of this paper is to clarify the setting and assumptions required for the construction to work and to show that the main ideas can essentially be turned into a fully fledged alternative approach to theory of stable manifolds. Moreover we show that the argument goes through under much weaker conditions than those which hold in the papers cited above.

1.5. Main Results

We shall consider dynamical systems given by maps

$$\varphi : M \to M,$$

where M is a two-dimensional Riemannian manifold with Riemannian metric d. The situation we have in mind is that of a piecewise C^2 diffeomorphism with singularities: there exists a set $\mathcal{S}$ of zero measure such that φ is a C^2 local diffeomorphism on $M \setminus \mathcal{S}$. The map φ may be discontinuous on $\mathcal{S}$ and/or the first and second derivatives may become unbounded near $\mathcal{S}$. The precise assumptions will be local and will be formulated below. First of all we introduce some notation. For $x \in M$ let

$$P_k(x) = \|D\varphi_{\varphi^k x}\|, \quad Q_k(x) = \|(D\varphi_{\varphi^k x})^{-1}\|, \quad \widetilde{P}_k(x) = \|D^2\varphi_{\varphi^k x}\|$$

and

$$\mathfrak{D}_k(x) = |\det D\varphi_{\varphi^k x}|, \quad \tilde{\mathfrak{D}}_k(x) = \|D(\det D\varphi_{\varphi^k x})\|.$$

Notice that all of these quantities depend only on the derivatives of φ at the point $\varphi^k(x)$. If φ is globally a C^2 diffeomorphism on a compact manifold, then they are all uniformly bounded above and below and play no essential role in the result.[1] On the other hand, if the contraction and/or expansion is unbounded near the singularity set $\mathcal{S}$ some control of the recurrence is implicitly given by some conditions which we impose on these quantities. We shall also use the notation

$$F_{j,k}(x) = \|D\varphi^{k-j-1}_{\varphi^{j+1}(x)}\|.$$

We now give a generalization of the notion of local stable manifold. For a sequence

$$\underline{\varepsilon} = \{\varepsilon_j\}_{j=0}^{\infty}$$

with $\varepsilon_j \geqslant \varepsilon_{j+1} > 0$ for all $j \geqslant 0$, we let

$$\mathcal{N}^{(k)} = \mathcal{N}^{(k)}_{\underline{\varepsilon}}(z) = \{\tilde{z} \in M : \|\varphi^j(\tilde{z}) - \varphi^j(z)\| \leqslant \varepsilon_j, \forall j \leqslant k-1\}.$$

[1] Many of the expressions which arise in the course of the proof can be significantly simplified if the quantities above are assumed to be bounded. Although we have given the general form of all calculations for generality, we suggest that a first reading of the technical parts of the paper be carried out by making this simplifying assumption.

This defines a nested sequence of neighbourhoods of the point z. We shall always suppose that for all $k \geqslant j \geqslant 1$ the restriction $\varphi^j|\mathcal{N}^{(k)}$ is a C^2 diffeomorphism onto its image. In the presence of singularities this may impose a strong condition on the sequence $\underline{\varepsilon}$ whose terms may be required to decrease very quickly. We then let $\{p_k, q_k, \tilde{p}_k\}_{k=1}^{\infty}$ be uniform upper bounds for the values of $P_k, Q_k, \tilde{P}_k$, respectively, in $\mathcal{N}^{(k)}(z)$:

$$p_k = \max_{x \in \mathcal{N}^{(k)}} P_k(x), \quad q_k = \max_{x \in \mathcal{N}^{(k)}} Q_k(x), \quad \tilde{p}_k = \max_{x \in \mathcal{N}^{(k)}} \tilde{P}_k(x).$$

These values may be unbounded. Then let $\{\gamma_k, \gamma_k^*, \delta_k\}_{k=1}^{\infty}$ be given by

$$\gamma_k = \max_{x \in \mathcal{N}^{(k)}} \{H_k\}, \quad \gamma_k^* = \max_{x \in \mathcal{N}^{(k)}} \{E_k\}$$

and

$$\delta_k = \max_{x \in \mathcal{N}^{(k)}} \left\{ \frac{E_k}{F_k^2} \sum_{j=0}^{k-1} \tilde{P}_j F_{j,k} F_j^2 + \frac{E_k}{F_k} \sum_{j=0}^{k-1} \mathfrak{D}_j^{-1} \tilde{\mathfrak{D}}_j F_j \right\}.$$

We are now ready to state our two hyperbolicity conditions. The first is a hyperbolicity condition

$$\sum_{k=1}^{\infty} p_k q_k \gamma_{k+1} + \tilde{p}_k q_k^5 p_k^3 \gamma_{k+1}^* + p_k^5 q_k^5 \delta_k + p_k^2 q_k^2 \delta_{k+1} < \infty. \tag{*}$$

Notice that if the norm of the derivative is bounded, such as in the absence of singularities, this reduces to the more "user-friendly" condition

$$\sum \gamma_k + \gamma_k^* + \delta_k < \infty.$$

The summability of $\{\gamma_k^*\}$ is not particularly crucial and is really only used to ensure that some minimal contraction is present, so that the presence of a contracting stable manifolds makes sense. The summability of $\{\gamma_k\}$ is simply the "summable hyperbolicity" assumptions stated above. The summability of $\{\delta_k\}$ is a quite important technical assumption related to the "monotonicity" of the estimates in k, it is not overly intuitive but it is easily verified in standard situations such as in the uniformly hyperbolic setting. Taking advantage of condition $(*)$ we define

$$k_0 = \min\{j : p_k q_k \gamma_{k+1} < 1/2, \ \forall\, k \geqslant j-1\} < \infty$$

and the sequence

$$\tilde{\gamma}_k = \gamma_k^* + 2 \max_{x \in \mathcal{N}^{(k)}} \{F_k\} \sum_{i=k}^{\infty} p_i q_i \gamma_{i+1} < \infty.$$

Our second assumption is that there exists some constant $\Gamma > 0$ such that

$$\tilde{\gamma}_j + 4 \max_{\mathcal{N}^{(j)}} \{\|F_j\|\} p_k q_k \gamma_{k+1} < \Gamma \varepsilon_j \tag{**}$$

for all $k \geqslant k_0$ and $j \leqslant k$.

This is not a particularly intuitive condition but thinking of it in the simplest setting can be useful. Supposing for example that we are in a uniformly hyperbolic situation and that all derivatives are bounded, we have that the left-hand side is $\lesssim E_k$ which specifies that in some sense, the images of the neighbourhoods of z under consideration should not shrink to fast relative to the contraction in these neighbourhoods. We now state our main result.

Main theorem. *Let $z \in M$ and suppose that there exists a sequence $\underline{\varepsilon}$ such that φ^k restricted to $\mathcal{N}^{(k)}$ is a C^2 diffeomorphism onto its image for all $k \geqslant 1$, and suppose also that conditions $(*)$ and $(**)$ hold. Then there exists $\varepsilon > 0$ and a C^{1+Lip} embedded one-dimensional submanifold $\mathcal{E}^{\infty}(z)$ of M containing z such that $|\mathcal{E}^{(\infty)}(z)| \geqslant \varepsilon$ and such that there exists a constant $C > 0$ such that $\forall z, z' \in \mathcal{E}^{\infty}(z)$ $\forall$ $k \geqslant k_0$ we have*

$$|\varphi^k(z) - \varphi^k(z')| \leqslant C\tilde{\gamma}_k |z - z'|.$$

In particular if $\tilde{\gamma}_k \to 0$ then $|\varphi^k(z) - \varphi^k(z')| \to 0$ as $k \to \infty$ and therefore

$$\mathcal{E}^{(\infty)}(z) \subseteq W^s_{\varepsilon}(z).$$

Moreover if $F_k \to \infty$ uniformly in k, then

$$\mathcal{E}^{(\infty)}(z) = \bigcap_{k \geqslant k_0} \mathcal{N}^{(k)}(z).$$

We divide the proof into several sections. In 3.1 we introduce some useful notation. In 3.2 we prove a technical estimate which shows that the summability condition on δ_k implies some uniform distortion bounds on the $\mathcal{N}^{(k)}$. In 3.3 we study the convergence of pointwise contracting directions and in 3.4 we use these to study the convergence of the local finite time stable manifolds. In 4.1 we show that the limit curve has positive length. This is not directly implied by the preceding convergence estimates which give convergence of the leaves on whichever domain they are defined. Here we need to make sure that such a domain of definition (i.e. length) of the leaves can be chosen uniformly. Thus we have to worry about the shrinking of the sets $\mathcal{N}^{(k)}(z)$. Condition $(**)$ is used crucially in this section. We remark that the lower bound ε for the length of the local stable manifold is determined in this section. In 4.2 we show that the limit curve is smooth and in 4.3 that it "contracts" and is therefore indeed part of the local stable manifold. Finally, in 4.4 we discuss uniqueness issues.

2. Hyperbolic fixed points

As an application of our abstract theorem, we consider the simplest case of a hyperbolic fixed point. The result is of course already well-known in this context, but we show that our conditions are easy to check and that it therefore follows almost immediately from our general result.

Let M be a two-dimensional Riemannian manifold with Riemannian metric d, and let $\varphi : M \to M$ be a C^2 diffeomorphism. Suppose that $p \in M$ is a fixed point. Recall from Section 1.1 that the local stable set of p is the subset of $W^s(p)$ of points which remain in a fixed neighbourhood of p for all forward iterations. Therefore, for some $\eta > 0$,

$$W^s_\eta(p) = W^s(p) \cap \mathcal{N}^{(\infty)}_\eta(p),$$

where

$$\mathcal{N}^{(k)}_\eta = \{x : d(\varphi^j(x), p) \leqslant \eta \ \forall\ 0 \leqslant j \leqslant k-1\}$$

and

$$\mathcal{N}^{(\infty)}_\eta = \bigcap_{k \geqslant 1} \mathcal{N}^{(k)}_\eta.$$

In this section, we shall focus on the simplest setting of a hyperbolic fixed point. We recall that the fixed point p is hyperbolic if the derivative $D\varphi_p$ has no eigenvalues on the unit circle.

Theorem. *Let $\varphi : M \to M$ be a C^2 diffeomorphism of a Riemannian surface and suppose that p is a hyperbolic fixed point with eigenvalues $0 < |\lambda_s| < 1 < |\lambda_u|$. Given $\eta > 0$, there exists a constant $\varepsilon(\eta) > 0$ such that the following properties hold*:
(1) $W^s_\eta(p)$ is C^{1+Lip} *one-dimensional submanifold of M tangent to E^s_p*;
(2) $|W^s_\eta(p)| \geqslant \varepsilon$ *on either side of p*;
(3) $W^s_\eta(p)$ *contracts at an exponential rate.*
(4)
$$W^s_\eta(p) = \bigcap_{k \geqslant 0} \mathcal{N}^{(k)}_\eta(p).$$

Proof. To prove this result, it suffices to verify the hyperbolicity conditions stated in Section 1.5. First of all, since φ is a C^2 diffeomorphism, all the first and second partial derivatives are continuous and bounded. Hence for all $k \geqslant 0$, there is a uniform constant $K > 0$ such that

$$p_k, q_k, \tilde{p}_k, \mathfrak{D}_k \tilde{\mathfrak{D}}_k \leqslant K.$$

To estimate expansion and contraction rates in $\mathcal{N}^{(k)}_\eta$ we have the following lemma:

Lemma 1. *There exists a constant $K > 0$ such that for all $\delta > 0$ there exists $\eta(\delta) > 0$ such that for all $k \geqslant j \geqslant 0$ and all $x \in \mathcal{N}_\eta^{(k)}$ we have*

$$K(\lambda_u + \delta)^j \geqslant F_j \geqslant (\lambda_u - \delta)^j \geqslant (\lambda_s + \delta)^j \geqslant E_j \geqslant K^{-1}(\lambda_s - \delta)^j \tag{2}$$

and

$$\sum_{j=0}^{k-1} F_j \leqslant K F_k; \quad F_j F_{j,k} \leqslant K F_k; \quad \text{and} \quad \sum_{i=j}^{\infty} H_i \leqslant K H_j. \tag{3}$$

In particular

$$\|D^2 \varphi_x^k\| \leqslant K F_k^2; \quad \text{and} \quad \|D(\det D\varphi_x^k)\| \leqslant K E_k F_k^2. \tag{4}$$

Proof. The estimates in (2) and (3) follow from standard estimates in the theory of uniform hyperbolicity. We refer to [32] for details and proofs. The estimates in (4) then follow from substituting (3) into the more general estimates which will be proved in Lemma 2 of Section 3.2. □

Next we verify hyperbolicity conditions (∗) and (∗∗). We estimate $\gamma_k, \tilde{\gamma}_k, \gamma_k^*$ and δ_k for each $k \geqslant 0$. For γ_k and γ_k^* we have

$$\gamma_k = \max_{x \in \mathcal{N}^{(k)}} \{H_k\} \leqslant K \frac{(\lambda_s + \delta)^k}{(\lambda_u - \delta)^k},$$

$$\gamma_k^* = \max_{x \in \mathcal{N}^{(k)}} \{E_k\} \leqslant K(\lambda_s + \delta)^k,$$

while for $\tilde{\gamma}_k$ we obtain

$$\tilde{\gamma}_k = \gamma_k + \max_{x \in \mathcal{N}^{(k)}} \{2F_k\} \sum_{i=k}^{\tilde{k}-1} p_i q_i \gamma_{i+1}$$

$$\leqslant 2(\lambda_s + \delta)^k + K\left[\frac{(\lambda_u + \delta)(\lambda_s + \delta)}{(\lambda_u - \delta)}\right]^k \leqslant K(\lambda_s + \tilde{\delta})^k,$$

where $\tilde{\delta}$ can be made small with δ small. To estimate δ_k, we just use Lemma 1 above to conclude that

$$\delta_k = \max_{x \in \mathcal{N}^{(\tilde{k})}} \left\{ \frac{E_k}{F_k^2} \sum_{j=0}^{k-1} \tilde{P}_j F_{j,k} F_j^2 + \frac{E_k}{F_k} \sum_{j=0}^{k-1} \mathfrak{D}_j^{-1} \tilde{\mathfrak{D}}_j F_j \right\}$$

$$\leqslant K(\lambda_s + \tilde{\delta})^k.$$

In the estimates above, the constant K is uniform and depends only on λ_s, λ_u and the bounds for the partial derivatives of φ.

Condition $(*)$ is now immediate, since for $\delta, \tilde{\delta}$ sufficiently small, the constants $\gamma_k, \gamma_k^*, \tilde{\gamma}_k$ and δ_k all decay exponentially fast. In particular there exists a constant $L > 0$ such that $\mathrm{Lip}(e^{(k)}) \leqslant L$ inside each $\mathcal{N}^{(k)}$.

Let k_0 be the constant defined in Section 1.5. To verify condition $(**)$, we just need to show that there is a $\Gamma > 0$ such that $\forall k \geqslant k_0$ we have:

$$K \frac{(\lambda_u + \delta)^j (\lambda_s + \delta)^{k+1}}{(\lambda_u - \delta)^{k+1}} + K(\lambda_s + \tilde{\delta})^k < \Gamma \eta, \quad \forall j \leqslant k+1. \tag{5}$$

The existence of Γ follows immediately if we choose δ sufficiently small so that

$$(\lambda_u + \delta)(\lambda_s + \delta)(\lambda_u - \delta)^{-1} < 1, \quad \text{and} \quad \lambda_s + \tilde{\delta} < 1.$$

The conclusions of the theorem now follow. In particular, the length ε of the limiting leaf $\mathcal{E}^{(\infty)}$ is determined by Eq. (21) in Section 4.1. □

3. Finite time local stable manifolds

In this section, we prove some estimates concerning the relationships between finite time local stable manifolds of different orders. In particular we prove that they form a Cauchy sequence of smooth curves. Throughout this and the following section we work under the assumptions of our main theorem. In particular we consider the orbit of a point z and are given a sequence of neighbourhoods $\mathcal{N}^{(k)} = \mathcal{N}^{(k)}(z)$ in which most contractive and most expanding directions are defined and thus, in particular, in which the finite time local stable manifolds $\mathcal{E}^{(k)}(z)$ are defined. The key problem therefore is to show that these finite time local stable manifolds converge, that they converge to a smooth curve, and that this curve has non-zero length!

3.1. Notation

We shall use K to denote a generic constant which is allowed to depend only on the diffeomorphism φ. For any $j \geqslant 1$ we let

$$e_j^{(k)}(x) = D\varphi_x^j(e^{(k)}(x)) \quad \text{and} \quad f_j^{(k)}(x) = D\varphi_x^j(f^{(k)}(x))$$

denote the images of the most contracting and most expanding vectors. To simplify the formulation of angle estimates we introduce the variable θ to define the position of the vectors. We write

$$e^{(n)} = (\cos\theta^{(n)},\ \sin\theta^{(n)}), \qquad f^{(n)} = (-\sin\theta^{(n)},\ \cos\theta^{(n)}).$$
$$e_n^{(n)} = E_n(\cos\theta_n^{(n)},\ \sin\theta_n^{(n)}), \qquad f_n^{(n)} = F_n(-\sin\theta_n^{(n)},\ \cos\theta_n^{(n)}).$$

Finally, we let

$$\phi^{(k)} = \measuredangle(e^{(k)}, e^{(k+1)}) \text{ and } \phi_j^{(k)} = \measuredangle(e_j^{(k)}, e_j^{(k+1)}).$$

We also identify any vector v with $-v$, or equivalently we identify an angle θ with the angle $\theta + \pi$. Important parts of the proof depend on estimating the derivative of various of these quantities with respect to the base point x. We shall write $D\phi^{(k)}$, $De^{(k)}$, $D\theta_j^{(n)}, \ldots$ to denote the derivatives with respect to the base point x. To simplify the notation we let

$$\Xi_k(x) := \frac{P_k(x)Q_k(x)H_{k+1}(x)}{(1 - P_k(x)Q_k(x)H_{k+1}(x))} \leqslant \frac{p_k q_k \gamma_{k+1}}{(1 - p_k q_k \gamma_{k+1})} := \xi_k. \tag{6}$$

Also, all statements hold uniformly for all $x \in \mathcal{N}^{(k)}$.

3.2. Distortion

The following distortion estimates follow from completely general calculations which do not depend on any hyperbolicity assumptions. The definition of δ_k is motivated by these estimates which will be used extensively in Section 3.3.

Lemma 2. *For all $k \geqslant 1$ and all x such that φ^k is C^2 at x, we have*

$$H_k \frac{\|D^2\varphi^k\|}{\|D\varphi^k\|} \leqslant \frac{E_k}{F_k^2} \sum_{j=0}^{k-1} \tilde{P}_j F_{j,k} F_j^2 \quad (\leqslant \delta_k) \tag{D1}$$

and

$$\frac{\|D(\det D\varphi_z^k)\|}{\|D\varphi^k\|^2} \leqslant \frac{E_k}{F_k} \sum_{j=0}^{k-1} \mathfrak{D}_j^{-1} \tilde{\mathfrak{D}}_j F_j \quad (\leqslant \delta_k). \tag{D2}$$

Proof. Let $A_j = D\varphi_{\varphi^j z}$ and let $A^{(k)} = A_{k-1}A_{k-2}\ldots A_1 A_0$. Let DA_j denote differentiation of A_j with respect to the space variables. By the product rule for differentiation we have

$$D^2\varphi_z^k = DA^{(k)} = D(A_{k-1}A_{k-2}\dots A_1A_0)$$
$$= \sum_{j=0}^{k-1} A_{k-1}\dots A_{j+1}(DA_j)A_{j-1}\dots A_0. \tag{7}$$

Taking norms on both sides of (7) and using the fact that $A_{k-1}\dots A_{j+1} = D\varphi^{k-j-1}_{\varphi^{j+1}z}$, $A_{j-1}\dots A_0 = D\varphi_z^j$ and, by the chain rule, $DA_j = D(D\varphi_{\varphi^j z}) = D^2\varphi_{\varphi^j z}D\varphi_z^j$, we get

$$\|D^2\varphi_z^k\| \leqslant \sum_{j=0}^{k-1}\|D\varphi^{k-j-1}_{\varphi^{j+1}z}\|\cdot\|D^2\varphi_{\varphi^j z}\|\cdot\|D\varphi_z^j\|^2 \leqslant \sum_{j=0}^{k-1}\|D^2\varphi_{\varphi^j z}\|F_{j,k}F_j^2.$$

The inequality (D1) now follows. For (D2) we argue along similar lines, this time letting $A_j = \det D\varphi_{\varphi^j z}$. Then we have, as in (7) above,

$$D(\det\varphi_z^k) = DA^{(k)} = \sum_{j=0}^{k-1} A_{k-1}\dots A_{j+1}(DA_j)A_{j-1}\dots A_0.$$

Moreover we have that $A_{k-1}\dots A_{j+1} = \det D\varphi^{k-j-1}_{\varphi^{j+1}z}$, $A_{j-1}\dots A_0 = \det D\varphi_z^j$, and by the chain rule, also:

$$DA_j = D(\det D\varphi_{\varphi^j z}) = (D\det D\varphi_{\varphi^j z})D\varphi_z^j.$$

This gives

$$D(\det\,\varphi_z^k) = \sum_{j=0}^{k-1}(\det D\varphi^{k-j-1}_{\varphi^{j+1}z})(D\det D\varphi_{\varphi^j z})(\det D\varphi_z^j)(D\varphi_z^j). \tag{8}$$

By the multiplicative property of the determinant we have the equality:

$$(\det D\varphi^{k-j-1}_{\varphi^{j+1}z})(\det D\varphi_z^j) = \det D\varphi_z^k/\det D\varphi_{\varphi^j z}.$$

Thus, taking norms on both sides of (8) gives

$$\|D(\det D\varphi_z^k)\| \leqslant |\det D\varphi_z^k| \sum_{j=0}^{k-1}\frac{\|D(\det D\varphi_{\varphi^j(z)})\|}{|\det D\varphi^{\varphi^j(z)}|}F_j$$

The inequality (D2) now follows from the fact that $\det D\varphi^k = E_k F_k$. □

3.3. Pointwise convergence

In this section, we prove two key lemmas showing that both the angle $\phi^{(k)}$ (Lemma 3) between consecutive most contracted directions and the norm of its spatial derivative $D\phi^k$ (Lemma 4) can be bounded in terms of the hyperbolicity. In particular, from the summability condition $(*)$, we obtain also that the norm $\|De^{(k)}\|$ of the spatial derivative of the contractive directions is uniformly bounded in k.

Lemma 3. *For all $k \geqslant k_0$ and $x \in \mathcal{N}^{(k)}$ we have*

$$|\phi^{(k)}| \leqslant |\tan\phi^{(k)}| \leqslant \frac{P_k Q_k H_{k+1}}{1 - P_k Q_k H_{k+1}} \quad (\leqslant \xi_k). \tag{9}$$

Moreover, for all $k \geqslant j \geqslant k_0$ we have

$$\|e_j^{(k)}(x)\| \leqslant E_j(z) + F_j(z) \sum_{i=j}^{k-1} \phi^{(i)}(z) \quad (\leqslant \tilde{\gamma}_j). \tag{10}$$

Notice that the estimate in (10) gives an upper bound for the contraction which depends only on j and not on k.

Proof. We claim first of all that for all $k \geqslant k_0$ we have

$$\|e_{k+1}^{(k)}\| / F_{k+1} \leqslant P_k Q_k H_{k+1} \leqslant 1/2. \tag{11}$$

To see this observe that $E_k \leqslant \|e_k^{(k+1)}\| \leqslant \|D\varphi_{z_k}^{-1} e_{k+1}^{(k+1)}\| \leqslant Q_k E_{k+1}$, $E_{k+1} \leqslant \|e_{k+1}^{(k)}\| = \|D\varphi_{z_k} e_k^{(k)}\| \leqslant P_k E_k$, $F_k = \|D\varphi_{z_k}^{-1} f_{k+1}^{(k)}\| \leqslant Q_k F_{k+1}$, $F_{k+1} = \|D\varphi_{z_k} f_k^{(k+1)}\| \leqslant P_k F_k$. Moreover $H_{k+1}/H_k = (E_{k+1}/F_{k+1})/(E_k/F_k)$. Combining these inequalities gives

$$E_{k+1}/E_k \in [Q_k^{-1}, P_k], \quad F_{k+1}/F_k \in [Q_k^{-1}, P_k], \quad H_{k+1}/H_k \in [(P_k Q_k)^{-1}, P_k Q_k]. \tag{12}$$

Therefore, writing write $e_{k+1}^{(k)} = D\varphi(z_k) e_k^{(k)}$ and applying the Cauchy–Schwarz inequality gives $\|e_{k+1}^{(k)}\| \leqslant E_{k+1} P_k Q_k$ which immediately implies the first inequality of (11). The second inequality follow simply by our choice of k_0.

Now write $e^{(k)} = \eta e^{(k+1)} + \varphi f^{(k+1)}$ where $\eta^2 + \varphi^2 = 1$ by normalization. Linearity implies that $e_{k+1}^{(k)} = \eta e_{k+1}^{(k+1)} + \varphi f_{k+1}^{(k+1)}$ and orthogonality implies that $\|e_{k+1}^{(k)}\|^2 = \eta^2 \|e_{k+1}^{(k+1)}\|^2 + \varphi^2 \|f_{k+1}^{(k+1)}\|^2 = \eta^2 E_{k+1}^2 + \varphi^2 F_{k+1}^2$ where $E_k = \|e_k^{(k)}\|$, $F_k = \|f_k^{(k)}\|$.

Since $\phi^{(k)} = \tan^{-1}(\varphi/\eta)$ we get

$$|\tan\phi^{(k)}| = \left(\frac{\|e^{(k)}_{k+1}\|^2 - E^2_{k+1}}{F^2_{k+1} - \|e^{(k)}_{k+1}\|^2}\right)^{\frac{1}{2}} \leqslant \frac{\|e^{(k)}_{k+1}\|/F_{k+1}}{\left(1 - \|e^{(k)}_{k+1}\|^2/F^2_{k+1}\right)^{\frac{1}{2}}} \leqslant \frac{\|e^{(k)}_{k+1}\|/F_{k+1}}{1 - \|e^{(k)}_{k+1}\|/F_{k+1}}.$$

In the last inequality we have used $\|e^{(k)}_{k+1}\| < F_{k+1}$ from (11) and then applied the inequality $\sqrt{(1-x^2)} > 1 - x$, valid for $x \in (0, 1)$. Using (11) again completes the proof of the first statement in the lemma.

To prove (10) we write $e^{(k)}_j = e^{(j)}_j + \sum_{i=j}^{k-1}(e^{(i+1)}_j - e^{(i)}_j)$. The first term is equal to $E_j(x)$ by definition. For the second we have, by linearity, $\|e^{(i+1)}_j - e^{(i)}_j\| = \|F_j(x)(e^{(i+1)} - e^{(i)})\| \leqslant \|F_j(x)\|\ |\phi^{(i)}|$. By (9) and the definition of $\tilde\gamma_j$ we get $\|e^{(k)}_j\| \leqslant \tilde\gamma_j$. □

Lemma 4. *For all $k \geqslant k_0$ and $x \in \mathcal{N}^{(k)}$ we have*

$$\|D\phi^{(k)}\| \leqslant 1597(p_k q_k)^2 \delta_{k+1} + 40(p_k q_k)^5 \delta_k + 40(p_k q_k)^3 q_k^2 \tilde p_k \gamma^*_{k+1}.$$

In particular, there exists a constant L independent of k such that

$$\|De^{(k)}\| \leqslant \sum_{j \leqslant k} \|D\phi^{(j)}\| \leqslant L.$$

Proof. Since $D\varphi^k$ is a linear map, we have

$$\tan\phi^{(k)} = H_{k+1} \tan\phi^{(k)}_{k+1}. \tag{13}$$

Differentiating (13) on both sides and taking norms we have

$$\begin{aligned} \|D\phi^{(k)}\| &\leqslant \|H_{k+1} \cdot D(\tan\phi^{(k)}_{k+1})\| + \|DH_k \cdot \tan\phi^{(k)}_{k+1}\| \\ &\leqslant \|H_{k+1}(1 + \tan^2\phi^{(k)}_{k+1}) D\phi^{(k)}_{k+1}\| + \|DH_{k+1} \cdot \tan\phi^{(k)}_{k+1}\|. \end{aligned} \tag{14}$$

In the next two sublemmas we obtain upper bounds for $\|D\phi^{(k)}_{k+1}\|$ and $\|DH_{k+1}\|$, respectively, and then substitute these bounds into (14).

Sublemma 4.1.

$$\|D\phi^{(k)}_{k+1}\| \leqslant \frac{2048\delta_{k+1}}{9H_{k+1}} + 8(P_k Q_k)^2 \frac{\delta_k}{H_k} + 8Q_k^2 P_k \tilde P_k F_k.$$

Proof. Writing $\phi_{k+1}^{(k)} = \theta_{k+1}^{(k+1)} - \theta_{k+1}^{(k)}$ we have

$$\|D\phi_{k+1}^{(k)}\| = \|D\theta_{k+1}^{(k+1)} - D\theta_{k+1}^{(k)}\| \leqslant \|D\theta_{k+1}^{(k+1)}\| + \|D\theta_{k+1}^{(k)}\|.$$

Our strategy therefore is to obtain estimates for the terms on the right-hand side. First of all we write

$$D\varphi^n(z) = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix},$$

where A_n, B_n, C_n and D_n are the matrix entries for the derivative $D\varphi^n$ evaluated at z. Since $\{e^{(n)}(z), f^{(n)}(z)\}$ correspond to (resp.) maximal contracting and expanding vectors under $D\varphi^n(z)$, i.e. solutions of the differential equation

$$\frac{d}{d\theta}\left\| D\varphi_z^n \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\| = 0.$$

By solving the differential equation above in θ we get

$$\tan 2\theta^{(k)} = \frac{2(A_k B_k + C_k D_k)}{A_k^2 + C_k^2 - B_k^2 - D_k^2} = \frac{2\mathcal{A}_k}{\mathcal{B}_k}$$

and by solving a similar one for the inverse map $D\varphi^{-n}$ we get

$$\tan 2\theta_k^{(k)} = \frac{2(B_k D_k + A_k C_k)}{D_k^2 + C_k^2 - A_k^2 - B_k^2} = -\frac{2\mathcal{C}_k}{\mathcal{D}_k}.$$

Notice the use of $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k$ as a shorthand notation for the expression in the quotients. Now $e_k^{(k)}$, $f_k^{(k)}$ are, respectively, maximally expanding and contracting for $D\Phi^{-k}$, and so we have the identity

$$D\Phi^{-k}(\Phi^k(\xi_0)) \cdot \det D\Phi^k(\xi_0) = \begin{pmatrix} D_k & -B_k \\ -C_k & A_k \end{pmatrix}.$$

Then, using the quotient rule for differentiation immediately gives

$$\|D\theta^{(k)}\| = \left\| \frac{\mathcal{A}_k'\mathcal{B}_k - \mathcal{A}_k\mathcal{B}_k'}{4\mathcal{A}_k^2 + \mathcal{B}_k^2} \right\| \text{ and } \|D\theta_k^{(k)}\| = \left\| \frac{\mathcal{D}_k'\mathcal{C}_k - \mathcal{D}_k\mathcal{C}_k'}{4\mathcal{C}_k^2 + \mathcal{D}_k^2} \right\|. \quad \square \tag{15}$$

Claim 4.1.1. $|\mathcal{A}_k|, |\mathcal{B}_k|, |\mathcal{C}_k|, |\mathcal{D}_k| \leqslant 4\|D\varphi^k\|^2$ *and* $\|\mathcal{A}'_k\|, \|\mathcal{B}'_k\|, |\mathcal{C}'_k\|, \|\mathcal{D}'_k\| \leqslant 16\|D\varphi^k\|$ $\|D^2\varphi^k\|$

Proof. For the first set of estimates observe that each partial derivative A_k, B_k, C_k, D_k of $D\varphi^k$ is $\leqslant \|D\varphi^k\|$. Then $|\mathcal{A}_k| = |A_k B_k + C_k D_k| \leqslant 2\|D\varphi^k\|^2$. The same reasoning gives the estimates in the other cases. To estimate the derivatives, write $\|\mathcal{A}'_k\| = |A'_k B_k + A_k B'_k + C'_k D_k + C_k D'_k|$. Now $|A'_k| \leqslant 2\|D^2\varphi^k\|$ and similarly for the other terms. □

Claim 4.1.2. $4\mathcal{C}_k^2 + \mathcal{D}_k^2 = 4\mathcal{A}_k^2 + \mathcal{B}_k^2 = (E_k^2 - F_k^2)^2$.

Proof. Notice first of all that E_k^2, F_k^2 are eigenvalues of

$$(D\Phi^k)^T D\Phi^k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \begin{pmatrix} A_k & C_k \\ B_k & D_k \end{pmatrix} = \begin{pmatrix} A_k^2 + B_k^2 & A_k C_k + D_k B_k \\ A_k C_k + D_k B_k & C_k^2 + D_k^2 \end{pmatrix}.$$

In particular E_k^2, F_k^2 are the two roots of the characteristic equation $\lambda^2 - \lambda(A_k^2 + B_k^2 + C_k^2 + D_k^2) + (A_k^2 + B_k^2)(C_k^2 + D_k^2) - (A_k C_k + B_k D_k)^2 = 0$ and therefore, by the general formula for quadratic equations, we have $F_k^2 + E_k^2 = A_k^2 + B_k^2 + C_k^2 + D_k^2$ and $E_k^2 F_k^2 = (A_k^2 + B_k^2)(C_k^2 + D_k^2) - (A_k C_k + B_k D_k)^2$. From this one can easily check that $4\mathcal{C}_k^2 + \mathcal{D}_k^2 = 4\mathcal{A}_k^2 + \mathcal{B}_k^2 = (E_k^2 - F_k^2)^2 = (E_k^2 + F_k^2)^2 - 4E_k^2 F_k^2$. □

Substituting the estimates of Claims 4.1.1–4.1.2 into (15) and using hyperbolicity and distortion conditions this gives

$$\|D\theta^{(k)}\|, \|D\theta_k^{(k)}\| \leqslant 128 \frac{\|D\varphi^k\|^3 \|D^2\varphi^k\|}{(E_k^2 - F_k^2)^2} \leqslant 128 \frac{\|D\varphi^k\|^3 \|D^2\varphi^k\|}{F_k^4 (1 - P_k Q_k H_k^2)^2} \leqslant \frac{2048}{9} \frac{\delta_k}{H_k}.$$

To estimate $D\theta_{k+1}^{(k)}$ we write $e_{k+1}^{(k)} = \tilde{E}_{k+1}(\cos\theta_{k+1}^{(k)}, \sin\theta_{k+1}^{(k)})$, so that

$$\tan\theta_{k+1}^{(k)} = \frac{C_1(z_k)\cos\theta_k^{(k)} + D_1(z_k)\sin\theta_k^{(k)}}{A_1(z_k)\cos\theta_k^{(k)} + B_1(z_k)\sin\theta_k^{(k)}} = \frac{\mathcal{M}_k}{\mathcal{N}_k}.$$

Then

$$\|D\theta_{k+1}^{(k)}\| = \left\| \frac{\mathcal{N}_k \mathcal{M}'_k - \mathcal{M}_k \mathcal{N}'_k}{\mathcal{M}_k^2 + \mathcal{N}_k^2} \right\| \quad \text{with} \quad \mathcal{M}_k^2 + \mathcal{N}_k^2 = \frac{\|e_{k+1}^{(k)}\|^2}{\|e_k^{(k)}\|^2} \geqslant \frac{1}{\|D\Phi^{-1}(z_k)\|^2}.$$

By inspecting this expression for $\|D\theta^{(k)}_{k+1}\|$ the following bound is obtained:

$$\begin{aligned}\|D\theta^{(k)}_{k+1}\| &\leqslant 2\|D\varphi^{-1}(z_k)\|^2\big\{2\|D\varphi(z_k)\|\big(2\|D\varphi^2(z_k)\|\cdot\|D\varphi^k(z)\| \\ &\qquad +2\|D\varphi(z_k)\|\cdot\|D\theta^{(k)}_k\|\big)\big\} \\ &\leqslant 8Q_k^2\big(P_k^2\|D\theta^{(k)}_k\|+P_k\tilde{P}_kF_k\big).\end{aligned}$$

Putting together the estimates for $\|D\theta^{(k)}_{k+1}\|$ and $\|D\theta^{(k+1)}_{k+1}\|$, we obtain

$$\|D\phi^{(k)}_{k+1}\|\leqslant\frac{2048\delta_{k+1}}{9H_{k+1}}+8(P_kQ_k)^2\frac{\delta_k}{H_k}+8Q_k^2P_k\tilde{P}_kF_k. \qquad \square$$

Sublemma 4.2.

$$\|DE_k\|,\|DF_k\|\leqslant\frac{2057}{9}\delta_kF_k/H_k \quad\text{ and }\quad \|DH_k\|\leqslant\frac{2066}{9}\delta_k.$$

Proof. We first estimate $D_zE_k = D\|e^{(k)}_k\|$. The corresponding estimate for D_zF_k is identical. By direct differentiation we have, $D_ze^{(k)}_k = D^2\varphi^k(z)e^{(k)}+D\varphi^k\cdot De^{(k)}$ and hence by Lemma 2 and the estimate for $\|D\theta^{(k)}\|$ we have:

$$\|D_ze^{(k)}_k\|\leqslant\|D^2\varphi^k(z)\|+\|D\varphi^k(z)\|\cdot\|D_ze^{(k)}\|\leqslant\frac{\delta_kF_k}{H_k}+\frac{2048F_k\delta_k}{9H_k}=\frac{2057\delta_kF_k}{9H_k}.$$

Since $D\|e^{(k)}_k\|=(e^{(k)}_k\cdot De^{(k)}_k)\|e^{(k)}_k\|^{-1}$ it follows that $\|D_zE_k\|\leqslant\|De^{(k)}_k\|$ and therefore $\|DE_k\|\leqslant 2057\delta_kF_k/9H_k$. Using the fact that $\det D\varphi^k=E_kF_k$ and the quotient rule for differentiation, we get

$$DH_k=D\left(\frac{E_k}{F_k}\right)=D\left(\frac{\det D\varphi^k}{F_k^2}\right)=\frac{D(\det D\varphi^k)}{F_k^2}-\frac{2E_kDF_k}{F_k^2}.$$

By the estimates for DF_k and Lemma 2 we then get $\|DH_k\|\leqslant 2066\delta_k/9$. $\square$

To complete the proof of Lemma 4, Eq. (13) and (14) give (for $k\geqslant k_0$):

$$\begin{aligned}\|D\phi^{(k)}\| &\leqslant |H_{k+1}|(1+\tan^2\phi^{(k)}_{k+1})\|D\phi^{(k)}_{k+1}\|+\|DH_{k+1}\|\cdot|\tan\phi^{(k)}_{k+1}| \\ &\leqslant |H_{k+1}|(1+\tan^2\phi^{(k)}_{k+1})\big(\|D\theta^{(k+1)}_{k+1}\|+\|D\theta^{(k)}_{k+1}\|\big) \\ &\qquad +\|DH_{k+1}\|\cdot|\tan\phi^{(k)}_{k+1}|\end{aligned}$$

$$\leqslant |H_{k+1}|\big(1+4P_k^2Q_k^2\big)\left(\frac{2048\delta_{k+1}}{9H_{k+1}}+8(P_kQ_k)^2\frac{\delta_k}{H_k}+8Q_k^2P_k\tilde{P}_kF_k\right)$$
$$+\frac{4132}{9}P_kQ_k\delta_{k+1}.$$

Collecting all the terms and using (12) together with the hyperbolicity assumptions we obtain

$$\|D\phi^{(k)}\|\leqslant 1597(p_kq_k)^2\delta_{k+1}+40(p_kq_k)^5\delta_k+40(p_kq_k)^3q_k^2\tilde{p}_k\gamma_{k+1}^*.$$

This gives us the required estimate for $\|D\phi^{(k)}\|$. To get the estimate for $\|De^{(k)}\|$ we use the fact that $\|De^{(k)}\|\approx\|D\theta^{(k)}\|$ with $\theta^{(k)}=\sum_{j=1}^{k-1}(\theta^{(j+1)}-\theta^{(j)})+\theta^{(1)}$. □

3.4. Global convergence

We have seen above that the contractive directions converge pointwise under some very mild hyperbolicity conditions. We now want to show that the curves $\mathcal{E}^{(k)}(z)$ converge to some limit curve $\mathcal{E}^{\infty}(z)$. Let $z_t^{(k)}$ and $z_t^{(k+1)}$ be parameterizations by arclength of the two curves $\mathcal{E}^{(k)}(z)$ and $\mathcal{E}^{(k+1)}(z)$ with $z_0^{(k)}=z_0^{(k+1)}=z$.

Lemma 5. *For every $k\geqslant k_0$ and t such that $z_t^{(k)}$ and $z_t^{(k+1)}$ are both defined, we have*

$$|z_t^{(k)}-z_t^{(k+1)}|\leqslant t\xi_ke^{Lt}.$$

Proof. By standard calculus we have

$$z_t^{(k)}=z_0+\int_0^t e^{(k)}(z_s)\,ds\quad\text{and}\quad z_t^{(k+1)}=\tilde{z}_0+\int_0^t e^{(k+1)}(z_s^{(k+1)})\,ds$$

and therefore

$$|z_t^{(k)}-z_t^{(k+1)}|=\int_0^t \|e^{(k)}(z_s^{(k)})-e^{(k+1)}(z_s^{(k+1)})\|\,ds. \tag{16}$$

By the Mean Value Theorem and Lemma 3 we have

$$\|e^{(k)}(z_s^{(k)})-e^{(k)}(z_s^{(k+1)})\|\leqslant\|De^{(k)}\||z_s^{(k)}-z_s^{(k+1)}|\leqslant L|z_s^{(k)}-z_s^{(k+1)}|. \tag{17}$$

By Lemma 4 we have

$$\|e^{(k)}(z_s^{(k+1)})-e^{(k+1)}(z_s^{(k+1)})\|\leqslant|\phi^{(k)}|\leqslant\xi_k. \tag{18}$$

By the triangle inequality, (17)–(18) give

$$\|e^{(k)}(z_s^{(k)}) - e^{(k+1)}(z_s^{(k+1)})\| \leqslant L|z_s^{(k)} - z_s^{(k+1)}| + \xi_k. \tag{19}$$

Substituting (19) into (16) and using Gronwall's inequality gives

$$|z_t^{(k)} - z_t^{(k+1)}| \leqslant t\xi_k + \int_0^t L|z_s^{(k)} - z_s^{(k+1)}|ds \leqslant t\xi_k e^{Lt}. \qquad \square \tag{20}$$

4. The infinite time local stable manifold

In this section we apply the convergence estimates obtained above to show that the local stable manifold converge to a smooth curve of positive length and on which we have some controlled contraction estimates.

4.1. Geometry

Lemma 5 gives a bound on the distance between finite time local stable manifolds of different order. However we have so far no guarantee that these manifolds all have some uniformly positive length. This depends on some delicate relationship between the geometry of the images of the neighbourhoods $\mathcal{N}^{(k)}$ and the position of the finite time local stable manifolds in $\mathcal{N}^{(k)}$. Here we show that we can find some uniform lower bound for the length of all local stable manifolds. For $\varepsilon > 0$ and $k \geqslant 1$, let

$$\omega_k = \omega_k(\varepsilon) = \varepsilon e^{L\varepsilon}\xi_k$$

and

$$\mathcal{E}^{(k)}(z, \varepsilon) = \{\xi \in \mathcal{E}^{(k)}(z) : d_{\mathcal{E}}(\xi, z) \leqslant \varepsilon\}.$$

Recall that the constant L is determined in Lemma 4. Here the distance $d_{\mathcal{E}}$ is defined to be the distance measure inside $\mathcal{E}^{(k)}(z)$ so that $\mathcal{E}^{(k)}(z, \varepsilon)$ is just a subset of $\mathcal{E}^{(k)}(z)$ which extends by a length of at most ε on both sides of z. If $\mathcal{E}^{(k)}(z)$ extends by a length of less than ε on one or both sides of z then $\mathcal{E}^{(k)}(z, \varepsilon)$ coincides with $\mathcal{E}^{(k)}$ on the corresponding sides. For simplicity we shall generally omit the ε from the notation and thus use the previous notation $\mathcal{E}^{(k)}(z)$ to denote the local stable manifold of order k restricted to a curve of length at most ε on either side of z. Let

$$\mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z)) = \{\xi : d(\xi, \mathcal{E}^{(k)}(z)) \leqslant \omega_k\}$$

denote a neighbourhood of $\mathcal{E}^{(k)}(z)$ of size ω_k. At this point we are ready to make explicit our choice of ε: we choose $\varepsilon > 0$ small enough so that

$$\varepsilon\Gamma < 1 \quad \text{and} \quad e^{\varepsilon L} < 2, \tag{21}$$

where Γ is the constant used in the definition of condition (∗∗), and such that for all $k \geqslant k_0$ we have

$$\mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z)) \subset \mathcal{N}^{(k_0)}. \tag{22}$$

Notice that (22) is possible because k_0 and $\mathcal{N}^{(k_0)}$ are fixed and $|\mathcal{E}^{(k)}(z)|$ and ω_k can be made arbitrarily small for $k \geqslant k_0$ by taking ε small and using the fact that $\xi_k \to 0$ by the summability condition (∗). With this choice of ε we can then state and prove the main result of this section.

Lemma 6. *For all $k \geqslant k_0$ we have*

$$|\mathcal{E}^{(k)}(z)| = \varepsilon.$$

It follows that each finite time local stable manifold $\mathcal{E}^{(k)}(z)$ can be parametrized by arclength as $z_t^{(k)}$ with $t \in [-\varepsilon, \varepsilon]$ and $z_0^{(k)} = z$. By Lemma 5, the pointwise limit

$$z_t^{(\infty)} = \lim_{k \to \infty} z_t^{(k)}$$

exists for each $t \in [-\varepsilon, \varepsilon]$ and defines the set

$$\mathcal{E}^{\infty}(z) = \{z_t^{(\infty)} : t \in [-\varepsilon, \varepsilon]\}.$$

In the following sections we will show that $\mathcal{E}^{(\infty)}(z)$ is a smooth curve, that $|\mathcal{E}^{\infty}(z)| \geqslant \varepsilon$, and that it belongs to the stable manifolds of z.

First of all we prove

Lemma 7. *For all $k \geqslant k_0$ we have*

$$\mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z)) \subset \mathcal{N}^{(k+1)}. \tag{23}$$

The proof of Lemma 7 is a crucial step in the overall argument and the only place in which condition (∗∗) is used. Compare (22) and (23): condition (22) follows immediately by taking ε small, without any additional geometrical considerations. On the other hand, (23) requires a non-trivial control of the geometry of $\mathcal{E}^{(k)}(z)$ in $\mathcal{N}^{(k+1)}$.

Proof. We prove (23) inductively by showing that for all $k \geqslant j \geqslant k_0$ we have the implication

$$\mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z)) \subset \mathcal{N}^{(j)} \;\Rightarrow\; \mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z)) \subset \mathcal{N}^{(j+1)}.$$

Together with (22), which provides the first step of the induction for $j = k_0$, this gives (23). Thus we need to prove that for all $x \in \mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z)) \subset \mathcal{N}^{(j)}$ we have

$d(\varphi^j(x), \varphi^j(z)) \leqslant \varepsilon_j$. We fix some point $y \in \mathcal{E}^{(k)}(z)$ with $d(x, y) \leqslant \omega_k$. and write $d(\varphi^j(z), \varphi^j(x)) \leqslant d(\varphi^j(z), \varphi^j(y)) + d(\varphi^j(y), \varphi^j(x))$. To estimate $d(\varphi^j(z), \varphi^j(y))$ we use the fact that both y and z are on $\mathcal{E}^{(k)}(z)$ and that $\mathcal{E}^{(k)}(z)$ is contracting under φ^j: by (10) we have (recall that $d(z, y) \leqslant \varepsilon$)

$$d(\varphi^j(z), \varphi^j(y)) \leqslant \max_{\xi \in \mathcal{E}^{(k)}} \{\|e_j^{(k)}(\xi)\|\}\ d(z, y) \leqslant \tilde{\gamma}_j \varepsilon.$$

To estimate $d(\varphi^j(y), \varphi^j(x)$ we simply use the fact that $d(y, x) \leqslant \omega_k$ by assumption and in particular x and the line segment joining x and y lies entirely in $\mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z))$ and therefore in $\mathcal{N}^{(j)}$ by our inductive assumption. A relatively coarse estimate using the maximum possible expansion in $\mathcal{N}^{(j)}$ thus gives

$$d(\varphi^j(y), \varphi^j(x)) \leqslant \max_{\xi \in \mathcal{N}^{(j)}} \{\|F_j(\xi)\|\}\ d(y, x) \leqslant \max_{\xi \in \mathcal{N}^{(j)}} \{\|F_j\|\}\ \omega_k.$$

For $k \geqslant k_0$ we have $\omega_k = \varepsilon e^{L\varepsilon} \xi_k \leqslant \varepsilon e^{L\varepsilon} 2 p_k q_k \gamma_{k+1}$ and therefore

$$\begin{aligned} d(\varphi^j(z), \varphi^j(x)) &\leqslant \varepsilon \tilde{\gamma}_j + 2\varepsilon e^{\varepsilon L} p_k q_k \gamma_{k+1} \max_{\mathcal{N}^{(j)}} \{\|F_j\|\} \\ &= \varepsilon \left(\tilde{\gamma}_j + 2e^{\varepsilon L} p_k q_k \gamma_{k+1} \max_{\mathcal{N}^{(j)}} \{\|F_j\|\} \right). \end{aligned}$$

By our choice of ε this is $\leqslant \varepsilon_j$. □

Proof of Lemma 6. By Lemma 7, $e^{(k+1)}$ is defined in $\mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z))$, and therefore, so is the integral leaf $\mathcal{E}^{(k+1)}$. Let $z_t^{(k+1)}$ denote a parameterization of $\mathcal{E}^{(k+1)}$ by arclength with $-t_0 \leqslant t \leqslant t_0$ where t_0 is chosen maximal so that $\{z_t^{(k+1)}\}_{t=-t_0}^{t_0} \subset \mathcal{T}_{\omega_k}(\mathcal{E}^{(k)}(z))$. We claim that $t_0 \geqslant \varepsilon$, which proves the statement in the Lemma. Indeed, by Lemma 5 and the definition of ω_k we have

$$|z_t^{(k)} - z_t^{(k+1)}| \leqslant t \xi_k e^{Lt} \leqslant \omega_k$$

for all $|t| \leqslant \varepsilon$. □

4.2. Smoothness

We now want to study the regularity properties of $\mathcal{E}^{\infty}(z)$.

Lemma 8. *The curve $\mathcal{E}^{(\infty)}(z)$ is C^{1+Lip} with*

$$|\mathcal{E}^{(\infty)}(z)| \geqslant \varepsilon.$$

The Lipschitz constant of the derivative is bounded above by L.

Proof. From (19) we have

$$\|e^{(k)}(z_t^{(k)}) - e^{(k+1)}(z_t^{(k+1)})\| \leqslant L|z_t^{(k)} - z_t^{(k+1)}| + \xi_k \tag{24}$$

and from (20) we have

$$|z_t^{(k)} - z_t^{(k+1)}| \leqslant t\xi_k e^{Lt}. \tag{25}$$

Thus, substituting (25) into (24), we get

$$\|e^{(k)}(z_t^{(k)}) - e^{(k+1)}(z_t^{(k+1)})\| \leqslant Lt\xi_k e^{Lt}.$$

The uniform summability condition therefore implies that the sequence of tangent directions $e^{(k)}(z_t)$ is uniformly Cauchy in t. Thus by standard convergence results they converge to the tangent directions of the limiting curve $\mathcal{E}^{(\infty)}(z)$ and this curve is C^1. To estimate the Lipschitz constant we let $x, x' \in \mathcal{E}^{\infty}(z)$ and write

$$\|e^{\infty}(x) - e^{\infty}(x')\| \leqslant \|e^k(x) - e^{\infty}(x)\| + \|e^k(x) - e^k(x')\| + \|e^{\infty}(x') - e^k(x')\|.$$

The middle term on the right-hand side is $\leqslant L|x - x'|$ by the mean value theorem and Lemma 4; the first and last term are bounded by $\sum_{j \geqslant k} |\phi^{(j)}| \leqslant \sum_{j \geqslant k} \xi_j$. Since ξ_k is summable, k is arbitrary, and L uniform, the result follows. □

4.3. Contraction

Let $z_t = z_t^{(\infty)}$ denote a parameterization by arclength of $\mathcal{E}^{\infty}(z)$, with $z_0 = z$.

Lemma 9. *For any $t_1, t_2 \in [\varepsilon, -\varepsilon]$ and $n \geqslant 1$ we have*

$$|\varphi^n(z_{t_1}) - \varphi^n(z_{t_2})| \leqslant \tilde{\gamma}_n |z_{t_1} - z_{t_2}|.$$

Proof. Write $e_n^{(\infty)} = e_n^{(n)} + (e_n^{(\infty)} - e_n^{(n)})$. Then

$$\|\varphi^n(z_{t_1}) - \varphi^n(z_{t_2})\| = \int_{t_1}^{t_2} \|e_n^{(\infty)}\| dt = \int_{t_1}^{t_2} \|D\varphi^n(e^{(n)}) + D\varphi^n(e^{(\infty)} - e^{(n)})\| dt.$$

Clearly $\|D\varphi^n(e^{(n)})\| \leqslant \gamma_n$ and, by Lemma 3,

$$\|e^{(\infty)}(z) - e^{(n)}(z)\| \leqslant \sum_{k \geqslant n} |\phi^{(k)}| \leqslant \sum_{k \geqslant n} \xi_k(z) \leqslant 2 \sum_{k \geqslant n} p_k q_k \gamma_{k+1}.$$

The definition of $\tilde{\gamma}_n$ thus implies the statement in the Lemma. □

4.4. Uniqueness

Here we show that the local stable manifold we have constructed is unique. That is, the set of points which remain in $\mathcal{N}^{(k)}(z)$ for all $k \geqslant 0$ must lie on the curve $\mathcal{E}^{(\infty)}(z)$.

Lemma 10. *The stable manifold through z is unique in the sense that*

$$\mathcal{E}^{(\infty)}(z) = \bigcap_{k \geqslant k_0} \mathcal{N}^{(k)}(z).$$

Proof. Suppose, by contradiction that there is some point $x \in B_{\varepsilon_0}(z)$ which belongs to $\bigcap_{k \geqslant k_0} \mathcal{N}^{(k)}(z)$ but not to $\mathcal{E}^{(\infty)}(z)$. We show that this point must eventually leave $\bigcap_{k \geqslant k_0} \mathcal{N}^{(k)}(z)$. That is, there exists $j \geqslant 1$ such that $x \notin \mathcal{N}^{(j)}(z)$. From the smoothness properties of the $e^{(k)}$ and $f^{(k)}$ vector fields, in particular their Lipschitz property, we may join x to a point $\tilde{x} \in \mathcal{E}^{\infty}(z)$ via a curve γ whose tangent direction has a strictly positive component in the $f^{(k)}$ direction. Hence we obtain

$$d(\varphi^k(z), \varphi^k(x)) \geqslant C \min_{x \in \mathcal{N}^{(k)}} \{F_k\} d(x, \tilde{x}) - \tilde{\gamma}_k d(z, \tilde{x})$$

with $d(z, \tilde{x}) < \varepsilon_0$. Since $F_k \to \infty$ as $k \to \infty$, there exists a $j \geqslant 1$ with $d(\varphi^j(z), \varphi^j(x)) > \varepsilon_j$. $\square$

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