A Theorem of Molien Type in Combinatorics

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In this note we give a simple proof of a special case of the Pólya enumeration theorem, and also a new proof of Burnside's lemma.

1. THE PROBLEM

Let $G$ be a finite group acting on a finite set $S$. Then $G$ also acts on the sets

$$S^{(k)} = \{\{s_1, \ldots, s_k\}; s_i \in S\}$$

by

$$g \cdot \{s_1, \ldots, s_k\} = \{g \cdot s_1, \ldots, g \cdot s_k\}.$$

Denote the number of orbits in this action by $p_k$ and put

$$P(S, G; t) = \sum_{k \geq 0} p_k t^k$$

(we let $p_0 = 1$).

This can, of course, easily be calculated using the Pólya enumeration theorem. Here we will use Burnside's lemma to calculate $P(S, G; t)$ in a very simple way.

2. THE THEOREM

When a group $G$ acts on a set $S$, we denote the set of orbits by $S/G$, and $\langle g \rangle$ is the subgroup of $G$ generated by $g$. The number of elements of a finite set $T$ will be denoted by $|T|$.

**Theorem.**

$$P(S, G; t) = \frac{1}{|G|} \sum_{g \in G} \prod_{O \in S/G} (1 + t^{|O|}).$$

The proof is simple and uses almost only

**Burnside's Lemma.** _When a finite group $G$ acts on a finite set $S$, denote by $\chi(g)$ the number of $s \in S$ such that $g \cdot s = s$. Then_

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

(It is probably more correct to call this Cauchy–Frobenius–Burnside's lemma.) The lemma can of course be proved in a completely elementary way, but let us give an 'invariant-theoretical' proof here.

**Proof of the Lemma.** Let $V$ be a complex vector space with basis $S$. Then the action of $G$ on $S$ gives an action also on $V$, so that $V$ is a $G$-module. For $O \in S/G$, let

$$v_O = \sum_{s \in O} s.$$

Obviously $v_O \in V^G$, the subspace of $V$ consisting of $G$-invariant elements.
Conversely, if $v = \sum_{s \in S} a_s s \in V^G$, then
\[
\sum_s a_s s = v = g^{-1} v = \sum_s a_s g^{-1} s = \sum_s a_{gs} s,
\]
so that $a_{gs} = a_s$ for all $s$ and $g$. Hence $v$ is a linear combination of the $v_0$. Since these are clearly linearly independent, we obtain
\[
|S/G| = \dim V^G.
\]
Let $\langle , , \rangle$ denote the scalar product on the space of central functions on $G$, i.e.
\[
\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.
\]
If $1_G$ is the trivial character of $G$ and $\chi_V$ the character of $V$, then
\[
\dim V^G = \langle 1_G, \chi_V \rangle.
\]
(For the details and the proofs, see, e.g., [3, Ch. 2].) But $\chi_V$ is precisely the $\chi$ in the lemma, whence
\[
|S/G| = \dim V^G = \langle 1_G, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g).
\]
This proves the lemma.

**Proof of the Theorem.** Let $\chi_k(g)$ be the number of fixed points of $g \in G$ in $S^{(k)}$. A subset $\{s_1, \ldots, s_k\}$ of $S$ is a fixed point of $g$ if and only if it is a union of orbits of $\langle g \rangle$ in $S$. Hence
\[
1 + \sum_{k \geq 1} \chi_k(g) = \prod_{\alpha \in S/\langle g \rangle} (1 + t^{\alpha}).
\]
Summing over $g$ proves the theorem, by the lemma.

We will now rewrite the right-hand side in a suggestive way. When $A$ is an $n \times n$ matrix with entries $a_{ij}$, we define its *permanent* to be
\[
\text{per } A = \sum_{\sigma \in S_n} a_{\sigma(1)} \cdots a_{\sigma(n)},
\]
i.e. as the determinant, but without sign changes.

As above, let $V$ be a vector space with basis $S$, and when $O \in S/\langle g \rangle$, let $V(O)$ be the subspace with basis $O$. As an endomorphism of $V(O)$, $g$ has the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\]
in a suitable ordering of the basis. It is easy to see that the permanent of $1 + tg$ (acting on $V(O)$) is $1 + t^{(O)}$. From [2, Ch. 2], Theorem 2.1 (the analogue of the Laplace expansion theorem for determinants) we conclude that on $V$, $1 + tg$ has the permanent
\[
\prod_{O \in S/\langle g \rangle} (1 + t^{(O)}).
\]
Hence the theorem can be rephrased as
\[
P(S, G; t) = \frac{1}{|G|} \sum_{g \in G} \text{per} \ (1 + tg).
\]

3. SOME REMARKS

Let \( V_k \) be the vector space with basis \( S^{(k)} \). Then, as in the proof of Burnside's lemma, we see that
\[
p_k = \dim(V_k)^G,
\]
where \((V_k)^G\) is the subspace consisting of \( G \)-invariant elements. In this sense, the theorem is an invariant-theoretical one. This becomes even more clear if one compares it with the classical Molien theorem.

When \( V \) is a finite-dimensional, complex vector space and \( G \) is a finite subgroup of \( \text{GL}(V) \), then \( G \) acts on the polynomial algebra \( \mathbb{C}[V] \) and also on the exterior algebra \( \Lambda(V) \). Molien's theorem (see e.g., [1], [3], or [5, Prop. 4.1.3]) states that the Hilbert series of the \( G \)-invariant sub-algebras are
\[
H(\mathbb{C}[V]^G, t) = \sum_{m \geq 0} (\dim \mathbb{C}[V]^G_m) t^m = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - tg)}
\]
and
\[
H(\Lambda(V)^G, t) = \sum_{m \geq 0} (\dim \Lambda(V)_m^G) t^m = \frac{1}{|G|} \sum_{g \in G} \det (1 + tg),
\]
where \( \mathbb{C}[V]^G_m \) and \( \Lambda(V)_m^G \) are the subspaces consisting of invariant, homogeneous polynomials of degree \( m \). Hence our theorem is a 'finite' Molien theorem.

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REFERENCES


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