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# Divergence bounded computable real numbers[☆]

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**Abstract**

A real $x$ is called *h-bounded computable*, for some function $h : \mathbb{N} \to \mathbb{N}$, if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ such that, for any $n \in \mathbb{N}$, at most $h(n)$ non-overlapping pairs of its members are separated by a distance larger than $2^{-n}$. In this paper we discuss properties of $h$-bounded computable reals for various functions $h$. We will show a simple sufficient condition for a class of functions $h$ such that the corresponding $h$-bounded computable reals form an algebraic field. A hierarchy theorem for $h$-bounded computable reals is also shown. Besides we compare semi-computability and weak computability with the $h$-bounded computability for special functions $h$.
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## 1. Introduction

In computable analysis, we often consider a computable sequence $(x_s)$ of rational numbers which converges to a real $x$ in order to discuss the effectiveness of $x$ (see, e.g., [12,14,15]). In the optimal situation, the sequence $(x_s)$ converges to $x$ *effectively* in the sense that $|x_s - x_{s+1}| \leqslant 2^{-s}$ for all $s \in \mathbb{N}$. In this case, the limit $x$ can be effectively approximated with an effective error estimation. According to Alan Turing [13], such kind of reals are called *computable*. We denote by **EC** (for Effectively Computable) the class of all computable reals. As shown by Robinson [8], $x$ is computable iff its Dedekind cut $L_x := \{r \in \mathbb{Q} : r < x\}$ is a computable set and iff its binary expansion[1] $x_A := \sum_{i \in A} 2^{-(i+1)}$ is computable (i.e., $A$ is a computable set). Of course, not every real is computable, because there are only countably many computable sequences of rational numbers and hence there are only countably many computable reals, while the set of reals is uncountable. Actually, as shown by Ernst Specker [12], there is an increasing computable sequence which converges to a non-computable real. The limit of an increasing computable sequence of rational numbers is called *left computable* (or *computably enumerable*, *c.e.*, for short, see [2,4]) and **LC** denotes the class of all left computable reals. Thus, we have $\mathbf{EC} \subsetneq \mathbf{LC}$. Similarly, the limit of a decreasing computable sequence of rational numbers is called *right computable*. Left and right computable reals are called *semi-computable*. The classes of right and semi-computable

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[1] In this case we consider only the reals from the unit interval [0; 1]. For other reals $y$, there are an $n \in \mathbb{N}$ and an $x \in [0; 1]$ such that $y = x \pm n$. $x$ and $y$ have obviously the same effectiveness in any reasonable sense.

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reals are denoted by **RC** and **SC**, respectively. The arithmetical closure of **LC** is denoted by **WC**, the class of *weakly computable* reals. It is shown by Ambos-Spies et al. [1], that $x$ is weakly computable iff there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ *weakly effectively* in the sense that $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leqslant c$ for some constant $c$. More generally, we call a real *computably approximable* if there is a computable sequence of rational numbers which converges to it and denote by **CA** the class of all computably approximable reals.

Non-computable reals can be classified further by, say, Turing reduction which can be defined by means of binary expansion (see e.g. [5,16]). Namely, for any $A, B \subseteq \mathbb{N}$, $x_A \leqslant_T x_B$ iff $A \leqslant_T B$, i.e. $A$ is Turing reducible to $B$. In computability theory, the Turing degree $\deg(A)$ of a set $A$ is defined as the class of all subsets of $\mathbb{N}$ which are Turing equivalent to $A$. For a real $x_A$, we can define its Turing degree simply by $\deg_T(x_A) := \deg_T(A)$. However, the classification of reals by Turing degrees is very rough and is not related to the analytical property of reals very well. For example, Zheng [16] has shown that there are reals $x, y$ of c.e. Turing degrees such that their difference $x - y$ does not have even an $\omega$-c.e. Turing degree. Here, a Turing degree is $\omega$-c.e. if it contains an $\omega$-c.e. set which is the limit of a computable sequence $(A_s)$ of finite sets such that $|\{s : n \in (A_s \setminus A_{s+1}) \cup (A_{s+1} \setminus A_s)\}| \leqslant f(n)$ for all $n$ and some computable function $f$.

A much finer classification of non-computable reals is introduced by so-called "Solovay reduction" [11] which can be applied to the class **LC**. Here, for any c.e. reals $x, y$, we say that $x$ is *Solovay reducible* to $y$ (denoted by $x \leqslant_S y$) if there are a constant $c$ and a partial computable function $f :\subseteq \mathbb{Q} \to \mathbb{Q}$ such that $(\forall r \in \mathbb{Q})(r < y \Longrightarrow c\cdot(y-r) > x - f(r))$. Very interestingly, Solovay reduction gives a natural description of the c.e. random reals. Namely, a real $x$ is c.e. random iff it is Solovay complete in the sense that $y \leqslant_S x$ for any c.e. real $y$ (see [2] for the details about this result).

Equivalently, a c.e. real $x$ is Solovay reducible to another c.e. real $y$ if and only if there are two computable increasing sequences $(x_s)$ and $(y_s)$ of rational numbers which converge to $x$ and $y$, respectively, and such that $c(y - y_n) \geqslant x - x_n$ for some constant $c$ and all $n$. In other words, Solovay reduction compares essentially the speed of convergence of the (increasing) approximations to different c.e. reals. Based on the approximation speed, Calude and Hertling [3] discussed the $c$-monotone computability of reals which is extended further to the *$h$-monotonic computability* of reals by Rettinger et al. [7,6] as follows. For any function $h : \mathbb{N} \to \mathbb{Q}$, a real $x$ is called $h$-monotonically computable ($h$-mc, for short) if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ $h$-monotonically in the sense that $h(n)|x - x_n| \geqslant |x - x_m|$ for all $n < m$. Obviously, if $h(n) \leqslant c < 1$, then $h$-mc reals are computable. For the constant function $h \equiv c \geqslant 1$, a dense hierarchy theorem is shown in [6]. Unfortunately, the classes of $h$-monotonically computable reals usually do not have nice analytic property. For example, even the class of $\omega$-monotonically computable reals, i.e., the $h$-mc reals for some computable function $h$, is not closed under addition and subtraction.

The speed of convergence of an approximation $(x_s)$ to $x$ can also be described by counting jumps of certain distance. In [17], a real is called *$h$-Cauchy computable* ($h$-cec, for short) if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ such that, for any $n \in \mathbb{N}$, there are at most $h(n)$ pairs of indices $(i, j)$ with $n \leqslant i < j$ and $2^{-n} \leqslant |x_i - x_j| < 2^{-n+1}$. Denote by $h$-c**EC** the class of all $h$-cec reals. Then, we have obviously that **EC** = 0-c**EC**. Furthermore, a hierarchy theorem of [17] shows that $g$-c**EC**$\nsubseteq f$-c**EC** for any computable functions $f, g$ such that $(\exists^\infty n)(f(n) < g(n))$. Intuitively, if $f(n) < g(n)$ for all $n \in \mathbb{N}$, then an $f$-cec real is easier to approximate than a $g$-cec number. Thus, $h$-Cauchy computability introduces a series of classes of non-computable reals which have different levels of (non)computability.

In this paper, we explore another approach to describe the approximation speed. For any sequence $(x_s)$ which converges to $x$, if the number of non-overlapping index pairs $(i, j)$ such that $|x_i - x_j| \geqslant 2^{-n}$ is bounded by $h(n)$, then we say that $(x_s)$ converges to $x$ *$h$-bounded effectively*. A real $x$ is *$h$-bounded computable* ($h$-bc, for short) if there is a computable sequence of rational numbers which converges to $x$ $h$-bounded effectively. Comparing with the $h$-Cauchy computability, $h$-bounded effective convergence consider all jumps which are larger than $2^{-n}$ instead of only jumps between $2^{-n}$ and $2^{-n+1}$ which appear after stage $n$. This tolerance introduces much better analytic properties of $h$-bounded computable reals. For example, a quite simple property about the class $C$ of functions guarantees that the class of all $C$-bc reals is a field, where a real is $C$-bc if it is $h$-bc for some $h \in C$. Obviously, a hierarchy theorem similar to that on $h$-cec reals does not hold any more. For example, for any constant function $h \equiv c$, only rational numbers are $h$-bc. Nevertheless, we can show another natural hierarchy theorem saying that there is a $g$-bc real which is not $f$-bc, if for any constant $c$, there exists an $n \in \mathbb{N}$ such that $f(n) + c < g(n)$. Also the weak computability of [1] can be well located in the hierarchy of $h$-bounded computable reals.

In the next section, we give the precise definition of $h$-bounded computability and discuss some of its basic properties. Especially, we show a simple condition on the class of functions such that corresponding $h$-bc reals form a field.

In Section 3 we prove the hierarchy theorem for the $h$-bounded computable reals. In Section 4, we compare the $h$-bounded computability with semi-computability and weak computability.

## 2. Divergence bounded computability

In this section, we introduce the definition of the $h$-bounded computability of reals and investigate the basic properties of $h$-bounded computable reals. Especially, we show a simple condition on the function class $C$ such that the corresponding $h$-bounded real class is closed under the arithmetical operations. In the following, two pairs $(i_1, j_1)$ and $(i_2, j_2)$ of indices are called *non-overlapping* if either $i_1 < j_1 \leqslant i_2 < j_2$ or $i_2 < j_2 \leqslant i_1 < j_1$.

**Definition 2.1.** Let $h : \mathbb{N} \to \mathbb{N}$ be a total function, $x$ be a real and let $C$ be a class of total functions $f : \mathbb{N} \to \mathbb{N}$.
(1) A sequence $(x_s)$ converges to $x$ *$h$-bounded effectively* if there are at most $h(n)$ non-overlapping pairs $(i, j)$ of indices such that $|x_i - x_j| \geqslant 2^{-n}$ for all $n \in \mathbb{N}$.
(2) $x$ is *$h$-bounded computable* ($h$-bc, for short) if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ $h$-bounded effectively.
(3) $x$ is *$C$-bounded computable* ($C$-bc, for short) if it is $h$-bc for some function $h \in C$.

The classes of all $h$-bc and $C$-bc reals are denoted by $h$-**BC** and $C$-**BC**, respectively. Especially, if $C$ is the class of all computable total functions, then $C$-**BC** is denoted also by $\omega$-**BC**. Notice that, if $x$ is $h$-bc, then it is also $h_1$-bc for the increasing function $h_1$ defined by $h_1(n) := \max\{h(i) : i \leqslant n\}$. Reasonably, we often consider only the $h$-bounded computability for non-decreasing functions $h : \mathbb{N} \to \mathbb{N}$. The next lemma is straightforward from the definition.

**Lemma 2.2.** *Let $x$ be a real and let $f, g : \mathbb{N} \to \mathbb{N}$ be total functions.*
(1) *$x$ is rational iff $x$ is $f$-bc and* $\liminf_{n\to\infty} f(n) < \infty$.
(2) *If $x$ is computable, then $x$ is id-bc for the identity function $id(n) := n$.*
(3) *If $f(n) \leqslant g(n)$ for almost all $n \in \mathbb{N}$, then $f$-**BC** $\subseteq$ $g$-**BC**.*

The next lemma shows that a constant distance between two functions $f$ and $h$ does not suffice to separate the class $f$-**BC** from $h$-**BC**.

**Lemma 2.3.** *Let $h : \mathbb{N} \to \mathbb{N}$ be a function and $c \in \mathbb{N}$ a constant. Then we have $(h + c)$-**BC** $=$ $h$-**BC**.*

**Proof.** By a simple induction, it suffices to show that $(h+1)$-**BC** $=$ $h$-**BC**. Suppose that $x$ is an $(h+1)$-bc real and $(x_s)$ is a computable sequence of rational numbers which converges to $x$ $(h+1)$-bounded effectively. If for all $n \in \mathbb{N}$, there are at most $h(n)$ non-overlapping index pairs $(i, j)$ such that $|x_i - x_j| \geqslant 2^{-n}$, then $x$ is in fact $h$-bc and we are done. Otherwise, choose a least $n \in \mathbb{N}$ such that there are $h(n) + 1$ pairs of indices $(i, j)$ with $|x_i - x_j| \geqslant 2^{-n}$. Let $(i_0, j_0)$ be the first of such kind of pairs and $i_0 < j_0$. Define a computable sequence $(y_s)$ of rational numbers by $y_s := x_{s+j_0}$ for any $s$. The sequence $(y_s)$ has at least one jump of size $\geqslant 2^{-m}$ less than the sequence $(x_s)$ for all $m \geqslant n$. Then $(y_s)$ converges to $x$ $h$-bounded effectively and hence $x \in h$-**BC**. □

The next theorem gives a sufficient condition for a class $C$ of functions such that $C$-**BC** is closed under the arithmetical operations.

**Theorem 2.4.** *Let $C$ be a class of total functions. If, for any $f, g \in C$ and $c \in \mathbb{N}$, there is a function $h \in C$ such that $h(n) \geqslant f(n + c) + g(n + c)$ for all $n$, then the class $C$-**BC** is an algebraic field.*

**Proof.** Let $f, g \in C$. If $(x_s)$ and $(y_s)$ are computable sequences of rational numbers which converge to $x$ and $y$ $f$- and $g$-bounded effectively, respectively, then by triangle inequalities the computable sequences $(x_s+y_s)$ and $(x_s-y_s)$ converge to $x + y$ and $x - y$ $h_1$-bounded effectively, respectively, for the function $h_1$ defined by $h_1(n) := f(n + 1) + g(n + 1)$.

For the multiplication, choose a natural number $N$ such that $|x_n|, |y_n| \leqslant 2^N$ and define $h_2(n) := f(N+n+1) + g(N+n+1)$ for any $n \in \mathbb{N}$. If $|x_i - x_j| \leqslant 2^{-n}$ and $|y_i - y_j| \leqslant 2^{-n}$, then we have

$$|x_i y_i - x_j y_j| \leqslant |x_i||y_i - y_j| + |y_j||x_i - x_j| \leqslant 2^N \cdot 2^{-n+1} = 2^{-(n-N-1)}.$$

This means that $(x_s y_s)$ converges to $xy$ $h_2$-bounded effectively.

Now suppose that $y \neq 0$ and w.l.o.g. that $y_s \neq 0$ for all $s$. Let $N$ be a natural number such that $|x_s|, |y_s| \leqslant 2^N$ and $|y_s| \geqslant 2^{-N}$ for all $s \in \mathbb{N}$. If $|x_i - x_j| \leqslant 2^{-n}$ and $|y_i - y_j| \leqslant 2^{-n}$, then we have

$$\begin{aligned}\left|\frac{x_i}{y_i} - \frac{x_j}{y_j}\right| &= \left|\frac{x_i y_j - x_j y_i}{y_i y_j}\right| \leqslant \frac{|x_i||y_i - y_j| + |y_j||x_i - x_j|}{|y_i y_j|} \\ &\leqslant 2^{3N} \cdot 2^{-n+1} = 2^{-(n-3N-1)}.\end{aligned}$$

That is, the sequence $(x_s/y_s)$ converges to $(x/y)$ $h_3$-bounded effectively for $h_3(n) := f(3N+n+1) + g(3N+n+1)$. Since the functions $h_1, h_2, h_3$ are bounded by some functions of $C$, the class $C$-**BC** is closed under arithmetical operations $+, -, \times$ and $\div$. $\square$

As a simple example, let $C$ be the class of all constant functions $f_c(n) = c$ for $c \in \mathbb{N}$. Then $C$-**BC** is a field. Actually, $C$-**BC** is the class of rational numbers in this case. Some other examples are listed in the following corollary.

**Corollary 2.5.** *The classes $C$-**BC** are fields for any classes $C$ of functions defined in the following*:
(1) $Lin := \{f : f(n) = c \cdot n + d$ *for some* $c, d \in \mathbb{N}\}$;
(2) $Log^{(k)} := \{f : f(n) = c \cdot \log^k(n) + d$ *for some* $c, d \in \mathbb{N}\}$;
(3) $Poly := \{f : f(n) = c \cdot n^d$ *for some* $c, d \in \mathbb{N}\}$;
(4) $Exp_1 := \{f : f(n) = c \cdot 2^n$ *for some* $c \in \mathbb{N}\}$.

## 3. Hierarchy theorem

In this section we will prove a hierarchy theorem for the $h$-bounded computable reals. By definition, the inclusion $f$-**BC** $\subseteq$ $g$-**BC** holds obviously, if $f(n) \leqslant g(n)$ for almost all $n$. On the other hand, as shown in Lemma 2.3, it does not suffice to separate the class $f$-**BC** from $g$-**BC** if the functions $f$ and $g$ are at most at a constant distance from each other. The next hierarchy theorem shows that more than a constant distance suffices for the separation.

**Theorem 3.1.** *Let $f, g : \mathbb{N} \to \mathbb{N}$ be two computable functions such that*

$$(\forall c \in \mathbb{N})(\exists m \in \mathbb{N})(c + f(m) < g(m)).$$

*Then there exists a $g$-bc real which is not $f$-bc, i.e., $g$-**BC** $\nsubseteq$ $f$-**BC**.*

**Proof.** We will construct a computable sequence $(x_s)$ of rational numbers which converges $g$-bounded effectively to a non-$f$-bc real $x$. That is, $x$ satisfies, for all $e \in \mathbb{N}$, the following requirements:

$$R_e: \quad (\varphi_e(s))_{s\in\mathbb{N}} \text{ converges } f\text{-bounded effectively to } y_e \Longrightarrow y_e \neq x,$$

where $(\varphi_e)$ is an effective enumeration of the partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$. The idea to satisfy a single requirement $R_e$ is easy. We choose an interval $I$ and a natural number $m$ such that $f(m) < g(m)$. Choose further two subintervals $I_e, J_e \subset I$ such that $I_e$ and $J_e$ are at least at a distance $2^{-m}$ apart. Then we can find a real $x$ either from $I_e$ or $J_e$ to avoid the limit $y_e$ of the sequence $(\varphi_e(s))$. To satisfy all requirements simultaneously, we use a finite injury priority construction. In the following construction, we use a second index $s$ to denote the parameters constructed up to stage $s$. For example, $I_{e,s}$ denotes the current value of $I_e$ at stage $s$; and $\varphi_{e,s}(n) = m$ means that the Turing machine $M_e$ which computes $\varphi_e$ outputs $m$ in $s$ steps with the input $n$. However, if it is clear from the context, we often drop the extra index $s$.

Formal construction of the sequence $(x_s)$:

*Stage $s = 0$*: We take the unit interval $[0; 1]$ as the base interval for $R_0$ and let $I_0 := [2^{-(m_0+1)}; 2 \cdot 2^{-(m_0+1)}]$, $J_0 := [4 \cdot 2^{-(m_0+1)}; 5 \cdot 2^{-(m_0+1)}]$ where $m_0 := \min\{m : m \geqslant 3 \;\&\; f(m) < g(m)\}$. Then define $x_0 := 3 \cdot 2^{-(m_0+2)}$.

Notice that the intervals $I_0$ and $J_0$ have the same length $2^{-(m_0+1)}$ and the distance between them is $2^{-m_0}$. The rational number $x_0$ is the middle point of $I_0$. We need another parameter $t_e$ to denote that $\varphi_e(t_e)$ is already used for our strategy. At this stage, let $t_{e,0} := -1$ for all $e \in \mathbb{N}$.

*Stage* $s+1$: Given $t_{e,s}$, $x_s$ and the rational intervals $I_0, I_1, \ldots, I_{k_s}$ and $J_0, \ldots, J_{k_s}$ for some $k_s \geqslant 0$ such that $I_e, J_e \subsetneq I_{e-1}$, $l(I_e) = l(J_e) = 2^{-(m_e+1)}$ and the distance between the intervals $I_e$ and $J_e$ is also $2^{-m_e}$, for all $0 \leqslant e \leqslant k_s$. We say that a requirement $R_e$ *requires attention* if $e \leqslant k_s$ and there is a natural number $t > t_{e,s}$ such that $\varphi_{e,s}(t) \in I_{e,s}$ and $\varphi_e$ does not make more than $f(m_e)$ jumps of distance larger than $2^{-m_e}$ so far. That is, $\max G_{e,s}(m_e, t) \leqslant f(m_e)$, where $G_{e,s}(n,t)$ denotes the following finite set

$$\{m\colon (\exists v_0 < \cdots < v_m \leqslant t)(\forall i < m)(|\varphi_{e,s}(v_i) - \varphi_{e,s}(v_{i+1})| \geqslant 2^{-n})\}.$$

Let $R_e$ be the requirement of highest priority (i.e., of minimal index) which requires attention and let $t$ be the corresponding natural number. Then we exchange the intervals $I_e$ and $J_e$, that is, define $I_{e,s+1} := J_{e,s}$ and $J_{e,s+1} := I_{e,s}$. All intervals $I_i$ and $J_i$ for $i > e$ are set to be undefined. Besides, define $x_{s+1} := \mathrm{mid}(I_{e,s+1})$, $t_{e,s+1} := t$ and $k_{s+1} := e$. In this case, we say that $R_e$ *receives attention* and the requirements $R_i$ for $e < i \leqslant k_s$ are *injured* at this stage.

Otherwise, suppose that no requirement requires attention at this stage. Let $e := k_s$ and let $n_s$ be the maximal $m_{i,t}$ which were defined so far for some $i \in \mathbb{N}$ and $t \leqslant s$. Denote by $j(s)$ the number of non-overlapping index pairs $(i, j)$ such that $i < j \leqslant s$ and $|x_i - x_j| \geqslant 2^{-n_s}$. Then define

$$m_{e+1} := (\mu m)(m \geqslant n_s + 3 \;\&\; j(s) + f(m) < g(m)). \tag{1}$$

Choose five rational numbers $a_i$ (for $i \leqslant 4$) by $a_0 := x_s - 2^{-(m_{e+1}+2)}$ and $a_i := a_0 + i \cdot 2^{-(m_{e+1}+1)}$ for $i := 1, 2, 3, 4$. Then define the intervals $I_{e+1,s+1} := [a_0; a_1]$, $J_{e+1,s+1} := [a_3; a_4]$ and let $x_{s+1} := x_s$. Notice that the intervals $I_{e+1}$ and $J_{e+1}$ have length $2^{-(m_{e+1}+1)}$ and the distance between them is $2^{-m_{e+1}}$. Furthermore, $x_{s+1}$ is the middle point of both intervals $I_e$ and $I_{e+1}$.

This ends the formal construction. To show that our construction succeeds, it suffices to prove the following claims.

**Claim 3.1.1.** *For any $e \in \mathbb{N}$, the requirement $R_e$ requires and receives attention only finitely many times.*

**Proof.** By induction hypothesis we suppose that there is a stage $s_0$ such that no requirement $R_i$ for $i < e$ receives attention after stage $s_0$. Then $m_{e,s} = m_{e,s_0}$ for all $s \geqslant s_0$. The intervals $I_e$ and $J_e$ may be exchanged after stage $s_0$ if $R_e$ receives attention. Notice that, if $R_e$ receives attention at stages $s_2 > s_1 (> s_0)$ successively, then we have $|\varphi_e(t_{e,s_1}) - \varphi_e(t_{e,s_2})| \geqslant 2^{-m_{e,s_0}}$, because the distance between the intervals $I_e$ and $J_e$ is $2^{-m_{e,s_0}}$. This implies that $R_e$ can receive attention after stage $s_0$ at most $f(m_{e,s_0}) + 1$ times because of the condition $\max G_{e,s}(m_e, t) \leqslant f(m_e)$ and hence $R_e$ receives attention finitely often totally. □

**Claim 3.1.2.** *The sequence $(x_s)$ converges $g$-bounded effectively to some $x$ and hence $x$ is $g$-bounded computable.*

**Proof.** By construction, if $x_s \neq x_{s+1}$, then there is an $e$ such that $R_e$ receives attention at stage $s+1$. In this case, we have $2^{-m_{e,s}} < |x_s - x_{s+1}| < 2^{-m_{e,s}+1}$. In addition, if $R_e$ receives attention according to the same $m_{e,s}$ at stage $s+1$ and $t+1 (> s+1)$ consecutively, then we have $|x_s - x_{t+1}| \leqslant 2^{-(m_{e,s}+1)}$ again because of $l(I_{e,s}) = 2^{-(m_{e,s}+1)}$. This means that, if a natural number $n$ has never been chosen as $m_{e,s}$ for some $e$ at some stage $s$, then there are no stages $s_1, s_2$ such that $2^{-n} \leqslant |x_{s_1} - x_{s_2}| \leqslant 2^{-n+1}$. Therefore, it suffices to show that, for any $m_{e,s}$, there are at most $g(m_{e,s})$ non-overlapping index pairs $(i, j)$ such that $|x_i - x_j| \geqslant 2^{-m_{e,s}}$.

Given any $m_{e,s}$, suppose that it is defined for the first time at stage $s$ according to condition (1). Then, there are only $j(s)$ non-overlapping index pairs $(i, j)$ such that $|x_i - x_j| \geqslant 2^{-m_{e,s}}$ up to stage $s$. After stage $s$, each of such jumps corresponds to a stage at which $R_e$ receives attention according to $m_{e,s}$. However, $R_e$ can receive attention at most $f(m_{e,s}) + 1$ times according to this same $m_{e,s}$ and $j(s) + f(m_{e,s}) < g(m_{e,s})$. Therefore, there are at most $g(m_{e,s})$ non-overlapping jumps of $(x_s)$ which are larger than $2^{-m_{e,s}}$. Thus, the computable sequence $(x_s)$ converges $g$-bounded effectively to a $g$-bc real $x$. □

**Claim 3.1.3.** *The real $x$ satisfies all requirements $R_e$. Therefore, $x$ is not $f$-bounded computable.*

**Proof.** For any $e \in \mathbb{N}$, suppose that $\varphi_e$ is a total function and $(\varphi_e(s))$ converges $f$-bounded effectively. By Claim 3.1.1, we can choose an $s_0$ such that $k_{s_0} \geqslant e$ and no requirement $R_i$ for $i \leqslant e$ requires attention after stage $s_0$. This means that $I_e := I_{e,s_0} = I_{e,s}$ and $t_e := t_{e,s_0} = t_{e,s}$ for any $s \geqslant s_0$. By definition of the sequence $(x_s)$, we have $x_s \in I_e$ for all $s \geqslant s_0$ and hence $x \in I_e$.

Assume by contradiction that $x = \lim_{s\to\infty} \varphi_e(s)$. Then there is a stage $s$ and a $t > t_e$ such that $\varphi_e(v)$ is defined for all $v \leqslant t$ and $\varphi_e(t) \in I_e$. Since $(\varphi_e(v))$ converges $f$-bounded effectively, $\max G_{e,s}(m_e, t) \leqslant f(m_e)$. That is, $R_e$ requires attention and will receive attention at stage $s+1$. This contradicts the choice of $s_0$. □

By Claims 3.1.2 and 3.1.3, the real $x$ is $g$-bounded computable but not $f$-bounded computable. This completes the proof of the theorem. □

**Corollary 3.2.** *If $f, g : \mathbb{N} \to \mathbb{N}$ are computable functions such that $f \in o(g)$, then $f$-**BC**$\subsetneq g$-**BC**.*

## 4. Semi-computability and weakly computability

This section discusses the relationship between $h$-bounded computability and other known computability notions of reals. Our first result shows that the classical computability notion of reals cannot be described directly by $h$-bounded computability for any monotone function $h$.

**Theorem 4.1.** *Let $h : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing computable function. Then* **EC**$\subsetneq h$-**BC**.

**Proof.** Suppose that the computable function $h$ is non-decreasing and unbounded. Then we can define a strictly increasing computable function $g : \mathbb{N} \to \mathbb{N}$ inductively by

$$\begin{cases} g(0) := 0, \\ g(n+1) := (\mu t)(t > g(n) \ \& \ h(t) > h(g(n))). \end{cases} \tag{2}$$

This implies that, for any natural numbers $n, m$, if $g(n) \leqslant m < g(n+1)$, then $n \leqslant h(g(n)) = h(m) < h(g(n+1))$.

If $x$ is a computable real, then there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ such that $|x_t - x_s| < 2^{-(s+1)}$ for all $t \geqslant s$. Suppose without loss of generality that $|x_0 - x| < 1$. Define a computable sequence $(y_s)$ by $y_s := x_{g(s)}$ for any $s \in \mathbb{N}$.

For any natural number $n$, we can choose an $i_0 \in \mathbb{N}$ such that $g(i_0) \leqslant n < g(i_0+1)$. Then we have $i_0 \leqslant hg(i_0) = h(n)$ by definition (2). If $(i, j)$ is a pair of indices such that $i < j$ and $|y_i - y_j| = |x_{g(i)} - x_{g(j)}| \geqslant 2^{-n}$, then, by the assumption on $(x_s)$, this implies that $g(i) < n$ and hence $i < i_0$. This means that there are at most $i_0$ non-overlapping pairs of indices $(i, j)$ such that $|y_i - y_j| \geqslant 2^{-n}$. Therefore, the sequence $(y_s)$ converges to $x$ $h$-bounded effectively and hence $x$ is an $h$-bc real.

To show the inequality, we can construct a computable sequence $(x_s)$ of rational numbers which converges $h$-bounded effectively to a non-computable real $x$, i.e., $x$ satisfies, for all $e \in \mathbb{N}$, the following requirements:

$$R_e: \quad (\forall s)(\forall t \geqslant s)(|\varphi_e(s) - \varphi_e(t)| \leqslant 2^{-s}) \Longrightarrow x \neq \lim_{s\to\infty} \varphi_e(s),$$

where $(\varphi_e)$ is an effective enumeration of partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$. This construction can be easily implemented by a finite injury priority technique. We omit the details here because this result can also be deduced directly from a more general result that $h$-**BC**$\not\subseteq$**SC** of Theorem 4.3. □

To prove $h$-**BC**$\not\subseteq$**SC**, we use a necessary condition of semi-computability as follows. Here $A \oplus B := \{2n : n \in A\} \cup \{2n+1 : n \in B\}$ is the join of two sets $A$ and $B$.

**Theorem 4.2** (Ambos-Spies et al. [1]). *If $A, B \subseteq \mathbb{N}$ are Turing incomparable c.e. sets, then the real $x_{A\oplus\overline{B}}$ is not semi-computable.*

Let $h : \mathbb{N} \to \mathbb{N}$ be a function. A set $A \subseteq \mathbb{N}$ is called *$h$-sparse* if, for any $n \in \mathbb{N}$, $A$ contains at most $h(n)$ elements which are less than $n$, namely, $|A\upharpoonright n| \leqslant h(n)$. Applying a finite injury priority construction similar to the original proof of the classical Friedberg-Muchnik Theorem (cf. [10, p. 118]) we can show that, if $h : \mathbb{N} \to \mathbb{N}$ is an unbounded and non-decreasing computable function, then there are Turing incomparable $h$-sparse c.e. sets $A, B \subseteq \mathbb{N}$, i.e., $A \not\leqslant_T B$ & $B \not\leqslant_T A$. Using this observation we can show that $h$-**BC**$\not\subseteq$**SC** for any unbounded and non-decreasing computable $h$.

**Theorem 4.3.** *Let $h : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing computable function. Then there exists an $h$-bc real which is not semi-computable.*

**Proof.** For any unbounded non-decreasing computable function $h$, there are c.e. sets $A, B \subseteq \mathbb{N}$ such that $A$ and $B$ are Turing incomparable and both $2A$ and $2B + 1$ are $h$-sparse. Then $x_{A \oplus \overline{B}}$ is not semi-computable. Furthermore, let $(A_s)$ and $(B_s)$ be the effective enumerations of $A$ and $B$, respectively. We define $x_s := x_{A_s \oplus \overline{B}_s}$. Then $(x_s)$ is a computable sequence of rational numbers which converges to $x_{A \oplus \overline{B}}$. If $i < j$ are two indices such that $|x_i - x_j| \geqslant 2^{-n}$, then there is some $m \leqslant n$ such that either $m/2$ enters $A$ or $(m-1)/2$ enters $B$ between stages $i$ and $j$. Because both $A$ and $B$ are $h$-sparse, there are at most $h(n)$ such non-overlapping index pairs $(i, j)$. Therefore, $x_{A \oplus \overline{B}}$ is $h$-bounded computable. □

Theorem 4.3 shows that the class **SC** does not contain all $h$-bc reals if $h$ is unbounded no matter how slowly the function $h$ increases. However, as observed by Soare [9], the set $A$ must be $\lambda n(2^n)$-c.e. if $x_A$ is a semi-computable real. Here, when a set $A \subseteq \mathbb{N}$ is called $h$-c.e. for some function $h$, this means that there is a computable sequence $(A_s)$ of finite sets such that $\lim_{s\to\infty} A_s = A$ and, for any $n \in \mathbb{N}$, there are at most $h(n)$ stages $s$ with $n \in A_{s+1} \setminus A_s$ or $n \in A_s \setminus A_{s+1}$. This implies immediately that $\mathbf{SC} \subseteq \lambda n(2^n)$-**BC**.

On the other hand, the next result shows that if $f$ is a computable function such that $f \in o(2^n)$, then **SC** is not contained completely in the class $f$-**BC** any more.

**Theorem 4.4.** *Let $o_e(2^n)$ be the class of all computable functions $h : \mathbb{N} \to \mathbb{N}$ such that $h \in o(2^n)$. Then* $\mathbf{SC} \not\subseteq o_e(2^n)$-**BC**.

**Proof.** We will construct an increasing computable sequence $(x_s)$ of rational numbers which converges to some real $x$ and $x$ satisfies, for all natural numbers $e = \langle i, j \rangle$, the following requirements:

$$R_e : \left.\begin{array}{c}\varphi_i \text{ and } \psi_j \text{ are total functions and } \psi_j \in o(2^n) \\ (\varphi_i(s)) \text{ converges } \psi_j\text{-bounded effectively}\end{array}\right\} \Longrightarrow x \neq \lim_{s\to\infty} \varphi_i(s),$$

where $(\varphi_e)$ and $(\psi_e)$ are effective enumerations of all partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$ and $\psi_e :\subseteq \mathbb{N} \to \mathbb{N}$, respectively.

To satisfy a single requirement $R_e$ ($e = \langle i, j \rangle$), we choose a rational interval $I_{e-1}$ of length $2^{-m_{e-1}}$ for some natural number $m_{e-1}$ and look for a "witness" interval $I_e \subseteq I_{e-1}$ such that every element of $I_e$ satisfies $R_e$.

Firstly, the interval $I_{e-1}$ is divided into four equidistant subintervals $J_e^t$ for $t < 4$ and let $I_e := J_e^1$ as the (default) candidate of witness interval of $R_e$. If the function $\psi_j$ is not a total function such that $\psi_j \in o(2^n)$, then $R_e$ is satisfied trivially and $I_e$ is already a correct witness interval. Otherwise, there exists a natural number $m_e > m_{e-1} + 2$ such that $2(\psi_j(m_e) + 2) \cdot 2^{-m_e} \leqslant 2^{-(m_{e-1}+2)}$. In this case, we divide the interval $J_e^3$ (which is of length $2^{-(m_{e-1}+2)}$) into subintervals $I_e^t$ of length $2^{-m_e}$ for $t < 2^{m_e-(m_{e-1}+2)}$ and let $I_e := I_e^1$ as the new candidate of witness interval of $R_e$. If the sequence $(\varphi_i(s))$ does not enter the interval $I_e^1$ at all, then we are done. Otherwise, suppose that $\varphi_i(s_0) \in I_e^1$ for some $s_0 \in \mathbb{N}$. Then we change the witness interval to be $I_e^3$. If $\varphi_i(s_1) \in I_e^3$ for some $s_1 > s_0$, then let $I_e := I_e^5$, and so on. This can happen at most $\psi_j(m_e)$ times if the sequence $(\varphi_i(s))$ converges $\psi_j$-bounded effectively. This means that a correct witness interval of $R_e$ can be eventually found in finitely many steps.

To satisfy all requirements $R_e$ simultaneously, we apply a finite injury priority construction described precisely as follows.

Formal construction of the sequence $(x_s)$:

*Stage $s = 0$*: Let $m_0 := 2$, $J_0^k := [k/4; (k+1)/4]$ for $k < 4$, $I_0 := J_0^1$ and $x_0 := \frac{1}{4}$. Set the requirement $R_0$ into the "default" state and all other requirements $R_e$ for $e > 0$ into the "waiting" state.

*Stage $s + 1$*: Given a natural number $e_s$ such that, for all $e \leqslant e_s$, the natural number $m_e$, the rational intervals $I_e$ and $J_e^k$ for $k < 4$ (if $R_e$ is in the "default" state) or $I_e^t$ for some $t$'s (if $R_e$ is in the "waiting" or "satisfied" state) are defined.

A requirement $R_e$ for $e = \langle i, j \rangle$ *requires attention* if $e \leqslant e_s$ and one of the following situations appears.

(R1) $R_e$ is in the "default" state and there is an $m \in \mathbb{N}$ such that

$$m > m_{e,s} + 2 \;\&\; (\psi_{j,s}(m) + 2) \cdot 2^{-m+1} \leqslant 2^{-m_{e,s}}. \tag{3}$$

(R2) $R_e$ is in the "ready" state and there is a $t \in \mathbb{N}$ such that $\varphi_{i,s}(t) \in I_e$.

If no requirement requires attention, then we define $e_{s+1} := e_s + 1$ and $m_{e_{s+1}} := m_{e_s} + 2$. Then divide the interval $I_{e_s}$ into four equidistant subintervals $J^k_{e_{s+1}}$ for $k < 4$ and let $I_{e_{s+1}} := J^1_{e_{s+1}}$. Finally, set $R_{e_{s+1}}$ into the "default" state.

Otherwise, let $R_e$ ($e = \langle i, j\rangle$) be the requirement of highest priority (i.e., of minimal index $e$) which requires attention and consider the following cases.

*Case* 1: The requirement $R_e$ is in the "default" state at stage $s$. Define $m_{e,s+1}$ as the minimal natural number $m$ which satisfies condition (3). Then we divide the interval $J^3_e$ into subintervals $I^t_e$ of length $2^{-m_{e,s+1}}$ for $t < 2^{m_{e,s+1}-m_{e,s}}$. Let $I_{e,s+1} := I^1_e$ be the new witness interval of $R_e$. The requirement $R_e$ is set into the "ready" state and all requirements $R_{e'}$ for $e' > e$ are set back into the "waiting" state.

*Case* 2: The requirement $R_e$ is in the "ready" state. If $I_{e,s} = I^t_{e,s}$ for some $t \in \mathbb{N}$ and $I^{t+1}_{e,s}$ is also defined, then let $e_{s+1} := e$ and $I_{e,s+1} := I^{t+1}_{e,s}$ and set all requirements $R_{e'}$ for $e' > e$ into the "waiting" state. Otherwise, if $I_{e,s} = I^t_{e,s}$ and $I^{t+1}_{e,s}$ is not defined any more, then set simply the requirement $R_e$ into the "satisfied" state and go directly to the next stage.

In both cases, we say that the requirement $R_e$ *receives attention*.

At the end of stage $s+1$, we define $x_{s+1}$ as the left endpoint of the rational interval $I_{e_{s+1}}$. This ends the construction. To show that our construction succeeds, it suffices to prove the following claims.

**Claim 4.4.1.** *Each requirement requires and receives attention only finitely many times and hence the limits $I_e := \lim_{s\to\infty} I_{e,s}$ exist.*

**Proof.** For any $e \in \mathbb{N}$, suppose by induction hypothesis that there is an $s_0$ such that no requirement $R_i$ for $i < e$ requires and receives attention after stage $s_0$. Assume w.l.o.g. that $e \leqslant e_{s_0}$, i.e., the natural number $m_{e,s_0}$ and an interval $I_{e,s_0}$ of the length $2^{-m_{e,s_0}}$ are defined.

*Case* A: $R_e$ is in the "default" state at stage $s_0$. Then the intervals $J^t_{e,s_0}$ for $t < 4$ are defined too. Suppose that the function $\psi_j$ is total and $\psi_j \in o(2^n)$ (otherwise $R_e$ is satisfied trivially). Then there is a (minimal) $s_1 > s_0$ and a natural number $m$ which satisfy condition (3). This means that $R_e$ requires, receives attention and is set into the "ready" state at stage $s_1 + 1$. It goes into case B.

*Case* B: $R_e$ is in the "ready" state at stage $s_0$. In this case, the intervals $I^t_e$ are already defined, say, at stage $s' + 1 \leqslant s_0$. Namely, at stage $s' + 1$, the interval $J^3_e$ is divided into subintervals $I^t_e$ of length $2^{-m_{e,s'+1}}$ for $t < T := 2^{m_{e,s'+1}-m_{e,s'}}$. Suppose that $I_{e,s_0} = I^{t_0}_{e,s_0}$ for some $t_0 = 2k + 1 < T$. After stage $s_0$, if $R_e$ receives attention at stage $s + 1$ with $I_{e,s} = I^t_{e,s_0}$ and $t + 2 < T$, then interval $I_e$ will be moved from some $I^t_{e,s_0}$ to $I^{t+2}_{e,s_0}$ and $R_e$ remains in the "ready" state. Of course, this can happen at most $T/2$ times. Namely, either $R_e$ will remain in the "ready" state after some stage and never require attention again, or it will be set into the "satisfied" state.

*Case* C: $R_e$ is in the "satisfied" state at stage $s_0$. Then $R_e$ will remain in this state and never require attention after stage $s_0$ any more.

In all above cases, the requirement $R_e$ requires and receives attention only finitely often totally. □

**Claim 4.4.2.** *The sequence $(x_s)$ is non-decreasing and the limit $x := \lim_{s\to\infty} x_s$ satisfies all requirements $R_e$.*

**Proof.** By construction, the sequence $(x_s)$ is obviously non-decreasing and hence the limit $x := \lim_{s\to\infty} x_s$ exists. Now we are going to show that $x$ satisfies all requirements $R_e$.

For any $e \in \mathbb{N}$, by Claim 4.4.1, there is an $s_0$ such that $R_e$ does not require attention after stage $s_0$. Suppose w.l.o.g. that $I_{e,s_0}$ is defined, i.e., $R_e$ is not in the "waiting" state. Then we have $I_{e,s} = I_{e,s_0}$ and $m_{e,s} = m_{e,s_0}$ for all $s \geqslant s_0$. Suppose that the assumptions on $R_e$ hold. Let us consider the following situations.

*Case* I: $R_e$ is in the "default" state. Since $\psi_j \in o(2^n)$, there must be some $s > s_0$ and $m \in \mathbb{N}$ which satisfy condition (3). Then $R_e$ requires attention at stage $s + 1$ and this contradicts the choice of $s_0$. Thus, this case cannot occur.

*Case* II: $R_e$ is in the "ready" state. From the construction it is easy to see that $x$ is an inner point of the interval $I_{e,s_0}$. Because $R_e$ never requires attention, the sequence $(\varphi_i(s))$ does not enter the interval $I_{e,s_0}$ and hence, $\lim_{s\to\infty} \varphi_i(s) \neq x$. Hence $R_e$ is satisfied at this case.

*Case* III: $R_e$ is in the "satisfied" state. Let $s_1$ be the last stage before stage $s_0$ at which the requirement $R_e$ is set into the "default" state. At stage $s_1$, we define a natural number $m_{e,s_1}$ and four intervals $J^k_{e,s_1}$ of length $2^{-m_{e,s_1}}$ for $k < 4$ and finally define $I_{e,s_1} := J^1_{e,s_1}$. Between stages $s_1$ and $s_0$, the requirement $R_e$ is set into the "ready" state at, say, stage $s_2+1$. At this stage, we define $m_{e,s_2+1}$ as the minimal natural number $m$ which satisfies condition (3) and divide the interval $J^3_{e,s_2}$ into subintervals $I^t_{e,s_2+1}$ for $t < T := 2^{m_{e,s_2+1}-m_{e,s_2}}$. Since $m_{e,s_2+1}$ satisfies the condition that $2(\psi_j(m_{e,s_2+1})+2)\cdot 2^{-m_{e,s_2+1}} \leqslant 2^{-m_{e,s_2}}$, the number of subintervals $I^t_{e,s_2+1}$ is at least $2\psi_j(m_{e,s_2+1})+2$ and hence $\psi_j(m_{e,s_2+1}) < T/2-1$. After stage $s_2+1$, $R_e$ will never be reset into "waiting" state, these intervals remain unchanged after stage $s_2+1$. Thus, we can denote them simply by $I^t_e := I^t_{e,s_2+1}$. At stage $s_2+1$, we define also $I_{e,s_2+1} := I^1_e$. Between stages $s_2+1$ and $s_0$, $R_e$ receives attention at, say, stages $v_0+1 < v_1+1 < \cdots < v_N+1 \leqslant s_0$. Notice that $I_{e,v_0} = I^1_e$. At any stage $v_t+1$ for $t < N$, we define $I_{e,v_t+1} = I^{k+2}_e$ if $I_{e,v_t} = I^k_e$ and $k+2 < T$. However, at stage $v_N+1$, $R_e$ should be set into the "satisfied" state. This means that $I_{e,v_N} = I^k_e$ for some $k$ such that $k < T \leqslant k+2$. Then, by a simple induction, we can show that $I_{e,v_t} = I^{2t+1}_e$ for any $t < N$ and $N = T/2-1$. Because of the requiring condition (R2), there are natural numbers $n_t$, for $t < N$, such that $\varphi_i(n_t) \in I_{e,v_t} = I^{2t+1}_e$ and hence $|\varphi_i(n_t) - \varphi_i(n_{t+1})| \geqslant l(I^{2t}_e) = 2^{-m_{e,s_2+1}}$. Since $N = T/2-1 > (\psi_j(m_{e,s_2+1}))$, the sequence $(\varphi_i(s))$ does not converge $\psi_j$-bounded effectively. This contradicts the hypothesis on $R_e$ and implies that this case does not occur actually either.

Therefore, $x$ satisfies all requirements $R_e$. □

By Claim 4.4.2, $x$ is left computable but not $\mathrm{o}_e(2^n)$-bounded computable. □

It is worth noting that the class $\mathrm{o}_e(2^n)$ is only the part of $\mathrm{o}(2^n)$ where only the computable functions are considered. For the class $\mathrm{o}(2^n)$ the situation is different as shown in the next results.

**Lemma 4.5.** *If $x$ is a semi-computable real, then there is a function $h \in \mathrm{o}(2^n)$ such that $x$ is $h$-bc. Thus,* **SC** $\subseteq \mathrm{o}(2^n)$-**BC**.

**Proof.** We consider only the left computable $x$. For right computable reals the proof is similar. Let $(x_s)$ be a strictly increasing computable sequence of rational numbers which converges to $x$. Define a function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) := |\{s \in \mathbb{N} : 2^{-n} \leqslant (x_{s+1} - x_s) < 2^{-n+1}\}|.$$

Then we have $\sum_{n\in\mathbb{N}} g(n)\cdot 2^{-n} \leqslant \sum_{s\in\mathbb{N}} |x_s - x_{s+1}| = x_0 - x$. This implies that $g \in \mathrm{o}(2^n)$. Especially, there is an $N_0 \in \mathbb{N}$ such that $g(n) \leqslant 2^n$ for all $n \geqslant N_0$. Let $c_1 := \sum_{i\leqslant N_0} g(i)$.

Let $h(n) := \sum_{i=0}^n g(i)$. Then the sequence $(x_s)$ converges $h$-bounded effectively. It remains to show that $h \in \mathrm{o}(2^n)$. Given any constant $c > 0$, there is an $N_1 \geqslant N_0$ such that $g(n) \leqslant c/4\cdot 2^n$ for all $n \geqslant N_1$. Thus, for any $n$ large enough such that $2^n \geqslant 2(c_1 + 2^{N_1+1})/c$, we have

$$\begin{aligned} h(n) &= \sum_{i=0}^{n} g(i) = \sum_{i\leqslant N_0} g(i) + \sum_{i=N_0+1}^{N_1} g(i) + \sum_{i=N_1+1}^{n} g(i) \\ &\leqslant c_1 + \sum_{i=N_0+1}^{N_1} 2^i + \sum_{i=N_1}^{n} c/4\cdot 2^i \leqslant c_1 + 2^{N_1+1} + c/4\cdot 2^{n+1} \\ &= 2^n(c_1\cdot 2^{-n} + 2^{(N_1+1)-n} + c/2) \leqslant c\cdot 2^n. \end{aligned}$$

Thus, $h \in \mathrm{o}(2^n)$ and the sequence $(x_s)$ converges $h$-bounded effectively. Hence $x$ is a $h$-bc real. □

By Theorem 2.4, class $\mathrm{o}(2^n)$-**BC** is a field which contains all semi-computable reals. But **WC** is the arithmetic closure of **SC**. Therefore, we have

**Corollary 4.6.** *Any weakly computable real is $h$-bounded computable for some function $h \in \mathrm{o}(2^n)$. Namely,* **WC** $\subseteq \mathrm{o}(2^n)$-**BC**.

Our next result shows that the inclusion $\mathbf{WC} \subseteq \mathrm{o}(2^n)$-$\mathbf{BC}$ is proper.

**Theorem 4.7.** *There is an* $\mathrm{o}(2^n)$*-bc real which is not weakly computable. That is,* $\mathbf{WC} \subsetneq \mathrm{o}(2^n)$-$\mathbf{BC}$.

**Proof.** We construct a computable sequence $(x_s)$ of rational numbers and a (not necessarily computable) function $h : \mathbb{N} \to \mathbb{N}$ such that the sequence $(x_s)$ converges $h$-bounded effectively to a non-weakly computable real $x$. That is, $x$ satisfies all the following requirements:

$$R_e: \left.\begin{array}{c}\varphi_e \text{ is a total function, and} \\ \sum_{s\in\mathbb{N}} |\varphi_e(s) - \varphi_e(s+1)| \leqslant 1\end{array}\right\} \Longrightarrow \lim_{s\to\infty} \varphi_e(s) \neq x,$$

where $(\varphi_e)$ is an effective enumeration of all partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$.

The strategy to satisfy a single requirement $R_e$ is quite simple. Namely, we choose two rational intervals $I_e$ and $J_e$ such that their distance is $2^{-m_e}$ for some natural number $m_e$. Then we choose the middle point of $I_e$ as $x$ whenever the sequence $(\varphi_e(s))$ does not enter the interval $I_e$. Otherwise, we choose the middle of $J_e$. If the sequence $(\varphi_e(s))$ enters the interval $J_e$ at a later stage, then define $x$ as the middle point of $I_e$ again, and so on. Because of the condition $\sum_{s\in\mathbb{N}} |\varphi_e(s) - \varphi_e(s+1)| \leqslant 1$, we need at most $2^{m_e}$ changes. By a finite injury priority construction, this works for all requirements simultaneously. However, the real $x$ constructed in this way is only a $2^n$-bounded computable real. To guarantee the $\mathrm{o}(2^n)$-bounded computability of $x$, we need several $m_e$'s instead of just one. That is, we choose at first a natural number $m_e > e$, two rational intervals $I_e$ and $J_e$ and implement the above strategy, but at most $2^{m_e-e}$ times. Then we look for a new $m'_e > m_e$ and apply the same procedure up to $2^{m'_e-e}$ times, and so on. This means that, in worst case, we need $2^e$ different $m_e$'s to satisfy a single requirement $R_e$. We can see that the finite injury priority technique can still be applied. More precisely, we have the following formal construction.

*Stage* $s = 0$: Set $k_0 := 0$, $I_0 := [7/16; 9/16]$, $J_0 := [13/16; 15/16]$, $m_0 := 2$, $m_{-1} := -1$, $c_0 := 0$ and $x_0 := \mathrm{mid}(I_0) = \frac{1}{2}$. Furthermore, we define $t_e := 0$ for all natural numbers $e$. Here, we use the counter $c_e$ to denote how many times the current parameter $m_e$ is used for $R_e$, and $t_e$ denotes that $\varphi_e(t_e)$ is just considered.

*Stage* $s+1$: Given a natural number $k_s \geqslant 0$ such that, for all $i \leqslant k_s$, the rational intervals $I_i$, $J_i$, the natural numbers $m_i$, $t_i$ and $c_i$ are defined. The lengths $l(I_i) = l(J_i) = 2^{-(m_i+1)}$ and the distance between the intervals $I_i$ and $J_i$ is $2^{-m_i}$.

A requirement $R_e$ *requires attention* if $e \leqslant k_s$ and there is a natural number $t > t_e$ such that

$$(\forall v \leqslant t)(\varphi_{e,s}(v)\downarrow) \ \&\ \varphi_{e,s}(t) \in I_e \ \&\ \sum_{v<t} |\varphi_{e,s}(v) - \varphi_{e,s}(v+1)| \leqslant 1. \tag{4}$$

Let $R_e$ be the requirement of highest priority which requires attention and $t$ the least natural number which satisfies condition (4). We consider the following cases.

*Case* 1: $c_e < 2^{m_e-e}$. We define $k_{s+1} := e$, exchange the intervals $I_e$ and $J_e$, i.e., define $I_{e,s+1} := J_{e,s}$ and $J_{e,s+1} := I_{e,s}$ and, furthermore, let $t_{e,s+1} := t$, and $c_{e,s+1} := c_{e,s} + 1$.

*Case* 2: $c_e = 2^{m_e-e}$. In this case, we have exchanged intervals $I_e$ and $J_e$ already $2^{m_e-e}$ times. Another exchange is not allowed in order to guarantee the sequence $(x_s)$ converges $\mathrm{o}(2^n)$-bounded effectively. Therefore, we have to define a new $m_e$. Thus, let $k_{s+1} := e$. We define $m_{e,s+1} := m_{k_s} + e + 3$, divide the interval $I_{k_s} = [a; b]$ equally by $a = a_0 < a_1 < \cdots < a_{16} = b$ and then define two new rational intervals $I_e$ and $J_e$ by $I_e := [a_7; a_9]$ and $J_e := [a_{13}; a_{15}]$ if $\varphi_{e,s}(t) \notin [a_7; a_9]$ and $J_e := [a_7; a_9]$ and $I_e := [a_{13}; a_{15}]$ otherwise. Finally, define $t_{e,s+1} := t$, and reset the counter $c_{e,s+1} := 0$.

In both cases, we say that the requirement $R_e$ *receives attention*, or more precisely, *receives* $m_{e,s+1}$*-attention*. For all $i > e$, we *initialize* the requirements $R_i$ by setting the intervals $I_i$, $J_i$ and parameters $m_i$, $t_i$, $c_i$ to be undefined. These requirements $R_i$ are said to be injured by $R_e$ if $e < i < k_s$.

If no requirement requires attention at this stage, then we define $k_{s+1} := k_s + 1$ and act similarly to case 2 above. Namely, for $e = k_{s+1}$, we define $c_{e,s+1} := 0$ and $m_{e,s+1} := n_s + e + 3$ where $n_s$ is the maximal natural number which is used as $m_{i,v}$ for some $i$ and $v \leqslant s$. Then we define two rational intervals $I_e := [a_7; a_9]$ and $J_e := [a_{13}; a_{15}]$ where $a = a_0 < a_1 < \cdots < a_{16} = b$ is an equidistant division of the interval $I_{k_s} = [a; b]$. In this case, we say that the requirement $R_e$ receives default attention.

In all cases, we define $x_{s+1} := \mathrm{mid}(I_{k_{s+1}})$ and all other parameters which are not explicitly defined remain the same as in stage $s$. This ends the construction. To show that our construction succeeds, we prove the following claims.

**Claim 4.7.1.** *For any $e \in \mathbb{N}$, the requirement $R_e$ requires and receives attention only finitely many times.*

**Proof.** We prove the claim by induction on $e \in \mathbb{N}$. Suppose by induction hypothesis that, for all $i < e$, the requirement $R_i$ requires and receives attention only finitely many often. Then there is a minimal stage $s_0$ such that no requirement $R_i$ for $i < e$ requires and receives (normal or default) attention after stage $s_0$. By the minimality of $s_0$, we have either $s_0 = 0$ or $k_{s_0} = e - 1$. Thus, at stage $s_0 + 1$, the requirement $R_e$ receives default attention. Namely, we define a new $m_e$, and two intervals $I_e$ and $J_e$ of length $2^{-(m_e+1)}$ such that they are separated by a distance $d\,(I_e, J_e) = 2^{-m_e}$. In this case, the counter $c_e$ is set to be 0. Every time, if $R_e$ receives attention with this $m_e$, then the counter $c_e$ increases by 1 until $c_e = 2^{m_e-e}$. This means that the requirement $R_e$ can receive attention with this $m_e$ at most $2^{m_e-e}$ times according to case 1. After that, if it is necessary, a new $m_e$ will be defined according to case 2 and the counter is set to be 0 again. However, if $R_e$ receives attention for the same $m_e$ at stages $v_0 < v_1 < \cdots < v_l$ for $l = 2^{m_e-e}$, then we have $\sum_{t=0}^{v_l} |\varphi_e(t) - \varphi_e(t+1)| \geqslant \sum_{i=0}^{l-1} |\varphi_e(t_{e,v_i}) - \varphi_e(t_{e,v_{i+1}})| \geqslant 2^{-m_e} \cdot l = 2^{-e}$. This implies that at most $2^e$ different $m_e$'s can be chosen after stage $s_0$ and hence $R_e$ requires and receives attention finitely many times totally. □

**Claim 4.7.2.** *For any $e$, the limits $m_e^* := \lim_{s\to\infty} m_{e,s}$ and $I_e^* := \lim_{s\to\infty} I_{e,s}$ exist and they satisfy the following conditions*:

$$l(I_e^*) = 2^{-(m_e^*+1)} \;\&\; I_{e+1}^* \subsetneq I_e^* \;\&\; m_e^* + e + 3 \leqslant m_{e+1}^*. \tag{5}$$

**Proof.** It follows immediately from Claim 4.7.1 and the definition of $m_{e,s+1}$ in the construction. □

By Claim 4.7.2, $(m_e)$ is a strictly increasing sequence of natural numbers. Thus, we define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) := 2^{m_e^*-e+1}$ for any $m_{e-1}^* < n \leqslant m_e^*$. Thus, $h \in o(2^n)$. Of course, the function $h$ is not necessarily computable.

**Claim 4.7.3.** *The sequence $(x_s)$ converges $h$-bounded effectively to some $x$, hence $x$ is $o(2^n)$-bounded computable.*

**Proof.** For any natural number $n$, there exists a minimal $e \in \mathbb{N}$ such that $n \leqslant m_e^*$. Let $m_{e_0,s_0} < m_{e_1,s_1} < \cdots < m_{e_k,s_k}$ be all natural numbers less than $m_e^*$ which are defined in the construction. Remember that we have $m_e^* \geqslant m_{e_k,s_k} + e + 3$. By construction, if a requirement $R_i$ requires $m_{i,s}$-attention at stage $s + 1$, then we have either $x_s = x_{s+1}$ (in case 2 for $\varphi_i(t) \in [a_7; a_9]$ or $R_i$ receives default attention) or $2^{-m_{i,s}} < |x_s - x_{s+1}| < 2^{-m_{i,s}+1}$. This means that the jumps of the sequence $(x_s)$ which are greater than $2^{-m_e^*}$ can only be caused when $R_e$ receives $m_e^*$-attention or $R_{e_i}$ receives $m_{e_i,s_i}$-attention for some $i \leqslant k$. Since for any fixed $m_{e_i,s_i}$, the requirement $R_{e_i}$ can receive $m_{e_i,s_i}$-attention at most $2^{m_{e_i,s_i}-e_i}$ times, the number of jumps of distance larger than $2^n$ is bounded by

$$\begin{aligned}\sum_{i=0}^{k} 2^{m_{e_i,s_i}-e_i} + 2^{m_e^*-e} &\leqslant \sum_{i=0}^{k} 2^{m_{e_i,s_i}} + 2^{m_e^*-e}\\ &\leqslant 2^{m_{e_k,s_k}+1} + 2^{m_e^*-e} \leqslant 2^{m_e^*-e+1} = h(n).\end{aligned}$$

That is, the sequence $(x_s)$ converges $h$-bounded effectively and the limit $x := \lim_{s\to\infty} x_s$ is $h$-bounded computable. Because $h \in o(2^n)$, $x$ is also $o(2^n)$-bounded computable. □

**Claim 4.7.4.** *The limit $x := \lim_{s\to\infty} x_s$ satisfies all requirements $R_e$ and hence it is not weakly computable.*

**Proof.** By construction we have $x_s \in I_{k_s} \subsetneq I_{e,s}$ for any $e \leqslant k_s$. This implies that $x \in I_e^*$ for any $e \in \mathbb{N}$. For any fixed $e \in \mathbb{N}$, by Claim 4.7.1, there is an $s_0$ such that the requirement $R_e$ does not require and receive attention after stage $s_0$. Therefore, $I_{e,s} = I_e^*$ for any $s \geqslant s_0$. If $\varphi_e$ is a total function such that $\sum_{s\in\mathbb{N}} |\varphi_e(s) - \varphi_e(s+1)| \leqslant 1$, then there is no $t > t_{e,s_0}$ such that $\varphi_e(t) \in I_e^*$. Otherwise, there is a stage $s_1 > s_0$ such that $\varphi_{e,s_1}(v)$ is defined for all $v \leqslant t$ and $\sum_{s\leqslant t} |\varphi_e(s) - \varphi_e(s+1)| \leqslant 1$. That is, condition (4) is satisfied and $R_e$ requires attention at stage $s_1$. This contradicts the choice of $s_0$. This means that the sequence $(\varphi_e(s))$ does not enter the interval $I_e^*$ and hence the limit $y_e = \lim_{s\to\infty} \varphi_e(s)$, if it exists, is not an inner point of $I_e^*$. On the other hand, $x \in I_{e+1}^* \subset I_e^*$ and $I_{e+1}^*$ consists only of the inner points of $I_e^*$. Therefore, $x \neq y_e$ and $R_e$ is satisfied. This implies that $x$ is not weakly computable. □

By Claims 4.7.3 and 4.7.4, the limit $x$ is an $o(2^n)$-bounded computable but not weakly computable real. This completes the proof of the theorem. □

Since the function $h$ constructed in the above proof is not necessarily computable, it is not clear whether the class $\mathrm{o}_e(2^n)$ is contained properly in **WC** or incomparable with **WC**.

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