ELSEVIER

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Discrete Mathematics 275 (2004) 339–346

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

On 3-cutwidth critical graphs☆

Yixun Lin, Aifeng Yang

Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450052, China

Received 13 September 2002; received in revised form 19 June 2003; accepted 24 June 2003

Abstract

The cutwidth of a graph G is the minimum congestion (the number of overlap edges) when G is embedded into a path. The cutwidth problem has been motivated from both applied and theoretical points of view. The characterization of forbidden subgraphs or critical graphs is always interesting in the study of a graph-theoretic parameter. In this paper we characterize the set of 3-cutwidth critical graphs by five specified elements.

Keywords: Graph labeling; Cutwidth; Critical graph

1. Introduction

The cutwidth problem has been motivated from both applied and theoretical aspects. Some application areas of the problem include the circuit layout design and the network communication [2,4]. And its theoretic interest comes up in connection with other graph-theoretic parameters such as bandwidth, pathwidth and treewidth (see [2,3,6]).

Let $G=(V,E)$ be a simple graph with vertex set V, $|V|=n$, and edge set E. A *labeling* of G is a bijection $f:V\to\{1,2,\ldots,n\}$, which can be regarded as an embedding of G into a path P_n. For a given labeling f of G, the cutwidth of G with respect to f is

$$c(G,f)=\max_{1\leqslant i<n}|\{uv\in E:\ f(u)\leqslant i<f(v)\}|,$$

which represents the congestion of the embedding. The *cutwidth* of G is defined by

$$c(G)=\min_{f} c(G,f),$$

☆ Project supported by NSFC (10071076).
E-mail address: linyixun@zzu.edu.cn (Y. Lin).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.06.012

where the minimum is taken over all labelings f. A labeling f attaining the above minimum value is called an optimal labeling.

In the embedding version, we may denote $u_i = f^{-1}(i)$ $(1 \leqslant i \leqslant n)$. Then the labeling f can be regarded as an ordering of vertices $u_1, u_2, \ldots, u_n$ arranged on a line. Let $S_i = \{u_1, u_2, \ldots, u_i\}$ be the set of the first i vertices. The cut $\nabla(S_i) = \{u_j u_k \in E : j \leqslant i < k\}$ is called the cut at $[i, i+1]$. Then the cutwidth $c(G, f)$ is the maximum size of all these cuts $\nabla(S_i)$, $i = 1, 2, \ldots, n-1$.

The cutwidth problem for general graphs is known to be NP-hard [5], while it has polynomial algorithms for trees [12]. Some exact results on special graphs, e.g., the complete graphs K_n, the complete bipartite graphs $K_{m,n}$, the n-cubes Q_n, the complete k-ary trees, the trees with diameter at most 4, and the meshes $P_m \times P_n, P_m \times C_n, C_m \times C_n$, have been obtained in the literature [2,7–10]. The relations between cutwidth and other graph-theoretic parameters were studied in various aspects [2,3,6].

In a theoretical point of view, the cutwidth has the following basic properties.

Proposition 1.1. (1) *If G' is a subgraph of G, then $c(G') \leqslant c(G)$.* (2) *If G' is homeomorphic to G (i.e., they can both be obtained from the same graph by inserting new vertices of degree two into its edges, called a subdivision of the graph), then $c(G') = c(G)$.*

In fact, the first property is obvious. The second is easy to show by the embedding version of cutwidth (refer to [13]).

Based on these properties, we may define the cutwidth critical graphs as follows. A graph G is said to be *k-cutwidth critical* if

(1) $c(G) = k$;
(2) for every proper subgraph G' of G, $c(G') < k$;
(3) G is homeomorphically minimal, that is, G is not a subdivision of any simple graph.

Proposition 1.2. *The unique* 1*-cutwidth critical graph is K_2. The only* 2*-cutwidth critical graphs are K_3 and $K_{1,3}$.*

In fact, the first assertion is trivial. For the second, let G be 2-cutwidth critical. If G is acyclic, then it is not a path and thus G is $K_{1,3}$; otherwise G has a cycle, thus G is K_3.

The main result of this paper is a characterization of the 3-cutwidth critical graphs. All of them are the five graphs illustrated in Fig. 1 (the numbers in each graph represent an optimal labeling). Note that H_1 is a star $K_{1,5}$; H_2 is a tree with diameter 4; H_3 is obtained from H_2 by replacing a claw $K_{1,3}$ by a triangle K_3; H_4 is a 'crown' made of a C_3 and three pendant edges; H_5 is a cycle C_4 with a chord.

From this, we obtain the forbidden subgraphs characterization of graphs with cutwidth two as follows: A graph G has cutwidth 2 if and only if it is not a path and it does not contain any subgraph homeomorphic to one of H_1, H_2, H_3, H_4 and H_5 in Fig. 1.

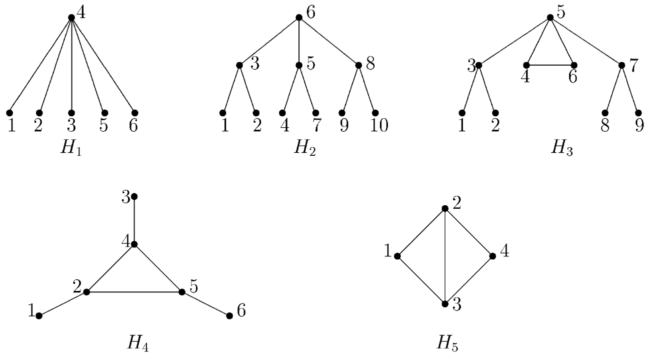

Fig. 1. The 3-cutwidth critical graphs.

A similar work has been done for the treewidth. A graph G is said to be k-treewidth critical if G has treewidth k and there is no proper minor G' of G having treewidth k. It is easy to see that for $1 \leqslant k \leqslant 3$, the unique k-treewidth critical graph is K_{k+1}. Further, [1,11] characterizes all 4-treewidth critical graphs (which are the complete graph K_5, the octahedron $K_{2,2,2}$, the Möbius ladder M_8 and the cyclic ladder $C_5 \times K_2$). The critical graphs for other parameters are worthy of further study.

The rest of this paper is organized as follows. In Section 2 we present some preliminary results. Section 3 is devoted to the proof of the main result. Section 4 gives a short summary.

2. Preliminaries

The following is an obvious lower bound.

Proposition 2.1. *Let $\Delta(G)$ denote the maximum degree of G. Then $c(G) \geqslant \lceil \Delta(G)/2 \rceil$.*

The bound is attainable by a caterpillar, a tree which yields a path (the spine) when all its pendant vertices are removed.

Proposition 2.2. *For any caterpillar T, $c(T) = \lceil \Delta(T)/2 \rceil$. In particular, $c(K_{1,n}) = \lceil n/2 \rceil$.*

In fact, it is easy to construct an embedding of T along the spine with cutwidth $\lceil \Delta(T)/2 \rceil$ (see [8] for details). The forbidden subgraph characterization of caterpillars is useful in the sequel. A well-known result is that a tree is a caterpillar if and only if it does not contain any "double claw" (the subdivision of $K_{1,3}$ by inserting a vertex in each edge). The following is a further result.

Proposition 2.3. *A tree T is homeomorphic to a caterpillar if and only if it does not contain any subgraph homeomorphic to H_2 (in Fig. 1).*

Proof. The *only if* part is due to the fact that H_2 is not homeomorphic to a caterpillar. We next show the *if* part. If T is not homeomorphic to a caterpillar, let T' be the minimal subtree of T containing all vertices of degree at least 3, then T' is not a path. Hence, T' has at least three pendant vertices x_1, x_2, x_3 and the minimal subtree T'' of T' containing x_1, x_2, x_3 must be homeomorphic to a star $K_{1,3}$ (with a center x_0). It is clear that the degree of x_i in T is at least three ($i = 1, 2, 3$). Therefore, the subtree T'' together with two neighbors of every x_i ($i = 1, 2, 3$) constitute a subgraph homeomorphic to H_2. This completes the proof. $\square$

For the sake of simplicity, when no confusion can arise in the context, two graphs will be regarded as the same if they are homeomorphic. For example, we may refer to the subdivision of a caterpillar as a caterpillar. Similarly, if G contains a subgraph homeomorphic to H, we may say that G contains H and write $H \subseteq G$. By this convention, Proposition 2.3 can be simplified as: A tree T is a caterpillar if and only if $H_2 \nsubseteq T$.

Recall that the only 2-cutwidth critical graphs are K_3 and $K_{1,3}$ (Proposition 1.2). Sometimes, a $K_{1,3}$-subgraph and a K_3-subgraph could play the same role in a cutwidth embedding scheme. This gives rise to the following equivalent transformations. Suppose that a $K_{1,3}$-subgraph S of G is comprised of a vertex x of degree three and its neighbors a, b, c at least one of which (say, c) is a pendant vertex. Then we can construct a new graph G' by replacing S to a K_3-subgraph T with vertices a, b, c such that the pendant vertices in G (namely, c and probably one more) correspond to the vertices of degree two in G' and the other(s) are keeping the same incident relation of G (see Fig. 2(a)). Since S contributes congestion 2 to G while T contributes congestion 2 to G', we can see that G and G' have the same cutwidth. Conversely, if a K_3-subgraph has at least one vertex of degree two, then it can be symmetrically replaced by a $K_{1,3}$-subgraph such that the vertices of degree two correspond to the pendant vertices. We will call these transformations the *triangle transformations*. A typical example is from H_2 to H_3 or conversely.

A graph W is called a *triangulated caterpillar* of $\Delta \leqslant 4$ if it is obtained from a caterpillar T of $\Delta(T) \leqslant 4$ by performing several triangle transformations, provided that the resultant graph W has $\Delta(W) \leqslant 4$. Some examples are illustrated in Figs. 2(b) and 3.

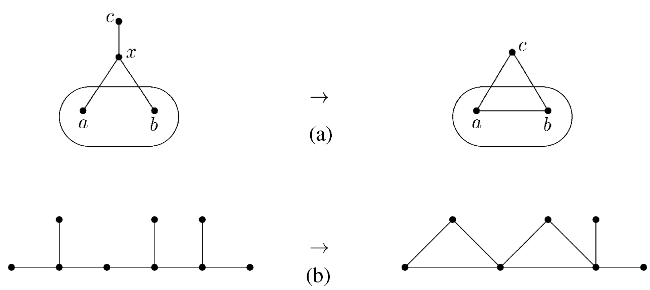

Fig. 2. Triangle transformation.

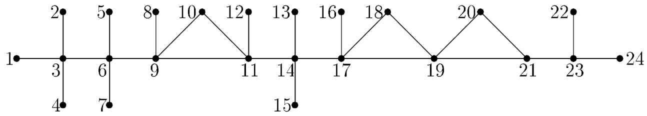

Fig. 3. A cutwidth 2 labeling.

Proposition 2.4. *Every triangulated caterpillar W of $\Delta \leqslant 4$ has cutwidth* 2.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices on the spine of W from left to right. For each u_i, let T_i be the star centered at u_i ($i = 1, 2, \ldots, n$). We may define a labeling f of W in the order of $T_1, T_2, \ldots, T_n$ such that the label of each center u_i is a median in star T_i (a median of k numbers means the one ranking $\lceil k/2 \rceil$ or $\lfloor k/2 \rfloor + 1$). An example is shown in Fig. 3. It is easy to show that the cutwidth with respect to f is $\lceil \Delta/2 \rceil = 2$. This completes the proof. □

3. 3-Cutwidth critical graphs

We consider the graphs H_1, H_2, H_3, H_4 and H_5 in turn.

Lemma 3.1. *A tree T is* 3*-cutwidth critical if and only if T is either H_1 or H_2.*

Proof. Note that H_1 is a star. By Proposition 2.2, we have $c(H_1) = 3$. Since H_2 is a tree with diameter 4, we can easily obtain that $c(H_2) = 3$ by the formula in [7]. In fact, the labeling in Fig. 1 implies that $c(H_2) \leqslant 3$ and the reverse inequality can be shown by the same method for H_3 in the next lemma. And any proper subgraph of H_1 or H_2 is homeomorphic to a caterpillar of $\Delta \leqslant 4$ whose cutwidth is at most 2. Also, they are homeomorphically minimal. Hence H_1 and H_2 are both 3-cutwidth critical.

Conversely, let T be a 3-cutwidth critical tree. If $\Delta(T) \geqslant 5$, then $H_1 \subseteq T$ and thus $T = H_1$ by the minimality of T. If $\Delta(T) \leqslant 4$, then T is not homeomorphic to a caterpillar (otherwise $c(T) \leqslant 2$ by Proposition 2.2); and thus $H_2 \subseteq T$ due to the characterization of caterpillars in Proposition 2.3. Again, by the minimality of T, we have $T = H_2$. This completes the proof. □

Lemma 3.2. *Graphs H_3 and H_4 are* 3*-cutwidth critical.*

Proof. The labeling of H_3 in Fig. 1 asserts that $c(H_3) \leqslant 3$. We next show that $c(H_3) \geqslant 3$. Denote the vertex of degree 4 in H_3 by x and denote its neighbors by a, b, y, z where xyz forms the triangle. In addition, let a_1 and a_2 be adjacent to a; and let b_1 and b_2 be adjacent to b (see Fig. 4).

For a given labeling f of H_3, if $f(x)$ is not the median of $\{f(x), f(a), f(y), f(z), f(b)\}$, then it is clear that $c(H_3, f) \geqslant 3$. So, there are essentially two cases to consider:

Case 1: $\max\{f(a), f(y)\} < f(x) < \min\{f(z), f(b)\}$. Let $i = f(x)$. Then $\{xz, xb, yz\} \subseteq \nabla(S_i)$, thus $c(H_3, f) \geqslant |\nabla(S_i)| \geqslant 3$.

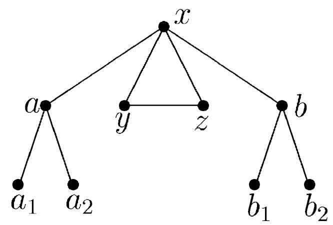

Fig. 4. Graph H_3.

Case 2: $f(a) < f(b) < f(x) < f(y) < f(z)$. Let $i = f(b)$. If $\max\{f(b_1), f(b_2)\} > i$, say $f(b_1) > i$, then $\{ax, bx, bb_1\} \subseteq \nabla(S_i)$, thus $c(H_3, f) \geqslant |\nabla(S_i)| \geqslant 3$; otherwise $f(b_1), f(b_2) < i$ and $\{ax, b_1b, b_2b\} \subseteq \nabla(S_{i-1})$, thus $c(H_3, f) \geqslant |\nabla(S_{i-1})| \geqslant 3$.

In this way, we show that $c(H_3) = 3$. On the other hand, any proper subgraph G' of H_3 is homeomorphic to either a caterpillar of $\Delta \leqslant 4$ or a triangulated caterpillar of $\Delta \leqslant 4$. It follows from Proposition 2.4 that $c(G') \leqslant 2$. Therefore H_3 is 3-cutwidth critical.

The labeling of H_4 in Fig. 1 also asserts that $c(H_4) \leqslant 3$. In order to show the lower bound $c(H_4) \geqslant 3$, we may examine the set $S_3 = f^{-1}(\{1,2,3\})$. It is easy to check that $|\nabla(S_3)| \geqslant 3$ for any labeling f in H_4. Hence $c(H_4, f) \geqslant |\nabla(S_3)| \geqslant 3$. Thus we have $c(H_4) = 3$. On the other hand, any proper subgraph G' of H_4 is either a caterpillar or a triangulated caterpillar of $\Delta \leqslant 3$, so $c(G') \leqslant 2$. Therefore H_4 is 3-cutwidth critical. The result follows. □

Lemma 3.3. *A unicyclic graph G is* 3*-cutwidth critical if and only if G is either H_3 or H_4.*

Proof. The '*if*' part has been shown in Lemma 3.2. We need only show the '*only if*' part. Let G be a 3-cutwidth critical unicyclic graph, in which the unique cycle is denoted by C. Since it is impossible that $H_1 \subseteq G$, we have $\Delta(G) \leqslant 4$. If C has three vertices of degree greater than 2, then G has a subgraph homeomorphic to H_4; and by the minimality of G, $G = H_4$. So, it is sufficient to consider the case that C has at most two vertices of degree 3 or 4. In this case, the cycle C can be homeomorphically contracted into a triangle. If $H_3 \subseteq G$, then $G = H_3$ by the minimality of G. Now, all we have left to show is the case that G satisfies the following conditions:

(a) $\Delta(G) \leqslant 4$;
(b) G contains a triangle C which has at least one vertex of degree two;
(c) $H_2 \nsubseteq G$ (due to the minimality of G);
(d) $H_3 \nsubseteq G$.

Let G' be obtained from G by replacing the triangle C by a star $K_{1,3}$ (performing a triangle transformation). Since G does not contain either H_2 or H_3, it follows that G' does not contain H_2. By Proposition 2.3, G' is homeomorphic to a caterpillar of $\Delta \leqslant 4$, thus G is in fact a triangulated caterpillar of $\Delta \leqslant 4$. Hence by Propositions 2.2

and 2.4, $c(G) = c(G') = 2$, which contradicts the assumption of G. Thus the desired result holds. □

By the cyclic rank (or cyclomatic number) of a graph G, we mean the number of independent cycles in G (i.e., $m - n + k$ for a graph with n vertices, m edges and k components).

Lemma 3.4. *A graph G with cyclic rank at least two is 3-cutwidth critical if and only if G is H_5.*

Proof. The labeling of H_5 in Fig. 1 shows that $c(H_5) \leqslant 3$. On the other hand, there are two vertices in H_5 which are connected by three internally-disjoint paths. This implies that $c(H_5) \geqslant 3$. So, we have $c(H_5)=3$. Moreover, it is clear that every proper subgraph of H_5 has cutwidth less than 3. Therefore, H_5 is 3-cutwidth critical.

Conversely, let G be a 3-cutwidth critical graph with cyclic rank at least two. That is to say, G has two or more independent cycles. If there are two cycles with common edges, then G has a subgraph homeomorphic to H_5, namely, $H_5 \subseteq G$. By the minimality of G, we see that $G=H_5$. Otherwise, all cycles in G are edge-disjoint. As argued before, G does not contain any subgraph homeomorphic to H_1, H_2, H_3 or H_4. This implies that G satisfies the following conditions:

(a) $\Delta(G) \leqslant 4$;
(b) for each cycle C in G, C has at most two vertices of degree greater than 2;
(c) in each cycle C in G, there is no vertex adjacent to two vertices outside C of degree greater than 2;
(d) in a path P connecting two cycles C_1 and C_2, there is no internal vertex adjacent to a vertex of degree greater than 2.

By (b), each cycle of G can be homeomorphically contracted into a triangle. By (c) and (d), all triangles (cycles) of G can be arranged in a sequence such that two consecutive triangles are connected by a caterpillar. Let G' be obtained from G by replacing each triangle by a star $K_{1,3}$. Then G' is a caterpillar of $\Delta \leqslant 4$, thus G is a triangulated caterpillar of $\Delta \leqslant 4$. Hence $c(G) = c(G') = 2$, a contradiction. Thus the lemma follows. □

To summarize Lemmas 3.1–3.4, we obtain the main result:

Theorem 3.5. *All 3-cutwidth critical graphs are H_1, H_2, H_3, H_4 and H_5.*

Corollary 3.6. *A graph G has cutwidth $\leqslant 2$ if and only if it does not contain any subgraph homeomorphic to H_1, H_2, H_3, H_4 or H_5.*

Corollary 3.7. *A graph G has cutwidth 2 if and only if it is homeomorphic to a caterpillar (other than a path) of $\Delta \leqslant 4$ or a triangulated caterpillar of $\Delta \leqslant 4$.*

4. Concluding remarks

In the foregoing discussion we characterize the set of 3-cutwidth critical graphs. As a result, it turns out that all graphs of cutwidth 2 can be generated from the caterpillars of $\Delta \leqslant 4$ via the triangle transformations. This suggests an algorithmic approach for determining the graphs of cutwidth 2. In fact, for a given graph G, we may first contract every vertex of degree two (unless it is in a triangle). Then, to decide if $c(G) \leqslant 2$, we need only check the following conditions one by one: (1) $\Delta(G) \leqslant 4$; (2) all vertices of degree greater than two induce a path P; (3) every pendant vertex is adjacent to a vertex in P; (4) every vertex of degree two is in a triangle with the other two vertices in P; (5) no two triangles have an edge in common. It is clear that each of these steps, and thus the algorithm, can be implemented in linear time.

A further task is to characterize the set of 4-cutwidth critical graphs. It is known that this set includes $K_4, K_{1,7}$, and some trees with diameter 4. More general properties of critical graphs are expected.

The counterpart for the bandwidth problem is worthwhile to study. However, the results are not as neat as those for the cutwidth.

Acknowledgements

The authors would like to thank the referees for their comments on improving the representation of the paper.

References

[1] S. Arnborg, A. Proskurowski, D.G. Corneil, Forbidden minors characterization of partial 3-trees, Discrete Math. 80 (1990) 1–19.
[2] F.R.K. Chung, Labeling of graphs, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory, Vol. 3, Academic Press, London, 1988, pp. 151–168.
[3] F.R.K. Chung, P.D. Seymour, Graphs with small bandwidth and cutwidth, Discrete Math. 75 (1985) 268–277.
[4] J. Diaz, J. Petit, M. Serna, A Survey of graph layout problems, ACM Comput. Surveys 34 (2002) 313–356.
[5] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, San Francisco, CA, 1979.
[6] E. Korach, N. Solel, Tree-width, path-width and cutwidth, Discrete Appl. Math. 43 (1993) 97–101.
[7] Y. Lin, The cutwidth of trees with diameter at most 4, Appl. Math. J. Chinese Univ. Ser. B 18 (2003), to appear.
[8] Y. Lin, X. Li, A. Yang, A degree sequence method for the cutwidth problem of graphs, Appl. Math. J. Chinese Univ. Ser. B 17 (2) (2002) 125–134.
[9] H. Liu, J. Yuan, Cutwidth problem on graphs, Appl. Math. J. Chinese Univ. Ser. A 10 (3) (1995) 339–348.
[10] J. Rolin, O. Sykora, I. Vrto, Optimal Cutwidth of Meshes, Lecture Notes in Computer Science, Vol. 1017, Springer, Berlin, 1995, pp. 252–264.
[11] A. Satyanarayana, L. Tung, A characterization of partial 3-trees, Networks 20 (1990) 299–322.
[12] M. Yannakakis, A polynomial algorithm for the min-cut arrangement of trees, J. ACM 32 (1985) 950–989.
[13] J. Yuan, The arrangement and fill-in of graphs, Ph.D. Thesis, Sichuan University, China, 1995.