Applied Mathematics Letters 22 (2009) 581–584

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Implicit Mann fixed point iterations for pseudo-contractive mappings

Ljubomir Ćirić [a,*], Arif Rafiq [b], Nenad Cakić [c], Jeong Sheok Ume [d]

[a] Faculty of Mechanical Engineering, Kraljice Marije 16, 11 000 Belgrade, Serbia

[b] Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

[c] Faculty of Electrical Engineering, Bul. Kralja Aleksandra 73, 11 000 Belgrade, Serbia

[d] Department of Applied Mathematics, Changwon National University, Changwon 641-773, Republic of Korea

ARTICLE INFO

Article history:
Received 25 February 2008
Received in revised form 21 April 2008
Accepted 3 June 2008

Keywords:
Banach space
Hilbert Space
Implicit-Mann-type iteration process
Pseudo-contractive mapping
Fixed point

ABSTRACT

Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ a continuous hemi-contractive map. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in [0, 1] such that $a_n + b_n + c_n = 1$, and $\{u_n\}$ and $\{v_n\}$ be sequences in K. In this paper we prove that, if $\{b_n\}$, $\{c_n\}$ and $\{v_n\}$ satisfy some appropriate conditions, then for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined iteratively by $x_n = a_n x_{n-1} + b_n T v_n + c_n u_n$; $n \geq 1$, converges strongly to a fixed point of T.

1. Introduction

Let E be a Banach space and K be a nonempty subset of E. A mapping $T : K \to E$ is said to be *pseudo-contractive* (see e.g., [1,2,11]) if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in K. \tag{1}$$

A mapping $T : K \to E$ is called *hemi-contractive* if $F(T) := \{x \in K : Tx = x\} \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2 \quad \text{for all } x^* \in F(T) \quad \text{and} \quad \text{for all } x \in K. \tag{2}$$

It is easy to see that the class of pseudo-contractions with fixed points is a subclass of the class of hemi-contractive maps. There are examples which show that a hemi-contraction is not necessarily a pseudo-contraction (see, for instance, [16,18]). For the importance of fixed points of pseudo-contractions the reader may consult [1].

The class of pseudo-contractive (and correspondingly accretive) operators has been studied extensively by various authors (cf. [3–9,13–20]).

Two effective methods for approximating a fixed point of a pseudo-contractive operator are the well known Mann [12] iterative and Ishikawa [10] iterative processes. These two iterative processes are equivalent in many aspects. In 1998 Xu introduced the following iteration process. For $T : K \to E$ and $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n' x_n + b_n' T x_n + c_n' v_n, \quad n \geq 0, \end{aligned} \tag{3}$$

* Corresponding author.
E-mail addresses: lciric@rcub.bg.ac.yu (L. Ćirić), arafiq@comsats.edu.pk (A. Rafiq), cakic@kondor.etf.bg.ac.yu (N. Cakić), jsume@changwon.ac.kr (J.S. Ume).

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.06.034

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ for all $n \geq 0$. If in (3), $b_n^{'} = 0 = c_n^{'}$, then we obtain the Mann iteration sequence in the sense of Xu. If in (3), $c_n = 0 = c_n^{'}$, then we obtain the usual Ishikawa iteration sequence.

In [15], the second author proved the following theorem.

Theorem 1. *Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ a hemi-contractive mapping. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2]$. For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined by $x_0 \in K$,*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n, \quad n \geq 1. \tag{4}$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

The purpose of this paper is to introduce and investigate the following modified Mann implicit iteration process. Let K be a closed convex subset of a real normed space H and $T : K \to K$ be a mapping. Define $\{x_n\}$ in K in the following way:

$$x_0 \in K,$$
$$x_n = a_n x_{n-1} + b_n T v_n + c_n u_n, \quad n \geq 1, \tag{5}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for each $n \in N$, and $\{u_n\}$ and $\{v_n\}$ are sequences in K.

We point out that the iterative processes, defined by (5), in which it is not necessary to compute a value of the given operator at x_n, but compute at an approximate point of x_n, are particularly useful in the numerical analysis.

In this paper we prove that, if K is a convex compact subset of a real Hilbert space H, $T : K \to K$ is a continuous hemi-contractive mapping and $\{b_n\}$, $\{c_n\}$ and $\{v_n\}$ satisfy some appropriate conditions, then the sequence $\{x_n\}$, defined by (5), converges strongly to some fixed point of T.

2. Mann-type iteration process for pseudo-contractive mappings in Hilbert spaces

We shall make use of the following results.

Lemma 2 ([19]). *Suppose that $\{\rho_n\}$, $\{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n \quad \forall n \geq N_0.$$

(a) *If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim \rho_n$ exists.*
(b) *If $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim \rho_n = 0$.*

Lemma 3 ([13]). *Let H be a Hilbert space, then for all $x, y, z \in H$*

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2,$$

where $a, b, c \in [0, 1]$ and $a + b + c = 1$.

Now we prove our main results.

Theorem 4. *Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ a continuous hemi-contractive map. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for each $n \in N$ and satisfying:*

(i) *$\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2}]$,*
(ii) *$\sum_{n=1}^{\infty} c_n < \infty$.*

For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ in K be iteratively defined by

$$x_n = a_n x_{n-1} + b_n T v_n + c_n u_n, \quad n \geq 1, \tag{6}$$

where $v_n \in K$ are chosen such that $\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty$ and $\{u_n\}_{n=1}^{\infty}$ is an arbitrary sequence in K. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of T.

Proof. Let $x^* \in K$ be a fixed point of T and $M = \text{diam}(K)$. Since T is hemi-contractive, then

$$\|Tv_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \|v_n - Tv_n\|^2 \tag{7}$$

for each $n \in N$. By virtue of (6), Lemma 3 and (7), we obtain the following estimates:

$$\begin{aligned}\|x_n - x^*\|^2 &= \|a_n x_{n-1} + b_n T v_n + c_n u_n - x^*\|^2 \\ &= \|a_n(x_{n-1} - x^*) + b_n(T v_n - x^*) + c_n(u_n - x^*)\|^2 \\ &= a_n \left\|x_{n-1} - x^*\right\|^2 + b_n \left\|T v_n - x^*\right\|^2 + c_n \left\|u_n - x^*\right\|^2 - a_n b_n \|x_{n-1} - T v_n\|^2 \\ &\quad - b_n c_n \|T v_n - u_n\|^2 - a_n c_n \|x_{n-1} - u_n\|^2 \\ &\le a_n \left\|x_{n-1} - x^*\right\|^2 + b_n \left\|T v_n - x^*\right\|^2 + c_n \left\|u_n - x^*\right\|^2 - a_n b_n \|x_{n-1} - T v_n\|^2 \\ &\le (1 - b_n) \left\|x_{n-1} - x^*\right\|^2 + b_n \left\|T v_n - x^*\right\|^2 + M^2 c_n - a_n b_n \|x_{n-1} - T v_n\|^2 \\ &\le (1 - b_n) \left\|x_{n-1} - x^*\right\|^2 + b_n \left(\|v_n - x^*\|^2 + \|v_n - T v_n\|^2\right) + M^2 c_n - a_n b_n \|x_{n-1} - T v_n\|^2 . \end{aligned} \tag{8}$$

We also have

$$\begin{aligned}\left\|v_n - x^*\right\|^2 &\le \|v_n - x_n\|^2 + \left\|x_n - x^*\right\|^2 + 2 \left\|x_n - x^*\right\| \|v_n - x_n\| \\ &\le \|v_n - x_n\|^2 + \left\|x_n - x^*\right\|^2 + 2M \|v_n - x_n\| ,\end{aligned} \tag{9}$$

$$\begin{aligned}\|v_n - T v_n\|^2 &\le \|v_n - x_n\|^2 + \|x_n - T v_n\|^2 + 2 \|x_n - T v_n\| \|v_n - x_n\| \\ &\le \|v_n - x_n\|^2 + \|x_n - T v_n\|^2 + 2M \|v_n - x_n\| ,\end{aligned} \tag{10}$$

and

$$\begin{aligned}\|x_n - T v_n\|^2 &= \|a_n x_{n-1} + b_n T v_n + c_n u_n - T v_n\|^2 \\ &= \|(1 - b_n)(x_{n-1} - T v_n) + c_n(u_n - x_{n-1})\|^2 \\ &\le [(1 - b_n) \|x_{n-1} - T v_n\| + c_n \|u_n - x_{n-1}\|]^2 \\ &\le [(1 - b_n) \|x_{n-1} - T v_n\| + M c_n]^2 \\ &\le (1 - b_n)^2 \|x_{n-1} - T v_n\|^2 + 3M^2 c_n .\end{aligned} \tag{11}$$

Substituting (11) in (10), and then (10) and (9) in (8), we get

$$\begin{aligned}\|x_n - x^*\|^2 &\le (1 - b_n) \left\|x_{n-1} - x^*\right\|^2 + b_n \left\|x_n - x^*\right\|^2 + 2b_n \|v_n - x_n\|^2 + 4M b_n \|v_n - x_n\| \\ &\quad + 4M^2 c_n - b_n \left[a_n - (1 - b_n)^2\right] \|x_{n-1} - T v_n\|^2 .\end{aligned} \tag{12}$$

From (i) we get

$$\begin{aligned}a_n - (1 - b_n)^2 &= 1 - b_n - c_n - (1 - b_n)^2 \\ &= b_n(1 - b_n) - c_n \\ &\ge \delta^2 - c_n .\end{aligned} \tag{13}$$

From (ii) it follows that there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we have $c_n \le \delta^3$, that is, $\delta^2 - c_n \ge \delta^2 (1 - \delta)$. Thus from (12) and (13), for all $n \ge n_0$ we have:

$$(1 - b_n)\|x_n - x^*\|^2 \le (1 - b_n) \left\|x_{n-1} - x^*\right\|^2 + 2 \|v_n - x_n\|^2 + 4M \|v_n - x_n\| + 4M^2 c_n - \delta^3 (1 - \delta) \|x_{n-1} - T v_n\|^2 .$$

Hence

$$\begin{aligned}\|x_n - x^*\|^2 &\le \left\|x_{n-1} - x^*\right\|^2 + \frac{2}{1 - b_n} \|v_n - x_n\|^2 + 4M \frac{1}{1 - b_n} \|v_n - x_n\| \\ &\quad + 4M^2 \frac{1}{1 - b_n} c_n - \frac{\delta^3 (1 - \delta)}{1 - b_n} \|x_{n-1} - T v_n\|^2 .\end{aligned}$$

Since $1/(1 - b_n) \le 1/\delta$ and $-1/(1 - b_n) \le -1/(1 - \delta)$, we have

$$\|x_n - x^*\|^2 \le \left\|x_{n-1} - x^*\right\|^2 - \delta^3 \|x_{n-1} - T v_n\|^2 + \sigma_n \tag{14}$$

for all $n \ge n_0$, where $\sigma_n = (1/\delta)[2 \|v_n - x_n\|^2 + 4M \|v_n - x_n\| + 4M^2 c_n]$. Under the hypotheses of Theorem 4, one obtains:

$$\sum_{j=n_0}^{\infty} \sigma_n < +\infty. \tag{15}$$

From (14) we get

$$\delta^3 \|x_{n-1} - T v_n\|^2 \le \left\|x_{n-1} - x^*\right\|^2 - \|x_n - x^*\|^2 + \sigma_n$$

and hence

$$\delta^3 \sum_{j=n_0}^{\infty} \left\|x_{j-1} - Tv_j\right\|^2 \leq \sum_{j=n_0}^{\infty} \sigma_j + \|x_{n_0-1} - x^*\|^2.$$

Hence by (15) we get $\sum_{j=1}^{\infty} \left\|x_{j-1} - Tv_j\right\|^2 < +\infty$. This implies that $\lim_{n\to\infty} \|x_{n-1} - Tv_n\| = 0$. From (11) and condition (ii) it further implies that $\lim_{n\to\infty} \|x_n - Tv_n\| = 0$. Also the condition $\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty$ implies $\lim_{n\to\infty} \|v_n - x_n\| = 0$. Thus, from (10),

$$\lim_{n\to\infty} \|v_n - Tv_n\| = 0. \tag{16}$$

By compactness of K there is a convergent subsequence $\{v_{n_j}\}$ of $\{v_n\}$, such that it converges to some point $z \in K$. By continuity of T, $\{Tv_{n_j}\}$ converges to Tz. Therefore, from (16) we conclude that $Tz = z$. Further, $\lim_{n\to\infty} \|v_n - x_n\| = 0$ implies

$$\lim_{j\to\infty} \left\|x_{n_j} - z\right\| = 0. \tag{17}$$

Since (14) holds for any fixed points of T, we have

$$\|x_n - z\|^2 \leq \|x_{n-1} - z\|^2 - \delta^3 \|x_{n-1} - Tv_n\|^2 + \sigma_n,$$

and in view of (15), (17) and Lemma 2 we conclude that $\|x_n - z\| \to 0$ as $n \to \infty$, i.e., $x_n \to z$ as $n \to \infty$. Thus we proved that $\{x_n\}$ converges strongly to some fixed point of T. ■

Corollary 5. *Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ a Lipschitz pseudo-contractive map. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{u_n\}$, $\{v_n\}$ and the sequence $\{x_n\}$ be as in Theorem 4. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From the Schauder fixed point theorem [17], T has a fixed point. Since any pseudo-contractive map with fixed points is hemi-contractive, we can apply Theorem 4. ■

Acknowledgements

The third author thanks the Ministry of Science and Environmental Protection of Serbia for their support.

This research is financially supported by Changwon National University in 2008.

References

[1] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, in: Proc. Symposia Pure Math., vol. XVIII, part 2, 1976.
[2] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197–228.
[3] C.E. Chidume, Chika Moore, Fixed point iteration for pseudocontractive maps, Proc. Amer. Math. Soc. 127 (4) (1999) 1163–1170.
[4] C.E. Chidume, S.A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc. 129 (8) (2001) 2359–2363.
[5] L.B. Ciric, S.N. Jesic, M.M. Milovanovic, J.S. Ume, On the steepest descent approximation method for the zeros of generalized accretive operators, Nonlinear Anal. 69 (2008) 763–769.
[6] L.B. Ćirić, J.S. Ume, Ishikawa process with errors for nonlinear equations of generalized monotone type in Banach spaces, Math. Nachr. 278 (10) (2005) 1137–1146.
[7] L. Deng, Iteration process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces, J. Math. Anal. Appl. 188 (1994) 128–140.
[8] T.L. Hicks, J.R. Kubicek, On the Mann iteration process in Hilbert space, J. Math. Anal. Appl. 59 (1977) 498–504.
[9] Z.Y. Huang, Weak stability of Mann and Ishikawa iterations with errors for phi-hemicontractive operators, Appl. Math. Lett. 20 (4) (2007) 470–475.
[10] S. Ishikawa, Fixed point by a new iteration method, Proc. Amer. Math. Soc. 4 (1) (1974) 147–150.
[11] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan 19 (1967) 508–520.
[12] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–610.
[13] M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comput. Math. Appl. 40 (2000) 559–567.
[14] J.W. Peng, Set-valued variational inclusions with T-accretive operators in Banach spaces, App. Math. Lett. 19 (3) (2006) 273–282.
[15] A. Rafiq, On Mann iteration in Hilbert spaces, Nonlinear Anal. 66 (10) (2007) 2230–2236.
[16] B.E. Rhoades, Comments on two fixed point iteration procedures, J. Math. Anal. Appl. 56 (1976) 741–750.
[17] J. Schauder, Der Fixpunktsatz in Funktionalräumen, Studia Math. 2 (1930) 171–180.
[18] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991) 407–413.
[19] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.
[20] Y. Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl. 224 (1998) 91–101.