# A bijection between ordered trees and 2-Motzkin paths and its many consequences 

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#### Abstract

A new bijection between ordered trees and 2-Motzkin paths is presented, together with its numerous consequences regarding ordered trees as well as other combinatorial structures such as Dyck paths, bushes, $\{0,1,2\}$-trees, Schröder paths, RNA secondary structures, noncrossing partitions, Fine paths, and Davenport-Schinzel sequences.


## Résumé

Une nouvelle bijection entre arbres ordonnés et chemins de Motzkin bicolorés est présentée, avec ses nombreuses conséquences en ce qui concerne les arbres ordonnés ainsi que d'autres structures combinatoires telles que chemins de Dyck, buissons, arbres de type $\{0,1,2\}$, chemins de Schröder, structures secondaires de type RNA, partitions non croisées, chemins de Fine, et enfin suites de Davenport-Schinzel. (c) 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

An ordered tree is an unlabeled rooted tree where the order of the subtrees of a vertex is significant. It is well known that the number of all ordered trees with $n$ edges is the Catalan number $C_{n}=[(1 / n+1)]\binom{2 n}{n}$. The first 10 terms are $1,1,2,5,14,42$, 132, 429, 1430, 4862; it is sequence M1459 in [26]].
By a 2-Motzkin path we mean paths starting and ending on the horizontal axis but never going below it, with possible steps $(1,1),(1,0)$, and $(1,-1)$, where the level

[^0]steps $(1,0)$ can be either of two kinds: straight and wavy, say. The length of the path is defined to be the number of its steps.
Ordinary Motzkin paths arise when the level steps are only of one kind. They are counted by the so-called Motzkin numbers $M_{n}=\sum_{k \geqslant 0}\binom{n}{2 k} C_{k}$ [10]. The first 10 terms are $1,1,2,4,9,21,51,127,323,835$; it is sequence M1184 in [26].
A Motzkin path with no level steps is called a Dyck path. Obviously, the length of a Dyck path is an even number. It can be shown in several ways that the number of Dyck paths of length $2 n$ is the Catalan number $C_{n}$.

We return to 2 -Motzkin paths. It is easy to show that the number of 2 -Motzkin paths with $n$ steps is given by the Catalan number $C_{n+1}$. A standard bijection between 2-Motzkin paths with $n$ steps and Dyck paths of length $2 n+2$ is as follows, where we denote an up (down) step of a Dyck path or a 2-Motzkin path by $U(D)$. For a given 2-Motzkin path $\mu$ we replace $U$ by $U U, D$ by $D D$, a straight level step by $U D$, and a wavy level step by $D U$. Note that the obtained path is a Dyck path except that it can go down to level -1 . We place a $U$ at the front and a $D$ at the end in order to obtain a valid Dyck path. The Dyck path obtained this way is the image of the 2-Motzkin path $\mu$ (see, for example [2], p. 179).

In this paper, we present a new bijection between ordered trees and 2-Motzkin paths. It has numerous consequences regarding ordered trees as well as other combinatorial structures such as Dyck paths, bushes, $\{0,1,2\}$-trees, Schröder paths, secondary structures, noncrossing partitions, and Fine paths. In the last section we present a new simple bijection between bushes and certain Davenport-Schinzel sequences and then, taking its composition with our main bijection, we obtain a bijection between Motzkin paths and Davenport-Schinzel sequences.

Remark. There is a basic bijection, $B$, between full binary trees and Dyck paths. Briefly, let $T$ be a full binary tree with $T_{1}$ and $T_{2}$ the left and right subtrees at the root. The bijection $B$ is defined recursively by $B(T)=U B\left(T_{1}\right) D B\left(T_{2}\right)$. A nonrecursive form of this bijection will follow as a special case of our bijection (see Section 5.1). At this point the reader could skip ahead to Section to Section 5.1 and develop results there by using the bijection $B$.

Consequently, as pointed out by one of the referees, one way to view the results in this paper is: start with the basic bijection $B$ between full binary trees and Dyck paths, add some new edges to the binary trees (to be called lonely and redundant), add some new kind of steps to the Dyck paths (namely level steps of two kinds), and extend $B$ to these enriched combinatorial structures, i.e. ordered trees and 2-Motzkin paths.

## 2. Tree terminology

As mentioned above, an ordered tree is an unlabeled rooted tree where the order of the subtrees of a vertex is significant. The subtrees of the root (having as roots the children of the root) are called the principal subtrees of the tree. The outdegree
of a vertex will be called its degree. A vertex of degree zero is called a leaf. By a planted tree we mean a tree having degree of the root equal to 1 . In this case, the edge emanating from the root is sometimes referred to as the planting stalk. Contrary to common usage, by a node we mean a vertex that is neither the root nor a leaf. Thus the vertices of a tree are partitioned into three sets: the root, the nodes, and the leaves. The nodes of a tree are of two kind: (i) branch node if its degree is at least 2 and (ii) lonely node if its degree is equal to 1 . By a branch of a tree we mean a path connecting either the root and a nearest branch node, or two nearest branch nodes, or a leaf and the nearest branch node. The branches form a partition of the set of edges. As far as edges are concerned, we introduce the following definitions:

- By a lonely edge we mean an edge emanating from a lonely node (edges $1,3,10$, 15, 18, 19, 21, 24, 26 in Fig. 1).
- By a redundant edge we mean either an edge that is emanating from the root, with the exception of the leftmost edge, or an edge emanating from a branch node, with the exception of the leftmost and rightmost edges (edges 7, 9, 12, 16, 17, 22, 25 in Fig. 1).
- By a left edge we mean the leftmost edge emanating from a branch node (edges 2, 4, 5, 13, 20 in Fig. 1).
- By a right edge we mean the rightmost edge emanating from a branch node (edges 6, 8, 11, 14, 23 in Fig. 1).
- By a right lonely edge we mean a lonely edge that is on the rightmost path of some principal subtree of the tree (edges 1, 15, 18, 19, 24, 26 in Fig. 1).

We have the following relation involving some of the newly defined terms:

$$
\begin{equation*}
\# \text { leaves }=1+\# \text { redundant edges }+\# \text { branch nodes. } \tag{1}
\end{equation*}
$$

To see this, we use the obvious identity

$$
\# \text { leaves }+\# \text { nodes }=\text { root degree }+\sum \text { node degree }
$$

from which we obtain

$$
\begin{aligned}
\# \text { leaves } & =\text { root degree }+\sum(\text { node degree }-1) \\
& =\text { root degree }+\sum(\text { branch node degree }-1) \\
& =\text { root degree }+\sum(\text { branch node degree }-2)+\# \text { branch nodes } \\
& =1+\# \text { redundant edges }+\# \text { branch nodes. }
\end{aligned}
$$

## 3. The bijection

We describe our bijection between ordered trees and 2-Motzkin paths. Let $\tau$ be any nonempty ordered tree. Traverse $\tau$ in preorder and for each edge encountered for the


Fig. 1. The bijection $\Phi$.
first time
(i) do nothing for the first edge;
(ii) draw an up step for a left edge;
(iii) draw a down step for a right edge;
(iv) draw a straight level step for a lonely edge;
(v) draw a wavy level step for a redundant edge.

It is easy to see that we have obtained a 2 -Motzkin path. More precisely, to an ordered tree with $n$ edges there corresponds a 2 -Motzkin path of length $n-1$. The described mapping will be denoted by $\Phi$.

Now we define the inverse mapping. Given a 2 -Motzkin path, we draw a tree in preorder by the following rule: we start with an edge and then, traversing the 2-Motzkin path from left to right, for each up step we draw a left edge, for each straight level step we draw a lonely edge, for each wavy level step we draw a redundant edge emanating from the appropriate vertex, and for each down step we draw a right edge emanating from the appropriate node. For an example see Fig. 1.
Bijective correspondences. The following correspondences under the bijection $\Phi$ follow more or less immediately from the definition of the bijection, excepting the last which


Fig. 2.
follows from equality (1):
ordered tree
left edge
right edge
lonely edge
right lonely edge redundant edge
redundant edge emanating from the root
redundant edge emanating from a node node of degree 2 whose left child is a leaf left edge ending in a branch node
right edge ending in a branch node \# leaves

2 - Motzkin path
up step
down step
straight level step
straight level step at level 0
wavy level step wavy level step at level 0 wavy level step at level $>0$ peak
doublerise, i.e. 2 consecutive up steps valley
$1+$ \# wavy level steps + \# up
steps

## 4. An involution on ordered trees

There is a trivial involution $\pi \rightarrow \pi^{\prime}$ on the 2-Motzkin paths: interchange the straight and wavy level steps. This involution and the bijection $\Phi$ from ordered trees onto 2-Motzkin paths induce in a straightforward manner an involution $\tau \rightarrow \tau^{\prime}$ on ordered trees. An example is given in Fig. 2.

This involution on ordered trees can be defined directly, without going through the 2-Motzkin paths. Roughly speaking, traverse the tree in preorder and replace lonely (redundant) edges by redundant (lonely) edges. For trees with 1, 2, and 3 edges the involution is shown in Fig. 3.


Fig. 3.
If $\tau$ is a tree and $\tau^{\prime}$ is its image under the tree involution, then

$$
\begin{equation*}
\# \text { leaves }(\tau)+\# \operatorname{leaves}\left(\tau^{\prime}\right)=n+1, \tag{2}
\end{equation*}
$$

where $n$ is the number of edges.
To prove this, let $\pi$ be the 2 -Motzkin path corresponding to $\tau$ under the bijection $\Phi$ and let $\pi^{\prime}$ be the 2 -Motzkin path obtained from $\pi$ by means of the trivial involution. Then

$$
\begin{aligned}
& \# \text { leaves }(\tau)=1+\# \text { wavy level } \operatorname{steps}(\pi)+\# \text { up } \operatorname{steps}(\pi) \\
& \# \text { leaves }\left(\tau^{\prime}\right)=1+\# \operatorname{straight} \text { level } \operatorname{steps}(\pi)+\# \text { down } \operatorname{steps}(\pi)
\end{aligned}
$$

Adding the last two equalities, we obtain (2). From (2) we have a bijective proof that the number of trees with $n$ edges and $k$ leaves is equal to the number of trees with $n$ edges and $n+1-k$ leaves (see [3]).

An immediate consequence of (2) is

$$
\begin{equation*}
\text { \# leaves }(\tau)=\# \text { internal } \operatorname{nodes}\left(\tau^{\prime}\right) \tag{3}
\end{equation*}
$$

where, the term "internal node" has the usual meaning (root or node). Thus, we have a bijective proof, of the known fact, that the statistics "number of leaves" and "number of internal nodes" are equidistributed, implying, in particular, that the total number of leaves in all ordered trees with $n$ edges is equal to the total number of internal nodes in all ordered trees with $n$ edges. This gives a new solution to a problem proposed by one of the authors [25].

From trivially equidistributed statistics on 2-Motzkin paths (obtained by interchanging the straight and wavy level steps) we obtain the following nontrivial equidistributions on ordered trees (see the list of correspondences in Section 3):
(i) the statistics "number of lonely edges" and "number of redundant edges" are equidistributed;
(ii) the statistics "number of right lonely edges" and "number of redundant edges at root level" are equidistributed.
Moreover, the number of ordered trees with $n$ edges and $k$ lonely edges (or $k$ redundant edges) is $\binom{n-1}{k} M_{n-1-k}$, where $M_{i}$ is the Motzkin number, defined in the
introduction. Indeed, if an ordered tree with $n$ edges has $k$ redundant edges, then the corresponding 2 -Motzkin path has length $n-1$ and $k$ wavy level steps. However, it is easy to see that the number of these is $\binom{n-1}{k} M_{n-1-k}$ (choose the $k$ positions of the wavy level steps and fill out the remaining positions by the steps of a possibly "interrupted" Motzkin path of length $n-1-k$ ).

Remark. Making use of various known bijections, we can obtain combinatorial proofs that $\binom{n-1}{k} M_{n-1-k}$ also counts
(i) the number of Dyck paths of length $2 n$ with $k$ peaks at even level;
(ii) the number of Dyck paths of length $2 n$ with $k$ DUD's (i.e. noninitial ascents of length 1);
(iii) the number of noncrossing partitions of $\{1,2,3, \ldots, n\}$ with $k$ singleton blocks other than $\{1\}$; we recall that a partition of $\{1,2,3, \ldots, n\}$ is said to be a noncrossing partition if for every four elements $1 \leqslant a<b<c<d \leqslant n$, the following condition is satisfied: if $a$ and $c$ lie in the same block, and $b$ and $d$ lie in the same block, then all four elements lie in the same block (see [27, p. 226]).
(iv) the number of noncrossing partitions of $\{1,2,3, \ldots, n\}$ with $k$ adjacent point pairs in the same block.

As far as the statistics "number of right lonely edges" and "number of redundant edges at root level" are concerned, clearly, they have the same distribution as the statistic "degree of root -1 ".

## 5. Special cases

### 5.1. Planted full binary trees

A full binary tree is a tree in which the root and the nodes have degree equal to 0 or 2. A planted full binary tree is obtained from a full binary tree by adding a planting stalk (the reader will note that a planted full binary tree is not a full binary tree). Thus, a planted full binary tree has neither lonely nor redundant edges. Consequently, their images under the bijection $\Phi$ are Dyck paths. The following correspondences between full binary trees and Dyck paths are immediate:

| full binary tree | Dyck path <br> left node <br> dight node <br> doublerise |
| ---: | :--- |
| valley |  |

From the obvious equidistribution of left and right statistics on full binary trees we obtain at once the nonobvious but well-known (see for example [5-7] also [1,11,17-19],
$1 \quad$.
$\wedge \quad-$
























Fig. 4. Bijection bushes $\leftrightarrow$ Motzkin paths.
[29-31]) equidistributions of the following pairs of statistics on Dyck paths:
(i) "number of doublerises" and "number of valleys";
(ii) "number of peaks" and " $1+$ number of doublerises";
(iii) "height of first peak" and "number of returns".

### 5.2. Bushes

A bush is an ordered tree whose nodes have degree at least two (see [9,10]). In other words, a bush is a tree with no lonely edges. It follows from here that the restriction of the bijection $\Phi$ to bushes yields a bijection between bushes and Motzkin paths. Namely, traverse the bush in preorder, do nothing for the first edge, draw an up step for each left edge, draw a level step for each redundant edge, and draw a down step for each right edge. The fact that the number of bushes with a given number of edges is a Motzkin number is well known (see [9,10]); this particular bijection may be new (see Fig. 4).

If, before we apply the bijection $\Phi$, we add a planting stalk to the bushes, then we obtain the following new manifestation of the Motzkin numbers: 2-Motzkin paths with no straight level steps, except possibly from $(0,0)$ to $(1,0)$, and no wavy level steps at level 0 . The $M_{4}=9$ such paths with five steps are given in Fig. 5.

## 5.3. $\{0,1,2\}$-Trees

A $\{0,1,2\}$-tree is an ordered tree all of whose vertices have degree not exceeding two (see $[9,10,14,24]$ ). In other words, a $\{0,1,2\}$-tree is a tree which, after a planting stalk is added, has no redundant edges. It follows from here that the restriction of



Fig. 5. 2-Motzkin paths with no straight level steps, except possibly from $(0,0)$ to $(1,0)$, and no wavy level steps at level 0 .


Fig. 6. Bijection $\{0,1,2\}$-trees $\leftrightarrow$ Motzkin paths (The planting stalk, added to the trees before the bijection, is not shown; nothing corresponds to it under the bijection).


Fig. 7. At most one wavy level step which, moreover, is at level zero.
the bijection $\Phi$ to $\{0,1,2\}$-trees to which a planting stalk has been added, yields a bijection between $\{0,1,2\}$-trees and Motzkin paths. Namely, traverse the $\{0,1,2\}$-tree in preorder, draw an up step for each left edge, draw a level step for each lonely edge, and draw a down step for each right edge. The fact that the number of $\{0,1,2\}$-trees with a given number of edges is a Motzkin number is well known (see $[9,10]$ ); this particular bijection may be new (see Fig. 6).

If we apply the bijection $\Phi$ directly to the $\{0,1,2\}$-trees, without adding first planting stalks, then we obtain the following new manifestation of the Motzkin numbers: 2Motzkin paths with at most one wavy level step which, moreover, is at level zero. The $M_{4}=9$ such paths with three steps are given in Fig. 7.

Remark. The tree involution from Section 4, restricted to bushes and followed by the removal of the edge emanating from the root, yields a bijection between bushes and


Fig. 8. Bijection bushes $\leftrightarrow\{0,1,2\}$-trees.
$\{0,1,2\}$-trees. Namely, given a bush, traverse it in preorder, replace each redundant edge by a lonely edge, and remove the edge emanating from the root (see Fig. 8).

### 5.4. Schröder paths

A Schröder path of length $2 n$ is a lattice path in the plane from $(0,0)$ to $(2 n, 0)$ with steps $(1,1),(2,0)$, and $(1,-1)$, that never go below the horizontal axis. We assume that the level steps of the Schröder paths, viewed as 2-Motzkin paths, are straight and we apply to them the bijection $\Phi^{-1}$. We obtain planted trees with nodes of degree at most two and having all branches of odd length. We will call these Schröder trees. The Schröder trees with 1, 3, and 5 edges are shown in Fig. 9. Now, after this new manifestation of the large Schröder numbers has been discovered, we sketch two proofs via generating functions. Let $G(z)$ be the generating function of the Schröder trees according to number of edges. Each Schröder tree is either a path consisting of an odd number of edges or it is such a path with two Schröder trees hanging at its end. Consequently, $G=P+P G^{2}$, where $P=z+z^{3}+z^{5}+\cdots=z /\left(1-z^{2}\right)$. This equation leads to the known generating function of the Schröder numbers. Alternatively, the Schröder trees can be obtained from the planted full binary trees by replacing edges by paths of odd length. Since the generating function of planted full binary trees is $z C\left(z^{2}\right)$, where $C(z)=(1-\sqrt{1-4 z}) / 2 z$, it follows that the generating function of the Schröder trees is $P C\left(P^{2}\right)$, leading again to the desired result. (For a systematic treatment of the derivation of generating functions from set-theoretic operations, see [12,24].)


Fig. 9. Schröder trees (planted, all nodes have degree $\leqslant 2$, and branches have odd length).


Fig. 10.

### 5.5. Secondary structures

A secondary ( $R N A$ ) structure is a graph (without loops and multiple edges) on the vertex set $[n]$ such that (i) $\{i, i+1\}$ is an edge for all $1 \leqslant i \leqslant n-1$; (ii) for all $i$, there is at most one $j$ such that $\{i, j\}$ is an edge and $|j-i| \neq 1$, and (iii) if $\{i, j\}$ and $\{k, l\}$ are edges with $i<k<j$, then $i<l<j$ [27] (see also [23,28]). For the sake of simplicity, in the graphical representation of a secondary structure we shall delete all the edges required by condition (i). The obtained graph is a noncrossing partition satisfying requirement (ii) from the definition of a secondary structure.

Secondary structures are in a simple bijection with Motzkin paths without peaks [21]. Indeed, given a secondary structure, we traverse it from left to right and for each isolated vertex we draw a level step, for each vertex where an edge starts we draw an up step, and for each vertex where an edge ends we draw a down step. For an example, see Fig. 10.

Now we apply the bijection $\Phi^{-1}$ to these Motzkin paths. We can consider two cases.
(i) The level steps of the Motzkin paths, viewed as 2-Motzkin paths, are considered to be straight. In this case we obtain planted trees with nodes of degree at most two and such that the left child of a branch node is not a leaf. Alternatively, removing the planting stalk, we have trees with vertices of degree at most two and such that the left child of a vertex of degree two is not a leaf. The eight such trees with five edges are given in Fig. 11. We remark that this result can be obtained also via a bijection due to Dershowitz and Zaks [4].
(ii) The level steps of the Motzkin paths, viewed as 2-Motzkin paths, are considered to be wavy. In this case we obtain bushes such that the left child of a node of degree 2 is not a leaf. The eight such trees with six edges are given in Fig. 12.

If we apply the Dershowitz-Zaks bijection [4] to these bushes, then we obtain noncrossing partitions satisfying the following two conditions: (a) there are no singletons,


Fig. 11. $\{0,1,2\}$-trees in which the left child of a vertex of degree two is not a leaf.


Fig. 12. Bushes in which the left child of a node of degree two is not a leaf.


- ゐ


Fig. 13. No singletons, except possibly $\{1\}$; no $\{i, i+1\}$, except possibly $\{1,2\}$.
except possibly the block $\{1\}$ and (b) there are no blocks of two consecutive integers, except possibly $\{1,2\}$. The eight such noncrossing partitions on six points are given in Fig. 13.

### 5.6. Fine paths

By a Fine path we mean a Dyck path without peaks of height 1. They are counted by the Fine numbers, having generating function $F(z)=(1-\sqrt{1-4 z}) / z(3-\sqrt{1-4 z})$ $\left(F(z)=1+z^{2}+2 z^{3}+6 z^{4}+18 z^{5}+57 z^{6}+186 z^{7}+\cdots\right)$. A survey of the Fine numbers can be found in [8]. Applying the bijection $\Phi^{-1}$ to the Fine paths, we obtain planted full binary trees. Removing the planting stalk, we obtain full binary trees. However, the absence of peaks of height one in the Fine paths implies that the full binary tree has no leaf as the left child of a node on the rightmost path. The six such trees with eight edges are given in Fig. 14.

Making use of a well-known bijection between Dyck paths and ordered trees, it follows at once that the Fine numbers count also the ordered trees with no leaves at


Fig. 14. No leaf is the left child of a vertex on the rightmost path.


Fig. 15. 2-Motzkin paths do not start or end with a wavy level step and do not have two consecutive wavy level steps at level zero.


Fig. 16. 2-Motzkin paths with an odd number of wavy level steps at level zero.
level 1. In order to see which 2-Motzkin paths correspond to these trees under the bijection $\Phi$, we prefer to look at the 2 -Motzkin paths obtained when we do have a leaf at level 1. If a leaf at level 1 is the endpoint of the leftmost edge emanating from the root, then the corresponding 2-Motzkin path is either empty or it starts with a wavy level step. If a leaf at level 1 is the endpoint of the rightmost edge emanating from the root, then the corresponding 2 -Motzkin path ends with a wavy level step. Finally, if a leaf at level one is the endpoint of an edge strictly between the leftmost edge and the rightmost edge emanating from the root, then the corresponding 2 -Motzkin path contains two consecutive wavy level steps at level zero. Consequently, 2-Motzkin paths that do not start or end with a wavy level step and do not have two consecutive wavy level steps at level zero are counted by the Fine numbers. The six such 2-Motzkin paths with three edges are given in Fig. 15.

It is known [8] that ordered trees having root of even degree are counted by the Fine numbers. Applying to these the bijection $\Phi$, we obtain immediately that 2-Motzkin paths with an odd number of wavy level steps at level zero are counted by the Fine numbers. The six such 2-Motzkin paths with three edges are given in Fig. 16.

It is also known [8] that 2-Motzkin paths with no level steps at level zero are counted by the Fine numbers. Let us apply the bijection $\Phi^{-1}$ to these paths. The absence of straight level steps at level zero implies that the corresponding tree has no right lonely edge, while the absence of wavy steps at level zero implies that the corresponding tree is planted. After removing the planting stalks, the obtained trees are characterized by the absence of lonely edges on the rightmost path. Consequently, trees with no vertices (root or node) of degree one on the rightmost path are counted by the Fine numbers. The six such trees with four edges are given in Fig. 17.


Fig. 17. Trees with no vertices of degree one on the rightmost path.

## 6. A bijection between Davenport-Schinzel sequences and bushes

In this paper by a Davenport-Schinzel sequence (more briefly, a DS sequence) of rank $n$ we shall mean a finite sequence selected from the set $[n] \stackrel{\text { def }}{=}\{1,2, \ldots, n\}$ and satisfying the following conditions: (a) each integer $i \in[n]$ occurs in the sequence; (b) for each pair $i, j \in[n], i<j$, the first appearance of $i$ in the sequence precedes that of $j$; (c) no two adjacent symbols in the sequence are identical; (d) for each pair $i, j \in[n]$, the sequence contains no subsequence of the form ijij. The number of symbols in a DS sequence is called the length of the sequence.

Given a bush, using the sequence of positive integers, we label the nodes and the leaves in preorder, except that each node and its youngest child (i.e. the rightmost child) have the same label. It is immediate that the sequence of labels is a DS sequence. Indeed, (i) there are no immediate repetitions in the sequence since a bush has no nodes of outdegree 1 and (ii) the preorder rule precludes subsequences of the form $i j i j$. The inverse mapping can be easily defined. An example illustrating the bijection is given in Fig. 18.

Obviously, the length of the DS sequence is equal to the number of edges. The rank of the DS sequence is equal to the number of leaves since each label occurs at exactly one leaf. From here it follows that DS sequences, grouped by length, are counted by the Motzkin numbers and, if grouped by rank, then they are counted by the Schröder numbers $[13,15,16,20,22]$. From relation (1) we can easily find that in a bush

$$
\# \text { edges }=2 \times \# \text { leaves }-\# \text { redundant edges }-1 .
$$

From here it follows that if the number of leaves is prescribed, then the number of edges is maximum if and only if the number of redundant edges is equal to zero, i.e. if and only if the bush is a planted full binary tree. We obtain, applying the above bijection, that the number of DS sequences with a prescribed rank and having maximal length is given by a Catalan number [20].

Now, if we take the composition of this bijection with the bijection of Section 3 or, more precisely, with its restriction to bushes (see Section 5.2), then we obtain a bijection between Motzkin paths and DS sequences. This can be described directly: given a Motzkin path, using the sequence of positive integers, label the step endpoints in sequence from left to right, except that points that can be connected by a horizontal line lying strictly under the path have the same label. It is immediate that the sequence of labels, read from left to right, is a DS sequence. The inverse mapping can be easily defined. An example illustrating the bijection is given in Fig. 19.


Fig. 18.


Fig. 19.
Obviously, the length of the obtained DS sequence is equal to the length of the Motzkin path increased by one unit and one can show that the rank of the DS sequence is equal to $1+($ length of Motzkin path + number of level steps $) / 2$.

Remark. This last bijection from Motzkin paths to DS sequences can be trivially modified to a bijection from Schröder paths to DS sequences, in which the rank of the DS sequence is equal to the length of the Schröder path increased by one unit.

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