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A characterization of partial metrizability: domains are quantifiable

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Abstract

A characterization of partial metrizability is given which provides a partial solution to an open problem stated by Künzi in the survey paper *Non-symmetric Topology* (in: Proceedings of the Szekszard Conference, Bolyai Soc. Math. Studies, Vol. 4, 1993, pp. 303–338; problem 7[1]). The characterization yields a powerful tool which establishes a correspondence between partial metrics and special types of valuations, referred to as Q-valuations (cf. also Theoret. Comput. Sci., to appear). The notion of a Q-valuation essentially combines the well-known notion of a valuation with a weaker version of the notion of a quasi-unimorphism, i.e. an isomorphism in the context of quasi-uniform spaces. As an application, we show that ω-continuous directed complete partial orders (dcpos) are quantifiable in the sense of O'Neill (in: S. Andima et al. (Eds.), Proceedings of the 11th Summer Conference on General Topology and Applications, Annals of the New York Academy of Sciences, Vol. 86, 1997, pp. 304–315), i.e. the Scott topology and partial order are induced by a partial metric. For ω-algebraic dcpos the Lawson topology is induced by the associated metric. The partial metrization of general domains improves prior Gpproaches in two ways:

- The partial metric is guaranteed to *capture the Scott topology* as opposed to e.g. Smyth (Quasi-uniformities: Reconciling Domains with Metric Spaces, Lecture Notes in Computer Science, Vol. 298, Springer, Berlin, 1987, pp. 236–253), Bonsangue et al. (Theoret. Comput. Sci. 193 (1998) 1), Flagg (Theoret. Comput. Sci., to appear) and Flagg (Theoret. Comput. Sci. 177 (1) (1997) 1), which in general yield a coarser topology.
- Partial metric spaces are *Smyth-completable* and hence their Smyth-completion reduces to the standard bicompletion. This type of simplification is advocated in Smyth (in: G.M. Reed, A.W. Roscoe, R.F. Wachter (Eds.), Topology and Category Theory in Computer Science, Oxford University Press, Oxford, 1991, pp. 207–229). Our results extend Smyth (1991)'s scope of application from the context of 2/3 SFP domains to general domains.

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[1] "Characterize those quasi-uniformities having a countable base which are induced by a weighted quasi-metric".

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The quantification of general domains solves an open problem on the partial metrizability of domains[2] stated in O'Neil (1997) and Heckmann (Appl. Categor. Struct. (1999) 71).

Our proof of the quantifiability of domains is novel in that it relies on the central notion of a semivaluation (Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci., to appear). The characterization of partial metrizability is entirely new and sheds light on the deeper connections between partial metrics and valuations commented on in [Bukatin and Shorina (in: M. Nivat (Ed.), Foundations of Software Science and Computation Structures, Lecture Notes in Computer Science, Vol. 1378, Springer, Berlin, 1998, pp. 125–139)]. Based on (Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci., to appear) and our present characterization, we conclude that the notion of a (semi)valuation is central in the context of Quantitative Domain Theory since it can be shown to underlie the various models arising in the applications.

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1. Introduction and related work

Recent developments in Domain Theory indicate that additional concepts are required in order to develop the corresponding applications. These developments include domain theoretic approaches to dataflow networks (e.g. [25,26]), logic programming (e.g. [40]), domain theoretic approaches to integration (e.g. [6]), models for probabilistic languages (e.g. [18,19]), models for real number computation [7] as well as models which incorporate complexity analysis (e.g. [31,35]).

Each of these application involve "real number measurements" in some sense, and hence the adjective *quantitative* is used as opposed to the adjective *qualitative* which indicates the traditional order theoretic approach. The terminology "Quantitative Domain Theory" was coined in [10].

At this point several foundations exist. The more abstract approaches include the Yoneda completion [2], the continuity spaces [9] and the topological quasi-uniform spaces [44]. These approaches are essentially equivalent (cf. [11,22]) and lead to complex completions, involving non-idempotency or subtle relations between two topologies and a quasi-uniformity. Moreover, they involve generalized metrics which typically lead to topologies coarser than the Scott topology, which for instance for the case of topological quasi-uniform spaces is resolved by the addition of a new topology.

In [42], the totally bounded spaces have been introduced, for which the notion of completion simplifies to the bicompletion and for which the induced topology is the Scott topology. Other approaches include the use of valuations (e.g. [6,18]) as well as the use of partial metrics (e.g. [25]).

In [28] the question is raised as to which domains are "quantifiable" in the sense that there exists a partial metric which induces the Scott topology. A similar question

[2] Pawel Waszkiewicz communicated recently to the author that he obtained similar results independently. A stronger result implying the quantifiability of general domains is reported in [23]. O'Neill obtained the partial metrizability of ω-algebraic domains in his thesis [29].

has been raised by Heckmann in [17]. It is shown in [28] that Scott domains are quantifiable. We improve on this result by showing that ω-continuous directed complete partial orders (dcpos) are quantifiable.

A connection between partial metrics and valuations has first been indicated in [28]. Valuation spaces $(X, \sqsubseteq, v)$ are introduced which are defined as consistent semilattices, i.e. semilattices for which every pair of elements which is bounded from above has a least upper bound, equipped with a strictly increasing *real*-valued valuation v. It is shown that these spaces can be equipped with a partial metric p_v defined by $p_v(x, y) = -v(x \sqcap y)$. This partial metric is generalized, in the sense that it assumes negative values. The relationship between valuations and partial metrics has been further discussed in [3] as well as in the recent [4]. These approaches all involve partial metrics generated from strictly increasing valuations.

According to [4], "the existence of deep connections between partial metrics nd valuations is well known in Domain Theory"; a claim which is supported in part by the examples discussed in [3,4,28].

It is well known that the topological characterization of partial metric spaces in general poses a hard problem. Some interesting partial results for restricted classes of spaces do exist, however, [21,23] and will be discussed below.

One of the reasons for the intractability of the problems stated in the survey paper [21], seems to be that partial metric spaces do not embody as yet enough of the structure of the examples arising in the applications. [3] Hence, in our study of these structures we have aimed at isolating a "mathematically nice" subclass of partial metric spaces, which is still sufficiently large to incorporate the domain theoretic examples involving partial metric spaces.

A suitable class of spaces was obtained in [38], where we focused on the class of quasi-metric semilattices. This class includes the Baire partial metric spaces of [25] as well as the complexity spaces of [34] (cf. also [31]). It also incorporates the Scott Domains, whether they be represented as totally bounded quasi-metric spaces, as in [42], or via 0–1 Valued quasi-metrics (e.g. [41] or [2]), and the interval domain [7].

To analyze partial metrizability, we study the slightly more general class of quasi uniform semilattices.

These structures are defined to be semilattices equipped with a quasi-uniformity with respect to which the semilattice operation is quasi-uniformly continuous. Quasi-uniform lattices are defined in a similar way, where the definitions generalize the classical definition of a uniform lattice (e.g. [46] or [47]).

One can show that the quasi-uniform continuity of the semilattice operation for each of the above mentioned examples is implied by one of the following invariance properties (e.g. [13]): $\forall x, y, z \in X. d(x \sqcap z, y \sqcap z) \leqslant d(x, y)$ and $d(x \sqcup z, y \sqcup z) \leqslant d(x, y)$. This has motivated the study of invariant quasi-metrics; which form a subclass of the quasi-uniform semilattices.

In order to study partial metric spaces, it is convenient to focus on their equivalent formulation as weightable quasi-metric spaces. A motivation for this choice is that an equivalence between weighting functions and strictly increasing valuations will be

[3] H.P. Künzi, private communication.

established, which will imply the equivalence between invariant partial metrics and strictly increasing valuations in this context. This approach has the additional benefit that the terminology is part of the standard theory of non-symmetric topology (e.g. [12,21]).

As remarked above, several open characterization problems on weightedness have been stated in the survey paper Nonsymmetric Topology ([21]): [4]

> "Characterize those quasi-uniformities having a countable base which are induced by a weighted quasi-pseudo-metric" (Problem 7)
> "Which topological spaces admit weightable quasi-pseudo-metrics?" (Problem 8)
> "Develop a concept of a weighted quasi-uniformity" (Problem 10).

Partial results in connection to Problems 7 and 8 are known [21,23]. The results concern restricted classes, as for instance the class of Alexandroff topologies in relation to a partial solution of Problem 8 [21]. In [23] an interesting sufficient condition is given in connection to Problem 7 for a large class of spaces: "any totally bounded quasi-uniform space with a countable base can be induced by a weighted quasi-pseudo-metric".

We recall [35] that Problem 10 cannot be solved via an axiomatization in terms of the entourages of the quasi-uniform space which would guarantee the weightability of all quasi-pseudo-metrics which induce the quasi-uniformity (for the case of quasi-uniformities with a countable base).

We present a partial solution to Problem 7 in the context of quasi-uniform semilattices and of quasi-uniform lattices. We characterize the class of quasi-uniform (semi) lattices which are induced by an invariant weightable quasi-metric. The solution is based on the notion of a Q-semivaluation. This notion intuitively is a semivaluation [38], which is an order quasi-unimorphism. Semivaluations have been introduced in [38] as a natural generalization of valuations from the context of lattices to the context of semilattices. Order-quasi-unimorphisms on quasi-uniform spaces $(X, \mathscr{U})$ generalize the definition of a quasi-unimorphism, where the condition of injectivity has been replaced by strict increasingness and where the quasi-unimorphism conditions are stated on the ordering relation $\leqslant_{\mathscr{U}}$. Order quasi-unimorphisms are "generalized" quasi-unimorphisms since, for the case of quasi-uniform spaces with an associated linear order, they reduce to a quasi-unimorphism.

The solution differs from prior work in two ways:

- the class of partial metric spaces for which he solution has been presented has been directly motivated by domain theoretic examples.
- the solution provides an equivalence, as opposed to prior results [21,23], which provide either necessary or sufficient conditions. The same remark holds for connections which have been obtained between partial metrics and valuations in [3,4,28], which have been restricted to partial metrics generated from strictly increasing valuations, but not conversely.

As an application of this result, we show that domains are partially metrizable.

[4] In [21] the problems are actually stated in terms of "quasi-metrics", which correspond to the "quasi-pseudo-metrics" as originally defined in [12].

The partial metrizability of domains has an interesting history. We obtained a solution to the problem during a visit at Imperial College in September 2000. Recently, Pawel Waszkiewicz communicated that he obtained similar results. On a later reading of O'Neill's thesis [29], we discovered that the result was obtained priorly in this work, but apparently never reported in the literature. O'Neil 's proof however fails to go through for ω-continuous domains, as illustrated by a counterexample in Section 5. Finally, we remark that the quantification of domains also follows from a more general result, obtained by Künzi and Vajner in [23], where it is shown that each T_0-space with a σ-point-bounded base admits a weighted quasi-pseudometric. So in particular each second-countable T_0-space admits a weighted quasi-pseudometric.

The partial metrizability of domains is interesting in two ways.

It extends Smyth's result on totally bounded spaces [42] to general domains, with preservation of the desirable property of Smyth-completability which guarantees that the completion simplifies to the bicompletion. Indeed, we recall that partial metric spaces are Smyth-completable [21].

The result also allows for the quantification of domains via a generalized metric which *does* induce the Scott topology. This should be contrasted with prior approaches which, for the case of arbitrary domains, only could guarantee that the generalized metric involved induced a topology coarser than the Scott topology. Typically, this was avoided through the introduction of a second topology.

Our work differs from O'Neill's in that we do not need the concept of a "generalized valuation space" [29] to obtain a partial metric from a valuation. Indeed, by using our characterization of partial metrizability, we can obtain our *invariant* partial metrics directly from Q-valuations. A similar remark applies to O'Neill's notion of an information measure, where his extra condition on the Borel measure "$\mu(x\downarrow - (x \sqcap y)\downarrow) = 0 \Rightarrow x \sqsubseteq y$" in our case is not required in order to generate a partial metric. Also, we show that the partial metric inducing the Scott topology can be generated from a function which has finite sum over all base elements. This may be contrasted with O'Neill's approach for domains with countable base $B = (a_n)_n$, which involves a specific choice of weights on base elements defined by $w(a_n) = 1/2^n$. Our characterization of partial metrizability for the case of quasi-uniform lattices, provides a much sharper result for which we not only can derive partial metrics from valuations, but also obtain an equivalence result between these notions.

In our proof, the partial metrizability of domains is obtained by an application of the characterization of partial metrizability to the lattice of Scott-closed sets. It is interesting to note that the partial metric on general domains is obtained from a *semivaluation* on this lattice, while this can be achieved for the case of ω-algebraic domains via a *valuation* on the lattice.

The characterization sheds light on the "deep connections" between partial metrics and valuations mentioned in [4] (cf. also [38]).

The proof that partial metrics on domains can be obtained from valuations on lattices, combined with [38], justifies the observation that valuations can be viewed as the central notion of Quantitative Domain Theory which allows one to quantify domains in a simple and elegant way.

Indeed, it is now clear that the notion of a (semi)valuation underlies varied and, on the face of it, entirely unrelated models such as the totally bounded spaces of [42], e.g. used as logic programming models [40], the partial metric spaces, e.g. used as models for dataflow networks, models for efficiency analysis and models for real number computation [7,25,34], and of course the models which are directly based on valuations, such as models for domain theoretic approaches to integration [6] and models for non-deterministic computation [18].

2. Background

The following notation is used throughout: $\mathcal{N}$ denotes the set of natural numbers, $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^+ = (0, \infty)$, $\mathcal{R}_0^+ = [0, \infty)$, while $\tilde{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}$, $\widetilde{\mathcal{R}^+} = \mathcal{R}^+ \cup \{\infty\}$ and $\widetilde{\mathcal{R}_0^+} = \mathcal{R}_0^+ \cup \{\infty\}$.

A function $d : X \times X \to \mathcal{R}_0^+$ is a *quasi-pseudo-metric* iff

(1) $\forall x \in X.\, d(x,x) = 0$

(2) $\forall x, y, z \in X.\, d(x, y) + d(y,z) \geqslant d(x,z)$.

A *quasi-pseudo-metric space* is a pair (X,d) consisting of a set X together with a quasi-pseudo-metric d on X.

In case a quasi-pseudo-metric space is required to satisfy the T_0-separation axiom, we refer to such a space as a *quasi-metric* space.

In that case, condition (1) and the T_0-separation axiom can be replaced by the following condition:

(1′) $\forall x, y.\, d(x, y) = d(y,x) = 0 \Leftrightarrow x = y$.

The *conjugate* d^{-1} of a quasi-pseudo-metric d is defined to be the function $d^{-1}(x, y) = d(y,x)$, which is again a quasi-pseudo-metric (e.g. [12]). The conjugate of a quasi-pseudo-metric space (X,d) is the quasi-pseudo-metric space (X,d^{-1}). The (*pseudo-*) *metric* d^* induced by a quasi-(pseudo-) metric d is defined by $d^*(x, y) = \max\{d(x, y), d(y,x)\}$.

A quasi-pseudo-metric space (X,d) is *totally bounded* iff $\forall \varepsilon > 0\, \exists x_1 \ldots x_n \in X \forall x \in X\, \exists i \in \{1,\ldots,n\}.\, d^*(x_i,x) < \varepsilon$.

A quasi-metric space (X,d) is *compact* iff the associated metric space (X,d^*) is compact.

The *associated preorder* $\leqslant_d$ of a quasi-pseudo-metric d is defined by $x \leqslant_d y$ iff $d(x, y) = 0$.

We write that a quasi-pseudo-metric space *encodes* a preorder when $\forall x, y \in X.\, d(x, y) \in \{0, 1\}$. In that case we also write that the *encoded preorder* is the preorder $(X, \leqslant_d)$. Conversely, for a given preorder $(X, \leqslant)$, one can define a quasi-pseudo-metric space $(X, d_{\leqslant})$ which encodes the preorder, in the obvious way.

Let $(P, \sqsubseteq_1)$ and $(Q, \sqsubseteq_2)$ be partial orders. A function $f : P \to Q$ is *increasing* (*decreasing*) $\Leftrightarrow \forall x, y \in P.\, x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y) (f(y) \sqsubseteq_2 f(x))$.

A function $f : (X,d) \to (X',d')$ is an *isometry* iff f is a bijection and $\forall x, y \in X.\, d'(f(x), f(y)) = d(x, y)$.

We recall [35, Lemma 5], that quasi-pseudo-metrics satisfy the following property, which we refer t as the "*Monotonicity Lemma*": if (X,d) is a quasi-pseudo-metric space then $\forall x,x',y,y' \in X. (x' \leqslant_d x$ and $y' \geqslant_d y) \Rightarrow d(x',y') \leqslant d(x,y)$.

We discuss a few examples of quasi-pseudo-metric spaces.

The function $d_1 : \mathscr{R}^2 \to \mathscr{R}_0^+$, defined by $d_1(x,y) = y - x$ when $x < y$ and $d_1(x,y) = 0$ otherwise, and its conjugate are quasi-pseudo-metrics. We refer to d_1 as the "left distance" and to its conjugate as the "right distance". These quasi-pseudo-metrics correspond to the non-symmetric versions of the standard metric m on the reals, where $\forall x,y \in \mathscr{R}. m(x,y) = |x - y|$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x,y \in \mathscr{R} x \leqslant_{d_1^{-1}} y \Leftrightarrow x \leqslant y$, while for the left distance we have $\forall x,y \in \mathscr{R}. x \leqslant_{d_1} y \Leftrightarrow x \geqslant y$.

The function d_2: $(\tilde{\mathscr{R}} - \{0\})^2 \to \mathscr{R}_0^+$, defined by $d_2(x,y) = 1/y - 1/x$ when $y < x$ and 0 otherwise, and its conjugate are quasi-pseudo-metrics.

The *complexity space* (C,d_C) has been introduced in [34] (cf. also [31,35]. Here

$$C = \left\{ f : \omega \to \widetilde{\mathscr{R}^+} \,\middle|\, \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty \right\}$$

and d_C is the quasi-pseudo-metric on C defined by

$$d_C(f,g) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right],$$

whenever $f,g \in C$. The complexity space (C,d_C) is a quasi-metric space with a maximum $\top$, which is the function with constant value ∞.

The *dual complexity space* is introduced in [31] as a pair (C^*, d_{C^*}), where $C^* = \{f : \omega \to \mathscr{R}_0^+ \mid \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\}$, and d_{C^*} is the quasi-metric defined on C^* by $d_{C^*}(f,g) = \sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0]$, whenever $f,g \in C^*$. We recall that (C,d_C) is isometric to (C^*, d_{C^*}) by the isometry $\Psi : C^* \to C$, defined by $\Psi(f) = 1/f$ (see [31]). Via the analysis of its dual, several quasi-metric properties of (C,d_C), in particular Smyth completeness and total boundedness, are studied in [31].

For a given set X, Δ is the identity relation. A *quasi-uniform space* is a pair $(X,\mathscr{U})$ consisting of a set X with a filter $\mathscr{U}$ on $X \times X$ such that

(1) $\forall U \in \mathscr{U}. \Delta \subseteq U$

(2) $\forall U \in \mathscr{U}\ \exists V \in \mathscr{U}. V \circ V \subseteq U$.

In that case, $\mathscr{U}$ is called a *quasi-uniformity* on X and its elements are referred to as *entourages*.

A *uniform space* is a quasi-uniform space $(X,\mathscr{U})$ such that

(3) $\forall U \in \mathscr{U}. U^{-1} \in \mathscr{U}$.

In that case, $\mathscr{U}$ is called a *uniformity* on X.

Given a quasi-uniform space $(X,\mathscr{U})$ then the *uniform space associated to* $(X,\mathscr{U})$ is defined to be the space $(X,\mathscr{U}^*)$ where $\mathscr{U}^* = \{V \subseteq X \times X \mid \exists U \in \mathscr{U}$ such that $V \supseteq U \cap U^{-1}\}$.

The *preorder associated* with a quasi-uniform space $(X,\mathscr{U})$ is the relation $\leqslant_{\mathscr{U}}$ defined to be the intersection of all the entourages of $\mathscr{U}$.

The *quasi-uniformity* $\mathscr{U}_d$ *generated by a quasi-pseudo-metric* d on a set X is the filter generated on $X \times X$ by the set of relations $(B_{\varepsilon>0})_\varepsilon$, where $\forall \varepsilon > 0. B_\varepsilon = \{(x, y) \mid d(x, y) < \varepsilon\}$. Two quasi-pseudo-metrics are *equivalent* iff they generate the same quasi-uniformity.

The topology $\mathscr{T}(\mathscr{U})$ associated to a quasi-uniformity $\mathscr{U}$ on a set X is the topology generated by the neighborhood filter base $\mathscr{U}[x] = \{U[x] \mid U \in \mathscr{U}\}$, where $\forall x \in X\, \forall U \in \mathscr{U}.\ U[x] = \{y \mid (x, y) \in U\}$.

If $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are quasi-uniform spaces, then a base of the *product quasi-uniformity* $\mathscr{U} \times \mathscr{V}$ is the set of all binary relations B on $X \times Y$, such that there is a $U \in \mathscr{U}$ and a $V \in \mathscr{V}$ such that for each (x, y) in $X \times Y$, $B[(x, y)] \supseteq U[x] \times V[y]$. The topology induced by the product quasi-uniformity is the product topology.

A function $f, : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is *quasi-uniformly continuous* iff $\forall V \in \mathscr{V}\, \exists U \in \mathscr{U}.\, f^2(U) \subseteq V$, where $f^2(U) = \{(f(x), f(y)) \mid xUy\}$. A *quasi-unimorphism* f: $(X, \mathscr{U}) \to (Y, \mathscr{V})$ is a bijection such that both f and f^{-1} are quasi-uniformly continuous. A function f: $(X, d) \to (Y, d')$ is *quasi-uniformly continuous* iff the function $f : (X, \mathscr{U}_d) \to (Y, \mathscr{V}_{d'})$ is quasi-uniformly continuous.

In case the associated order of a quasi-pseudo-metric (quasi-uniform) space is a linear order we refer to the space as a *linear* quasi-pseudo-metric (quasi-uniform) space.

A *join* (*meet*) *semilattice* is a partial order $(X, \leqslant)$ such that every two elements $x, y \in X$ have a supremum $x \sqcup y$ (infimum $x \sqcap y$) in X. A *lattice* is a partial order which is both a join and a meet semilattice.

We write that a quasi-pseudo-metric (quasi-uniform) space is a (semi)lattice iff the associated order is a (semi)lattice.

In fact, with slight abuse of terminology, we will refer to quasi-metric spaces for which the associated order is a semilattice, simply as *semilattices*, and a similar convention holds for the case of lattices.

In this context, the quasi-metric space is referred to as the *underlying quasi-metric space*.

The terminology of *quasi-pseudo-metric* (*quasi-uniform*) (*semi*)*lattice* is reserved for quasi-pseudo-metric (quasi-uniform) spaces which are (semi)lattices for which the operations are quasi-uniformly continuous with respect to the product quasi-uniformity $\mathscr{U}_d \times \mathscr{U}_d$ ($\mathscr{U} \times \mathscr{U}$). This is in accordance with the terminology used for the theory of uniform lattices (e.g. [46,47]).

A join semilattice (X, d) is *invariant* iff $\forall x, y, z \in X.\, d(x \sqcup z, y \sqcup z) \leqslant d(x, y)$. In that case we also write that the quasi-pseudo-metric d is invariant. The notions of an *invariant meet semilattice* and of an *invariant lattice* are defined in the obvious way. One can easily verify that invariant join semilattices are quasi-pseudo-metric join semilattices and that similar results hold for the case of invariant meet semilattices and for invariant lattices.

We recall that a quasi-pseudo-metric (quasi-uniform) space is T_0 iff the associated order of the space is a partial order (e.g. [12]). We will work under the assumption that all spaces satisfy the T_0 separation axiom; that is we will solely refer to quasi-metric spaces in the following.

A non-negative real valued function f on a set X is *bounded* iff $\exists K \geqslant 0\, \forall x \in X.\, f(x) \leqslant K$.

A function $f : X \to \mathcal{R}_0^+$ is *fading* iff $\inf_{x\in X} f(x) = 0$. We recall the definition of a valuation on a lattice $(L, \sqsubseteq)$.

A function $f : L \to \mathcal{R}_0^+$ is a *valuation* iff
(1) f is increasing.
(2) $\forall x, y \in L. f(x \sqcap y) + f(x \sqcup y) = f(x) + f(y)$.
In case the function f is decreasing and satisfies (2), we refer to f as a *co-valuation*.

If f only satisfies (2) we say that f satisfies the *modularity law*, or also that f is *modular*.

There does not seem to be a consistent terminology in the literature. Valuations, also called evaluations, as used in computer science (e.g. [3] or [19]) typically satisfy (1) and (2) above. In the classical mathematical literature a valuation only needs to satisfy (2) (e.g. [1]).

It is convenient for matters of presentation to reserve the definition given above for a valuation in order to state results on connections between partial metrics and valuations as they occur in Computer Science.

Finally, a (co)valuation f on a lattice $(L, \sqsubseteq)$ is *strictly increasing* (*strictly decreasing*) if $\forall x, y \in L. x \sqsubset y \Rightarrow f(x) < f(y) (f(x) > f(y))$. What we call strictly increasing corresponds to the strongly non-degenerate requirement of [4] and strictly increasing valuations are exactly the dimension functions as defined in [5].

A partial order $(X, \leqslant)$ is *directed* iff $\forall x, y \in X\, \exists z \in X. z \geqslant x$ and $z \geqslant y$.

A dcpo is a partially ordered set $(, \sqsubseteq)$ with a least element $\perp$ and such that every directed subset has a supremum. The set of elements below an element x is denoted by $x\downarrow$.

Suppose that x and y are elements of a dcpo $(P, \sqsubseteq)$, then x is way below y iff for all directed subsets A, $y \sqsubseteq \sqcup \sup A \Rightarrow \exists a \in A. x \sqsubseteq a$. The set of elements way below a given element x is denoted by $x\Downarrow$. For any set $A \subseteq P$, $A\Downarrow = \{x \in P \mid \exists a \in A. x \ll a\}$. A compact element is an element which is way below itself. The set of compact elements of P is denoted by $K(P)$.

A subset B of a dcpo P is a basis for P iff for all $x \in P$, the set $B_x = B \cap (x\Downarrow)$ is directed with supremum x.

A dcpo P is called continuous if it has a basis and it is called algebraic if it has a basis of compact elements.

An equivalent characterization of a continuous dcpo is that $x = \sqcup(x\Downarrow)$ for any element x of the continuous dcpo. Every algebraic domain is a continuous domain. A continuous dcpo with a countable basis is called ω-continuous. An ω-continuous dcpo is also simply referred to as a domain. Similarly, we use the terminology of an ω-algebraic dcpo P, when $K(P)$ is countable.

A basis characterizes the ordering since $x \leqslant y \Leftrightarrow B_x \subseteq B_y$. Of course $x \ll y \Rightarrow x \sqsubseteq y$.

A dcpo in which each pair of elements with an upper bound has a supremum, is called bounded-complete. The ω-algebraic bounded-complete dcpos are called Scott domains. An example of a continuous dcpo which is not algebraic is the unit interval $[0, 1]$, with its usual order, where $x \ll y \Leftrightarrow (x < y$ or $x = 0)$. We discuss an example of a Scott domain below.

Example. As in Example 4 of [42], let Σ^∞ denote the set of all finite and infinite sequences ("words") over a countable alphabet Σ. For any subset A of Σ, let A^* denote the set of all finite sequences over A. Given a sequence $s \in \Sigma^\infty$, say of length $L \geqslant 1$, then for any natural number n such that $1 \leqslant n \leqslant L$, $s(n)$ denotes the nth element. The prefix order $\sqsubseteq$ on Σ^∞ is defined as follows: for any two sequences, $s, s' \in \Sigma^\infty$: $s \sqsubseteq s' \Leftrightarrow s$ is an initial subsequence of s'. Then $(\Sigma^\infty, \sqsubseteq)$ is an example of a Scott Domain, where the set of finite elements is Σ^*.

We recall some basic information from [30] on 2/3 SFP domains.

Definition 1. Given a subset A of a partial order P, then an upper (lower) bound of A is an element $x \in P$ such that $\forall y \in A. x \sqsupseteq y\,(x \sqsubseteq y)$. A minimal upper bound of A is an upper bound for A which has no other upper bounds of A below it. The set of minimal upper bounds of A is *complete* iff every upper bound of A has a minimal upper bound below it.

Definition 2. An ω-algebraic domain is 2/3 SFP iff its Lawson topology is compact.

Theorem 3 (The 2/3 SFP Theorem of Plotkin [30]). *An ω-algebraic domain is* 2/3 *SFP iff for any pair of compact elements of the domain, the set of minimal upper bounds is complete and finite.*

A quasi-metric space (X, d) is *weightable* iff there exists a function $w : X \to \mathscr{R}_0^+$ such that $\forall x, y \in X. d(x, y) + w(x) = d(y, x) + w(y)$. The function w is called a *weighting function*, $w(x)$ is the *weight* of x and the quasi-metric d is *weightable by the function* w. A *weighted* space is a triple (X, d, w) where (X, d) is a quasi-metric space weightable by the function w.

A quasi-metric space (X, d) is *co-weightable* iff its conjugate (X, d^{-1}) is weightable. A *co-weighting function* of a quasi-metric space is a weighting function of its conjugate. A *co-weighted* space (X, d, w) is a triple consisting of a set X, a quasi-metric d on X and a co-weighting function w.

A quasi-metric space (X, d) is *bi-weightable* iff it is weightable and co-weightable. We remark that any weighted space (X, d, w) of bounded weight, where say $\forall x \in X. w(x) \leqslant K$, is co-weighted by the weighting function $K - w$ [21]. Hence any weighted space of bounded weight is bi-weightable. Similarly, one obtains that any co-weighted space of bounded co-weight is bi-weightable.

For a detailed discussion of weightable quasi-metric spaces and the equivalent partial metric spaces we refer the reader to Section 4 of [38]. We recall in the following the definition of a semivaluation and its natural interpretation as a generalization of the notion of a valuation to the context of semilattices (cf. also [38]).

Definition 4. If $(X, \preccurlyeq)$ is a meet semilattice then a function $f : (X, \preccurlyeq) \to \mathscr{R}_0^+$ is a meet valuation iff

$$\forall x, y, z \in X. f(x \sqcap z) \geqslant f(x \sqcap y) + f(y \sqcap z) - f(y)$$

and f is meet co-valuation iff

$$\forall x, y, z \in X.\ f(x \sqcap z) \leqslant f(x \sqcap y) + f(y \sqcap z) - f(y).$$

Definition 5. If $(X, \preccurlyeq)$ is a join semilattice then a function $f : (X, \preccurlyeq) \to \mathscr{R}_0^+$ is a join valuation iff

$$\forall x, y, z \in X.\ f(x \sqcup z) \leqslant f(x \sqcup y) + f(y \sqcup z) - f(y)$$

and f is join co-valuation iff

$$\forall x, y, z \in X.\ f(x \sqcup z) \geqslant f(x \sqcup y) + f(y \sqcup z) - f(y).$$

Definition 6. A function is a semivaluation if it is either a join valuation or a meet valuation. A join (meet) valuation space is a join (meet) semilattice equipped with a join (meet) valuation. A semivaluation space is a semilattice equipped with a semivaluation.

Proposition 7. *Let L be a lattice.*

(1) *A function $f : L \to \mathscr{R}_0^+$ is a join valuation if and only if it is increasing and satisfies join-modularity, i.e.:*

$$f(x \sqcup z) + f(x \sqcap z) \leqslant f(x) + f(z).$$

(2) *A function $f : L \to \mathscr{R}_0^+$ is a meet valuation if and only if it is increasing and satisfies meet-modularity, i.e.*

$$f(x \sqcup z) + f(x \sqcap z) \geqslant f(x) + f(z).$$

Corollary 8. *A function on a lattice is a valuation iff it is a join valuation and a meet valuation. A function on a lattice is a co-valuation iff it is a join co-valuation and a meet co-valuation.*

The last result clearly motivates the fact that semivaluations provide a natural generalization of valuations from the context of lattices to the context of semilattices. We refer the reader to [38] for the correspondence theorems which link partial metrics to semivaluations. We include Theorem 10 of [38] to which we will refer extensively in the following.

Theorem 9 ([38, Theorem 10]). *For every join semilattice $(X, \preccurlyeq)$, there exists a bijection between invariant weighted quasi-metrics d on X with $\leqslant_d = \preccurlyeq$ and fading strictly decreasing join co-valuations $f : (X, \preccurlyeq) \to (\mathscr{R}_0^+, \leqslant)$. The map $f \mapsto d_f$ is defined by $d_f(x, y) = f(y) - f(x \sqcup y)$. The inverse is the function which to each weighted space (X, d) associates its unique fading weighting. Similarly one can show that for every join semilattice $(X, \preccurlyeq)$, there exists a bijection between invariant co-weighted quasi-metrics d on X with $\leqslant_d = \preccurlyeq$ and fading strictly increasing join valuations $f : (X, \preccurlyeq) \to (\mathscr{R}_0^+, \leqslant)$. The map $f \mapsto d_f$ is defined by $d_f(x, y) = f(x \sqcup y) - f(y)$. The inverse is the function which to each co-weighted space (X, d) associates its unique fading co-weighting.*

We end the section with the remark that the totally bounded quasi-metric Scott domains of [42] are neither weightable nor co-weightable in general. The fact that the spaces are not co-weightable, prevents a straightforward representation of such a domain as a meet valuation space via the dual version of Theorem 9 (cf. Theorem 11 of [38]). However, the problem will be eliminated, since we will show that domains allow for a bi-weightable quasi-metric which induces the Scott topology and their partial order (Proposition 7).

Counterexample. We recall that any weightable quasi-metric space (X,d,w) is order convex [35], i.e. $\forall x,y,z \in X. x \geqslant_d y \geqslant_d z \Rightarrow d(x,z) = d(x,y) + d(y,z)$. It is easy to show that the same holds for co-weightable quasi-metric spaces. We show that in general, Scott Domains are not order convex and hence not co-weightable nor weightable.

Let Σ be a countable alphabet, say $\Sigma = (a_n)_{n \geqslant 0}$. We consider a Scott domain $(\Sigma^\infty, \sqsubseteq)$ as in the example at the end of Section 2. We equip the domain with a quasi-metric d_r, defined by $d_r(x,y) = \inf\{2^{-n} \mid e \sqsubseteq x \Rightarrow e \sqsubseteq y$ for every finite e of rank $\leqslant n\}$. Here the rank of a finite element is determined by the function $r : \Sigma^* \to \mathcal{N}$ where $\forall w \in \Sigma^*. r(w) =$ the minimal n such that $w \in \{a_0, \ldots, a_n\}^*$ and $\text{length}(w) \leqslant n$.

Let $w_1 = (a_0, \ldots, a_k)$, $w_2 = (a_0, \ldots, a_l)$ and $w_3 = (a_0, \ldots, a_m)$, where $k > l > m$. Then $w_1 \geqslant_{d_r} w_2 \geqslant_{d_r} w_3$, but $d_r(w_1,w_2) = 2^{-l}$, $d_r(w_2,w_3) = 2^{-m}$, while $d_r(w_1,w_3) = 2^{-m}$. So (Σ^∞, d_r) is not order convex, and thus not co-weightable nor weightable.

3. Quasi-uniform semilattices

We recall the following useful characterization of invariance from [38].

Lemma 10. *A join semilattice* (X,d) *is invariant iff* $\forall x,y \in X. d(x \sqcup y, y) = d(x,y)$. *A meet semilattice* (X,d) *is invariant iff* $\forall x,y \in X. d(x, x \sqcap y) = d(x,y)$.

We say that a partial metric on a join semilattice is *invariant* iff its corresponding weightable quasi-metric is invariant. The definitions for the case of meet semilattices and lattices are similar.

We will discuss several examples of quasi-uniform (semi)lattices which arise in Domain Theory (cf. [38]). In each case, the quasi-uniform continuity of the (semi)lattice operations follows from the fact that the quasi-uniformity is generated by an invariant quasi-metric.

Example 1. Any quasi-metric space which encodes a semilattice is invariant with respect to the semilattice operation. This is in particular the case for quasi-metrics which encode a Scott domain, since any bounded-complete algebraic cpo is a semilattice (e.g. [14]).

Not only straightforward encodings of Scott domains give rise to quasi-uniform semilattices. We recall that a main example of [42], regarding totally bounded spaces as domains of computation (Example 2), as well as the Baire partial metric spaces of

[26] (Example 3), the complexity space of [34] and its dual (Example 4) correspond to quasi-uniform semilattices [38].

Example 2. As in [42], let $(D, \sqsubseteq)$ be a Scott domain equipped with a rank function $r : F_D \to \mathcal{N}$, where $\forall n \in \mathcal{N} . r^{-1}(n)$ is a finite non-empty set and F_D is the set of finite elements of D. Then the following function defines a totally bounded quasi-metric on D:

$$d_r(x, y) = \inf\{2^{-n} \mid e \sqsubseteq x \Rightarrow e \sqsubseteq y \text{ for every finite } e \text{ of rank } \leqslant n\}.$$

The resulting structure is a quasi-metric meet semilattice.

Example 3. Any Baire partial metric space (Σ^∞, p) (cf. [38]) gives rise to a quasi-uniform meet semilattice induced by the corresponding weightable quasi-pseudo-metric meet semilattice (Σ^∞, b).

We remark that the weighting function of a Baire quasi-metric space is bounded and thus such a space is bi-weightable.

Example 4. The interval domain $(I([0,1]), p)$ consisting of the closed bounded intervals of $[0,1]$ ordered by reverse inclusion and equipped with the partial metric p (see [29]) defined by

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}.$$

One can easily verify that the associated weighted quasi-metric space $(I([0,1]), d_p)$ is a quasi-metric meet semilattice with a bounded weighting function.

Example 5. The complexity space (C, d_C) and its dual (C^*, d_{C^*}) are examples of invariant join and meet lattices, respectively. We refer the reader to [35], where the invariance (optimality) of the complexity space is shown and where more general examples of invariant semilattices involving upper weighted function spaces are discussed.

We remark that neither the weighting function of the complexity space nor of its dual is bounded. However, as discussed in [31], complexity functions of programs computing a given problem frequently can be shown to possess a complexity lower bound. A theoretical justification for the existence of lower bounds has been given in [32] based on Levin's theorem (e.g. [20]). It is remarked in [20] that "for an important class of problems that can occur in practice an optimal algorithm *does* exist", by Levin's theorem, and hence one does obtain a lower bound in general. So it is reasonable to restrict the complexity space to complexity functions respecting a given lower bound. It is easy o verify that the complexity distance is bounded on such restricted spaces and hence the fading weighting is bounded. So, we obtain that the restricted spaces are bi-weightable.

A similar argument can be given for the dual complexity space. For more information on complexity spaces with a lower bound, we refer the reader to [31,32].

Example 6. Any quasi-metric space for which the associated order is linear is invariant with respect to its lattice operations. We leave the verifications to the reader. Some examples are the quasi-metric space (I, d_1^{-1}) considered in [42], where I is the unit interval $[0,1]$, as well as the spaces $(\mathscr{R}_0^+, d_1)$ and $(\widetilde{\mathscr{R}^+}, d_2)$.

The following theorem provides a partial solution to Problem 7 of [21], for the class of quasi-uniform join semilattices. We first define the useful notion of an order quasi-unimorphism.

Definition 11. If $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are quasi-uniform spaces then a function $f:(X,\mathscr{U}) \to (Y,\mathscr{V})$ is an order quasi-unimorphism iff
(1) f is strictly increasing with respect to the associated orders,
(2) $\forall V \in \mathscr{V}\ \exists U \in \mathscr{U}\ \forall x, y \in X. x \geqslant_{\mathscr{U}} y \Rightarrow (xUy \Rightarrow (fx)V(fy))$,
(3) $\forall U \in \mathscr{U}\ \exists V \in \mathscr{V}\ \forall x, y \in X. x \geqslant_{\mathscr{U}} y \Rightarrow (f(x)Vf(y) \Rightarrow xUy)$.

If (X,d) and (Y,d') are quasi-metric spaces then a function $f:(X,d) \to (Y,d')$ is an order isometry iff
(1) f strictly increasing with respect to the associated orders,
(2) $\forall x, y \in X. x \geqslant_d y \Rightarrow (d'(f(x), f(y)) = d(x,y))$.

We also refer to f as an order isometry on the space (X,d).

Clearly, every quasi-unimorphism $f:(X,\mathscr{U}) \to (Y,\mathscr{V})$ is a surjective order quasi-unimorphism. We remark that for the case where the domain $(X,\mathscr{U})$ is linear, the notion of a surjective order quasi-unimorphism and that of a quasi-unimorphism are equivalent.

We will focus in the following on order quasi-unimorphisms with range space $(\mathscr{R}_0^+, \mathscr{U}_{d_1})$ and $(\mathscr{R}_0^+, \mathscr{U}_{d_1^{-1}})$, respectively. These are referred to as *left order quasi-unimorphisms* and *right order quasi-unimorphisms* respectively. A similar terminology is used for order isometries.

Definition 12. A Q-join valuation on a quasi-uniform join semilattice is a join valuation which is a right order quasi-unimorphism. A Q-join co-valuation on a quasi-uniform join semilattice is a join co-valuation which is a left order quasi-unimorphism.

Remark. The fact that a join co-valuation is decreasing, while a Q-join co-valuation is increasing with respect to the associated orders is of course consistent, since the associated order of the left distance d_1 is the opposite of the standard ordering on the reals.

Theorem 13. *If $(X,\mathscr{U})$ is a quasi-uniform join semilattice then $\mathscr{U}$ is generated by a weightable invariant quasi-metric $\Leftrightarrow$ there exists a Q-join co-valuation on $(X,\mathscr{U})$.*

Proof. Let $(X,\mathscr{U})$ be a quasi-uniform join semilattice generated by a weightable invariant quasi-metric, say d. Let w be a weighting function for the space (X,d). By Theorem 9, w is a join co-valuation and strictly increasing with respect to the associated orders.

We show that $w:(X,\mathscr{U})\to(\mathscr{R}_0^+,\mathscr{U}_{d_1})$ is a left order quasi-unimorphism. For this it suffices to show that $w:(X,d)\to(\mathscr{R}_0^+,d_1)$ is an order isometry. Since w is strictly increasing with respect to the associated orders, we only need o verify that $\forall x,y\in X.\,x\geqslant_d y \Rightarrow (d_1(w(x),w(y))=d(x,y))$.

Let $x,y\in X$ be such that $x\geqslant_d y$, then, by the weighting equality, we obtain that $d(x,y)=w(y)-w(x)=d_1(w(x),w(y))$.

To show the converse implication, we assume that there exists a Q-join co-valuation f on $(X,\mathscr{U})$. Since $f:(X,\leqslant_{\mathscr{U}})\to(\mathscr{R}_0^+,\leqslant)$ is a strictly decreasing join co-valuation and $(X,\leqslant_{\mathscr{U}})$ is a join semilattice, by Theorem 9, we obtain the weighted space (X,d_f,f) where $\forall x,y\in X.\,d_f(x,y)=f(y)-f(x\sqcup y)$ and where d_f is invariant. We show that d_f induces $\mathscr{U}$.

For the proof, note that $d_f(x,y)=f(y)-f(x\sqcup y)=d_1(f(x\sqcup y),f(y))$.

For $\mathscr{U}_{d_f}\subseteq\mathscr{U}$, we must show $\forall\varepsilon>0\,\exists\,U\in\mathscr{U}.\,xUy\Rightarrow d_f(x,y)<\varepsilon$. Let $\varepsilon>0$. Since f is a left-order quasi-unimorphism, there is a $U'\in\mathscr{U}$ with $[z\geqslant_{\mathscr{U}} y$ and $zU'y]\Rightarrow d_1(f(z),f(y))<\varepsilon$. Since $(X,\mathscr{U})$ is a quasi-uniform join semilattice, there is $U\in\mathscr{U}$ with $aUb\Rightarrow (a\sqcup y)U'(b\sqcup y)$. Hence xUy implies $(x\sqcup y)U'y$. Since $x\sqcup y\geqslant_{\mathscr{U}} y$ holds, xUy implies $\varepsilon>d_1(f(x\sqcup y),f(y))=d_f(x,y)$.

For $\mathscr{U}\subseteq\mathscr{U}_{d_f}$, we must show that $\forall U\in\mathscr{U}\,\exists\,\varepsilon>0.\,d_f(x,y)<\varepsilon\Rightarrow xUy$. Let $U\in\mathscr{U}$. By the definition of quasi-uniformities, there is a $V\in\mathscr{U}$ such that $V\circ V\subseteq U$. Since f is a left-order quasi-unimorphism, there is $\varepsilon>0$ with $[z\geqslant_{\mathscr{U}} y$ and $d_1(f(z),f(y))<\varepsilon]\Rightarrow zVy$. We remark that $d_f(x,y)<\varepsilon$ implies $d_1(f(x\sqcup y),f(y))<\varepsilon$. Since $x\sqcup y\geqslant_{\mathscr{U}} y$ holds, this implies $(x\sqcup y)Vy$ and of course $xV(x\sqcup y)$. Hence $xV(x\sqcup y)Vy$ and thus xUy. □

Definition 14. A Q-meet valuation on a quasi-uniform meet semilattice is a meet valuation which is a right order quasi-unimorphism. A Q-meet co-valuation is a meet co-valuation which is a left order quasi-unimorphism. A Q-semi-valuation is a function which is either a Q-join valuation or a Q-meet valuation.

Finally, we give a dual version of Theorem 13.

Theorem 15. *If $(X,\mathscr{U})$ is a quasi-uniform meet semilattice then $\mathscr{U}$ is generated by a co-weigh able invariant quasi-metric $\Leftrightarrow$ there exists a Q-meet valuation on $(X,\mathscr{U})$.*

4. Quasi-uniform lattices

Definition 16. A Q-valuation on a quasi-uniform lattice is a modular right order quasi-unimorphism. A Q-co-valuation on a quasi-uniform lattice is a modular left order quasi-unimorphism.

We will obtain a version of Theorems 13 and 15 for the case of quasi-uniform lattices. We focus on the case of bounded valuations since all of the above discussed examples involve bounded semivaluations.

Theorem 17. *Let $(X,\mathscr{U})$ be a quasi-uniform lattice. There exists a bounded Q-(co-)valuation f on $(X,\mathscr{U})$ iff $\mathscr{U}$ is generated by a bi-weightable invariant quasi-metric.*

Proof. Let $(X,\mathscr{U})$ be a quasi-uniform lattice.

Let f be a bounded Q-valuation on $(X,\mathscr{U})$, say with a bound K. By Corollary 8, the function f is a meet valuation.

Thus f is a Q-meet valuation and hence we obtain by (the proof of) Theorem 15 (cf. also Theorem 11 of [38]) a quasi-metric d_f which is co-weightable, invariant with respect to the meet operation and which induces $\mathscr{U}$, where $\forall x,y\in X.\, d_f(x,y)=f(x)-f(x\sqcap y)$.

Since f is also join-modular and strictly increasing, we obtain that $\bar{f}=K-f$ is co-join-modular and strictly decreasing. Hence $\bar{f}$ is a Q-join co-valuation and by (the proof of) Theorem 13 (cf. also Theorem 9), we obtain that $\mathscr{U}$ is induced by the co-weightable quasi-metric $d_{\bar{f}}$ which is invariant with respect to the join operation and where $d_{\bar{f}}=\bar{f}(y)-\bar{f}(x\sqcup y)$.

By Proposition 13 of [38], we obtain that $\forall x,y\in X.\, d_f(x,y)=d_{\bar{f}}(x,y)$. We denote these identical quasi-metrics by d.

Clearly, the quasi-metric d is bi-weightable, invariant and induces the quasi-uniformity $\mathscr{U}$.

To show the converse, we assume that there exists a bi-weightable invariant quasi-metric d on X which induces $\mathscr{U}$. Let f be the unique fading weighting and g the unique fading co-weighting of d. Then $\forall x,y\in X.\, d(x,y)+f(x)=d(y,x)+f(y)\Rightarrow d(x,y)-d(y,x)=f(y)-f(x)$ and $d(x,y)+g(y)=d(y,x)+g(x)\Rightarrow d(x,y)-d(y,x)=g(x)-g(y)$. Hence $\forall x,y\in X.\, f(y)-f(x)=g(x)-g(y)\Rightarrow f(x)+g(x)=f(y)+g(y)$. Thus, $f+g$ is a constant function, say K. This implies that f and g are bounded by K, and $g=K-f$.

By meet-invariance, $d(x,y)=d(x,x\sqcap y)=f(x\sqcap y)-f(x)$, and by join-invariance, $d(x,y)=d(x\sqcup y,y)=f(y)-f(x\sqcup y)$. Together, this shows that f is modular and thus a co-valuation. Therefore, $g=K-f$ is a valuation.

By (the proof of) Theorem 13, f is a left order quasi-unimorphism, hence a Q-co-valuation. Similarly, g is a Q-valuation. □

We recall (cf. the counterexample following Corollary 8) that Scott domains are not co-weightable in general. As an application of Theorem 17, we will show in the next section that domains are partially metrizable. In particular, we will show that domains can be equipped with a bi-weightable invariant quasi-metric which induces the Scott topology.

5. Domains are quantifiable

Definition 18. A domain $(P,\sqsubseteq)$ is quantifiable iff there exists a weighted quasi-metric d on P which induces the Scott topology on the domain and for which the associated order coincides with the domain order.

We introduce the Smyth quasi-metric related to a quasi-metric discussed in [42].

Definition 19. For a domain $(X, \sqsubseteq)$ with a basis $B=(a_n)_n$, the Smyth-quasi-metric $d_{\rm S}$ is defined by

$$d_{\rm S}(x,y) = \inf\left\{\frac{1}{2^n}\,|\forall i\leqslant n.\, a_i \ll x \Rightarrow a_i \ll y\right\}.$$

We leave the proof of the following proposition as an exercise.

Proposition 20. *Domains $(P,\sqsubseteq)$ are quasi-metrizable by the quasi-metric $d_{\rm S}$; i.e. $d_{\rm S}$ induces the Scott topology on $(P,\sqsubseteq)$.*

For the case of ω-algebraic domains, the quasi-metric $d_{\rm S}$ can be simplified by replacing the way below inequality "$\ll$" by "$\sqsubseteq$".

Proposition 21. *Two b SFP ω-algebraic domains $(P,d_{\rm S})$, equipped with the Smyth quasi-metric $d_{\rm S}$, are totally bounded.*

Proof. Let P be a 2/3 SFP ω-algebraic domain. We show that $(P,d_{\rm S})$ is totally bounded, where $(a_n)_{n\in\omega}$ is the enumeration of the base elements which determines $d_{\rm S}$.

Given $\varepsilon>0$, then consider a natural number N such that $1/2^N<\varepsilon$.

Let the finitely many elements which satisfy the total boundedness requirement consist of the union of the sets of the minimal upper bounds of all combinations of finite elements with an index $\leqslant N$. We denote this set by $X=\{x_1,\ldots,x_n\}$.

Given an element $x\in X$, we consider the base elements below x with index $\leqslant N$. Note that for this set there exists an upper bound x and hence a minimal upper bound below x by the completeness condition. This minimal upper bound is an element of X, say some x_i. It is easy to verify that $d(x,x_i)<\varepsilon$ and hence $d^*(x,x_i)<\varepsilon$. $\square$

In the following P denotes a domain $(P,\sqsubseteq)$ unless otherwise stated.

Definition 22. Let $\mathscr{C}(P)$ denote the lattice of Scott-closed subsets of P, ordered by inclusion.

We remark that $P\downarrow=\{x\downarrow x\in P\}\subseteq\mathscr{C}(P)$.

Definition 23. The Smyth quasi-metric is extended to the lattice of Scott-closed sets $\mathscr{C}(P)$ by

$$D^c_{\rm S}(C,C') = \inf{}_n\left\{\frac{1}{2^n}\,|\,\forall i\leqslant n.\, a_i\in C\Downarrow\Rightarrow a_i\in C'\Downarrow\right\}.$$

The restriction $d^c_{\rm S}$ of $D^c_{\rm S}$ to P is defined by $d^c_{\rm S}(x,y)=D^c_{\rm S}(x\downarrow,y\downarrow)$.

We leave the straightforward verifications of the following three lemmas to the reader.

Lemma 24. *D_S^c is a join-invariant quasi-metric and hence $(\mathscr{C}(P), \mathscr{U}_{D_S^c})$ is a quasi-uniform join semilattice, equipped with the subset order, i.e. $C \leqslant_{D_S^c} C' \Leftrightarrow C \subseteq C'$.*

For the following lemma we use the fact that $(x\downarrow)\Downarrow = x\Downarrow$.

Lemma 25. *d_S^c coincides with the Smyth quasi-metric d_S.*

Lemma 26. *Let $(P, \sqsubseteq)$ be a domain with countable base $B = (a_n)_n$. We use the following notation for $A \subseteq P : \forall n. A[n] = A \cap \{a_i \mid i \leqslant n\}$. Then, for any two Scott-closed sets, C and C' such that $C \supseteq C'$, we have*

$$D_S^c(C, C') = \inf_n \left\{ \frac{1}{2^n} \mid (C\Downarrow)[n] = (C'\Downarrow)[n] \right\}.$$

In the following proposition, on the quantifiability of domains $(P, \sqsubseteq)$ with a countable base $B = \{a_n\}_n$, we use the notation d^w to distinguish this quasi-metric from the quasi-metric d_w defined in Theorem 9. In fact, d^w is d_{K-w} (using the notation of Theorem 9), where $K = \sum_{a_n} w(a_n)$.

Proposition 27. *Let $(P, \sqsubseteq)$ be a domain with a countable base $B = (a_n)_n$ and let $w : B \to \mathscr{R}^+$ denote a function such that*[5] *$\forall n. w(a_n) > 0$ and $\sum_{a_n \in B} w(a_n) < \infty$. Then $(P, \sqsubseteq)$ is quantifiable by the following bi-weightable quasi-metric:*

$$d^w(x, y) = \sum_{a_n \in x\Downarrow - y\Downarrow} w(a_n).$$

If $(P, \sqsubseteq)$ is ω-algebraic, then $(P, \sqsubseteq)$ is quantifiable by the following bi-weightable quasi-metric:

$$d^w(x, y) = \sum_{a_n \in x\downarrow - y\downarrow} w(a_n).$$

In this case, the associated metric d^* induces the Lawson topology.

Proof. Consider a function $w : B \to \mathscr{R}^+$ such that $\sum_n w(a_n)$ has finite value, say K.

We apply Theorem 13 to the lattice of Scott-closed sets $\mathscr{C}(P)$ in order to show that $\mathscr{U}_{d_S^c}$ is generated by a bi-weighted invariant quasi-metric. We extend w to $\mathscr{C}(P)$ as follows: for any Scott-closed set C,

$$W(C) = \sum_{a_n \in C\Downarrow} w(a_n).$$

By Theorem 13, we need to show that $W' = K - W$ is a Q-join co-valuation on the quasi-uniform lattice $(\mathscr{C}(P), \mathscr{U}_{d_S^c})$.

[5] In case the domain has a least element $\perp$, one can allow $w(\perp) = 0$.

We remark that

$$\forall C_1, C_2 \in \mathscr{C}(P). (C_1 \cup C_2) \Downarrow = C_1 \Downarrow \cup C2 \Downarrow.$$

However, in general one only has

$$\forall C_1, C_2 \in \mathscr{C}(P). (C_1 \cap C_2) \Downarrow \subseteq C_1 \Downarrow \cap C_2 \Downarrow.$$

Hence, $W(C \cup C') + W(C \cap C') = \sum_{a_n \in (C \cup C')\Downarrow} w(a_n) + \sum_{a_n \in (C \cap C')\Downarrow} w(a_n) \leqslant \sum_{a_n \in (C\Downarrow \cup C'\Downarrow)} w(a_n) + \sum_{a_n \in (C\Downarrow \cap C'\Downarrow)} w(a_n) = \sum_{a_n \in C\Downarrow} w(a_n) + \sum_{a_n \in C'\Downarrow} w(a_n) = W(C) + W(C')$.

So W is a join valuation and hence W' is a join co-valuation on the lattice $\mathscr{C}(P)$ with the operations of intersection and union. To verify that W' is strictly decreasing, it suffices to verify that $W : (\mathscr{C}(P), \subseteq) \to (\mathscr{R}^+, \leqslant)$ is strictly increasing.

Let C, C' be Scott-closed sets such that C is strictly included in C'. Then $W(C) \leqslant W(C')$. We show by way of contradiction that $W(C) < W(C')$. Indeed, if $W(C) = W(C')$ then for any base element a_n, we obtain: $a_n \in C \Downarrow \Leftrightarrow a_n \in C' \Downarrow$ from the fact that $\forall n. w(a_n) > 0$.

Hence $C \Downarrow \cap B = C' \Downarrow \cap B$. If x is an element of C' then, since P is a domain, we have that $x = \sqcup B_x$. Since $x \in C'$ and C' is downwardly closed, $B_x \subseteq C'$. Hence $B_x \subseteq C$ and thus $x \in C$. So we obtain $C' \subseteq C$ which is a contradiction. Hence W' is a strictly decreasing join co-valuation.

We verify that W' is a left order quasi-unimorphism. $W' : (\mathscr{C}(P), \mathscr{U}_{d_S^c}) \to (\mathscr{R}^+, d_1)$ is strictly increasing since $W' : (\mathscr{C}(P), \subseteq) \to (\mathscr{R}^+, \leqslant)$ is strictly decreasing. So it remains to be verified that

(1) $\forall \varepsilon \, \exists \delta \, \forall C, C' \in \mathscr{C}(P). C \geqslant_{D_S^c} C' \Rightarrow (D_S^c(C, C') < \delta \Rightarrow d_1(W'(C), W'(C')) < \varepsilon)$.

(2) $\forall \varepsilon \, \exists \delta \, \forall C, C' \in \mathscr{C}(P). C \geqslant_{D_S^c} C' \Rightarrow (d_1(W'(C), W'(C')) < \delta \Rightarrow D_S^c(C, C') < \varepsilon)$.

We remark that for any two Scott-closed sets C, C' with $C \supseteq C'$, we have

$$d_1(W'(C), W'(C')) = \sum_{a_n \in C\Downarrow - C'\Downarrow} w(a_n).$$

In order to prove the result, we can assume that the basis B is infinite. Indeed, in case B is finite, it is easy to verify that for $C' \subseteq C$ in (1), requiring that $D_S^c(C, C') < \delta$, and for $C' \subseteq C$ in (2), requiring that $d_1(W'(C), W'(C')) < \delta$, for δ sufficiently small, implies that $C = C'$ and hence (1) and (2) are satisfied.

We assume in what follows that B is infinite.

To show (1), assume that $\varepsilon > 0$ is given, C and C' are Scott-closed sets such that $C \supseteq C'$ and $d_S^c(C, C') < \delta$.

If N is the largest natural number such that $1/2^N \leqslant \delta$ then $\forall n \leqslant N. (C \Downarrow)[n] = (C' \Downarrow)[n]$. Thus

$$d_1(W'(C), W'(C')) = \sum_{a_n \in C\Downarrow - C'\Downarrow} w(a_n) = \sum_{a_n \in C\Downarrow - C'\Downarrow \text{ and } n \geqslant N} w(a_n).$$

Finally, since B is infinite and the series $\sum_n w(a_n)$ converges, we can choose δ small enough and hence N large enough such that $\sum_{n \geqslant N} w(a_n) < \varepsilon$ and thus

$$d_1(W'(C), W'(C')) \leqslant \sum_{n \geqslant N} w(a_n) < \varepsilon.$$

To show (2), we assume that $\varepsilon > 0$ is given, C and C' are Scott-closed sets such that $C \supseteq C'$ and $d_1(W'(C), W'(C')) < \delta$. Let N_0 be the largest number such that $1/2^{N_0} \leqslant \varepsilon$. We will show that $d_S^c(C, C') \leqslant 1/N_0$ for sufficiently small value of δ. I.e. we need to verify that $\forall n \leqslant N_0. (C \Downarrow)[n] = (C' \Downarrow)[n]$. Let $w_0 = \min\{w(a_i) | a_i \in (C \Downarrow)[N_0]\}$.

Pick $\delta < w_0$. Then $d_1(W'(C), W'(C')) < \delta \Rightarrow \sum_{a_n \in C\Downarrow - C'\Downarrow} w(a_n) < w_0 \Rightarrow \forall a_i \in (C \Downarrow)[N_0]. a_i \notin C \Downarrow - C' \Downarrow \Rightarrow \forall a_i \in (C \Downarrow)[N_0]. a_i \in C' \Rightarrow (C \Downarrow)[N_0] = C' \Downarrow)[N_0] \Rightarrow d_S^c(C, C') \leqslant 1/2^{N_0}$.

So W' is a Q-join co-valuation on the quasi-uniform lattice $(\mathscr{C}(P), \mathscr{U}_{D_S^c})$.

Hence by (the proof of) Theorem 13, we obtain that $\mathscr{U}_{D_S^c}$ is induced by the weighted quasi-metric $D_{W'}$ defined by

$$D_{W'}(C, C') = W'(C') - W'(C \cup C').$$

We remark that

$$D_{W'}(C, C') = \sum_{a_n \in C\Downarrow - C'\Downarrow} w(a_n).$$

Since $\mathscr{U}_{D_S^c} = \mathscr{U}_{D_{W'}}$, we obtain that the restrictions of these quasi-uniformities to the product $P\downarrow \times P\downarrow$ coincide.

We now consider the weighted quasi-metric $d_{W'}$, defined on P by $d_{W'}(x, y) = D_{W'}(x\downarrow, y\downarrow)$.

We recall that the Smyth quasi-metric d_S coincides with the distance d_S^c, i.e. the restriction of the quasi-metric D_S^c to the product $P\downarrow \times P\downarrow$.

Hence we obtain a (trivial) quasi-unimorphism $i_1 : (X, \mathscr{U}_{d_S}) \Rightarrow (X\downarrow, \mathscr{U}_{d_S^c})$, which maps x to $x\downarrow$. Similarly, we obtain a quasi-unimorphism $i_2 : (X, \mathscr{U}_{d_{w'}}) \Rightarrow (X\downarrow, \mathscr{U}_{d_{W'}})$, where

$$d_{w'}(x, y) = \sum_{a_n \in x\Downarrow - y\Downarrow} w(a_n).$$

Since the restrictions of $\mathscr{U}_{D_S^c}$ and $\mathscr{U}_{D_{W'}}$ to the product $P\downarrow \times P\downarrow$ coincide, we obtain, via the quasi-unimorphisms i_1 and i_2, that $\mathscr{U}_{d_S}$ and $\mathscr{U}_{d_{w'}}$ coincide. Since the Smyth quasi-metric d_S generates the Scott topology on the domain P, we obtain in particular that $\mathscr{U}_{d_{w'}}$ induces the Scott topology and that its associated order is the domain order. So it suffices to chose $d^w = d_{w'}$.

Finally, we show that ω-algebraic dcpos $(P, \sqsubseteq)$ can be quantified as a weighted quasi-metric space (P, d), where $d(x, y) = \sum_{a_n \in x\downarrow - y\downarrow} w(a_n)$ and where the associated metric d^* induces the Lawson topology.

For the case of ω-algebraic dcpos $(P, \sqsubseteq)$, one can easily verify that the above proof simplifies since one does not need to refer to the way below relation. In particular, one can define W on $\mathscr{C}(P)$ as follows: $W(C) = \sum_{a_n \in C} w(a_n)$. It is easy to verify that this function is a strictly increasing valuation. Once can then show that W' is a Q-co-valuation and apply Theorem 17 (rather than Theorem 13) in order to obtain a bi-weightable invariant quasi-metric. Finally, via a restriction to $P\downarrow \times P\downarrow$, one obtains that $(P,\sqsubseteq)$ is quantifiable via the weighted quasi-metric d^w, where $d^w(x,y) = \sum_{a_n \in x\downarrow - y\downarrow} w(a_n)$.

The proof that d^* induces the Lawson topology is a straightforward generalization of the one given in [29], so we only provide a sketch. First, remark that for quasi-metric spaces (X,d) in general $\mathscr{T}_d^k \subseteq \mathscr{T}_{d^{-1}}$, where $\mathscr{T}_d^k$ is the co-compact topology. Hence the Lawson topology is included in the associated metric topology. So it suffices to verify for the quantified domain (P,d_w) that $B_\varepsilon^*[x] = \{y \mid d^*(x,y) < \varepsilon\}$ is Lawson-open for every $\varepsilon > 0$ and for every $x \in P$. Pick N such that $1/2^N = \sum_{n \geqslant N+1} 1/2^n < \varepsilon$ and let $\delta = 1/2^N$. Consider the set $A = \{a_n \mid a_n \not\sqsubseteq x, 1 \leqslant n \leqslant N\}$ and let $B_\delta[x] = \{y \mid d_w(x,y) < \delta\}$ and $O = B_\delta[x] \cap (\bigcap_{a_n \in A}(P - a_n\uparrow))$. Then O is Lawson-open since $B_\delta[x]$ is Scott-open. We show $O \subseteq B_\varepsilon^*[x]$. Consider $y \in O$. Then, by definition of O, $d_w(x,y) < \delta < \varepsilon$. Finally, note that $d_w(y,x) = \sum_{a_n \in y\downarrow - x\downarrow} w(a_n)$. If $a_{n \in y\downarrow - x\downarrow}$ then $a_n \sqsubseteq y$ but $a_n \not\sqsubseteq x$ and thus $n > N$. Hence $d_w(y,x) \leqslant \sum_{n \geqslant N+1} 1/2^n < \varepsilon$. □

The case of ω-continuous domains which are not ω-algebraic. We remark that the quantifiability for general domains cannot be obtained via an argument on the Smyth-quasi-metric d_S defined via the partial order domain relation $\sqsubseteq$, i.e. $d_S(x,y) = \inf\{1/2^n \mid \forall i \leqslant n.\, a_i \sqsubseteq x \Rightarrow a_i \sqsubseteq y\}$, nor from the weighted quasi-metric $d^w(x,y) = \sum_{a_n \in x\downarrow - y\downarrow} w(a_n)$, since neither distance induces the Scott topology in general. Consider for instance the ω-continuous lattice $[0,1]$ with as basis the dyadic rationals in [0,1). Let the basis be enumerated by: $a_1 = 0, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, a_4 = \frac{3}{4}, a_5 = \frac{1}{8}, a_6 = \frac{3}{8}, \ldots$. Then the set $\{y \mid d_S(\frac{1}{2}, y) < \frac{1}{2}\} = [\frac{1}{2}, 1] = \{y \mid d^w(\frac{1}{2}, y) < \frac{1}{4}\}$ is d_S-open and d^w-open, but not Scott open. Hence the argument for the quantifiability of ω-continuous domains reported in [29], based on $d^w(x,y) = \sum_{a_n \in x\downarrow - y\downarrow} w(a_n)$, only holds for the ω-algebraic case.

Definition 28. Given a domain $(P,\sqsubseteq)$ with countable base B. A function $w : B \to \mathscr{R}^+$ which has finite sum over B is called a basic valuation. The valuation

$$W(C) = \sum_{a_n \in C} w(a_n)$$

is called the valuation generated by w and the weighted quasi-metric

$$d^w(x,y) = \sum_{a_n \in x\Downarrow - y\Downarrow} w(a_n)$$

is called the quasi-metric generated by w. The corresponding partial metric

$$p^w(x,y) = \sum_{a_n \notin x\Downarrow \cap y\Downarrow} w(a_n)$$

is called the partial metric generated by w.

For any domain $(P,\sqsubseteq)$, we refer to the partial metric space (P, p^w) as a quantification of the domain.

We obtain the following immediate corollary of Proposition 27.

Corollary 29. *Quantifications* (P, p^{w_1}) *and* (P, p^{w_2}) *of a domain* $(P, \sqsubseteq)$ *are equivalent, i.e. the quasi-metrics* d^{w_1} *and* d^{w_2} *generate the same quasi-uniformity.*

Corollary 30. *For* ω*-algebraic dcpos P the following holds: P is* 2/3 *SFP iff any of its quantifications* (P, p^w) *is compact.*

Proof. We remark that for ω-algebraic dcpo's the associated metric induces the Lawson topology and 2/3 SFP ω-algebraic dcpo's have a compact Lawson topology. Hence the result follows. □

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