Convergence and Stability of the Lax–Friedrichs Scheme for a Nonlinear Parabolic Polymer Flooding Problem

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We prove convergence and stability of the Lax–Friedrichs scheme for a nonlinear parabolic system of partial differential equations. The system models a polymer flooding process in enhanced oil recovery. The properties of the approximate solutions are used to obtain existence, uniqueness, and stability results for the solution of the system. We illustrate by a numerical example that the solution of the parabolic system converges towards the solution of the corresponding hyperbolic system as the dispersion coefficient tends to zero.

1. INTRODUCTION

The purpose of this paper is to study the convergence and stability of the Lax–Friedrichs scheme applied to the pure initial value problem of the following $2 \times 2$ system of nonlinear partial differential equations,

\begin{align}
s_t + f(s, c)_x &= \varepsilon s_{xx}, \\
(sc + a(c))_t + (cf(s, c))_x &= \varepsilon (sc + a(c))_{xx},
\end{align}

where $\varepsilon > 0$ is a constant. Here $(s, c)$ is the unknown state vector and $f = f(s, c)$ and $a = a(c)$ are given functions. We will give the precise assumptions on the model in Section 2. Our main results are that the Lax–Friedrichs scheme converges to a unique classical solution of the system (1.1) for all relevant initial data having bounded total variation, and that the scheme is stable with respect to perturbations both in the initial data and in the function $f$. By using the convergence and stability of the approximate solutions, we obtain stability results for the system (1.1).

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When \( \varepsilon = 0 \), (1.1) degenerates to the system

\[
\begin{align*}
  s_t + f(s, c)_x &= 0 \\
  (sc + a(c))_t + (cf(s, c))_x &= 0
\end{align*}
\]  

(1.2)

of hyperbolic conservation laws. This system is a well-known model of a polymer flooding process in enhanced oil recovery; cf. Johansen and Winther [6] and references therein.

Consider a fluid in a one-dimensional homogenous porous medium consisting of two immiscible phases, an aqueous phase and an oleic phase. The water is used in order to displace the oil in the medium. The effect of this process can be increased by adding some polymer to the water and thereby increase the viscosity of the aqueous phase. We assume that all the polymer remains in the aqueous phase, and that the polymer is totally miscible in water. Let \( s = s(x, t) \) denote the saturation of the aqueous phase (\( 1 - s \) denotes the saturation of the oleic phase), and let \( c = c(x, t) \) denote the concentration of the polymer in the aqueous phase. Then the system (1.2) models the displacement process when all dispersive effects are neglected. The function \( a = a(c) \) models the adsorption of the polymer on the rock. The function \( f \) is called the fractional flow function, and it is determined by the relative permeabilities and the viscosities of the aqueous phase and the oleic phase and by the influence of the gravitation.

The polymer model (1.2) has been studied in a series of papers. The Riemann problem of the system with \( a = 0 \) was solved by Isaacson [3] and by Keyfitz and Kranzer [8]. The Cauchy problem of (1.2) with \( a = 0 \) was studied by Temple [12], who established the existence of a weak solution of the system by using Glimm’s method [2]. In the work of Johansen and Winther [6], the Riemann problem of (1.2) with the adsorption term was solved. In the present paper we study the system (1.1), where some dispersive effects in the process are taken into account. The physical dispersion is approximated by assuming that the dispersion matrix is \( \varepsilon > 0 \) times identity.

In the next section we give the details of the mathematical model that we want to study, and in Section 3 we define a finite difference approximation of the model. Sections 4, 5, and 6 are devoted to the analysis of the finite difference scheme. In Section 4 we prove that the scheme converges to a unique classical solution of (1.1), provided that the initial data has bounded total variation. In Section 5 we state an error estimate for the approximate solutions, and in Section 6 we prove stability of the approximate solutions with respect to perturbations in the initial data and in the fractional flow function \( f \). We also prove the corresponding stability of the solutions of (1.1) by applying the discrete stability and the convergence of the approximate solutions.
We remark that most of the bounds obtained in this paper include a term which tends to infinity as $T/\varepsilon$ tends to infinity; here $T$ denotes time. Hence, we cannot pass to the limit in $\varepsilon$ in order to obtain results for the hyperbolic problem. But having established the existence of a smooth solution of (1.1) for all $\varepsilon > 0$, we might, by using some other technique, obtain results independent of $\varepsilon$ for the system (1.1) and then pass to the limit. However, this is not likely to be possible without some restrictions on the initial data, since the total variation of the solution of the system (1.2) can increase from being small to being infinite in a very short time, cf. Temple [12].

Despite the fact that we are unable to let $\varepsilon$ tend to zero in our estimates, we can, of course, try to do it numerically. In Section 7 we present a numerical example which indicates that the solution of (1.1) with Riemann initial data converges to the solution of the corresponding Riemann problem of (1.2) as $\varepsilon$ tends to zero.

In the final section of the paper we mention some generalizations of the results obtained in the previous sections.

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2. The Mathematical Model

In this section we give a precise description of the mathematical model we want to study. Let $s = s(x, t)$ be the water saturation and let $c = c(x, t)$ be the concentration of polymer in the aqueous phase. Defining $b = sc + a(c)$, our mathematical model takes the form

$$
\begin{align*}
s_t + f(s, c)_x &= \varepsilon s_{xx} \\
b_t + (cf(s, c))_x &= \varepsilon b_{xx}
\end{align*}
$$

for $x \in \mathbb{R}$, $t > 0$, and $\varepsilon > 0$. We assume throughout the paper that $\varepsilon > 0$ is fixed. The initial conditions are supposed to be such that

$$
0 \leq s^0(x), c^0(x) \leq 1 \quad \forall x \in \mathbb{R} \quad (2.2)
$$

$$
s^0, c^0 \in BV, \quad (2.3)
$$

where $BV = BV(\mathbb{R})$ denotes the class of functions having bounded total variation.
The fractional flow function $f$ is assumed to be a smooth function satisfying the following properties:

(i) $f(0, c) = 0, f(1, c) = 1 \forall c \in [0, 1]$ 

(ii) $\exists K_1 < \infty$ such that

$$|f(s_1, c_1) - f(s_2, c_2)| \leq K_1(|s_1 - s_2| + |c_1 - c_2|)$$

$\forall s_1, s_2, c_1, c_2 \in [0, 1]$. 

The function $a = a(c)$ models adsorption of the polymer on the rock. We assume that the adsorption function is a smooth function on $[0, 1]$ satisfying

(i) $a(0) = 0$

(ii) $H_0 \leq \frac{da(c)}{dc} \defeq h(c) \leq H_1,$ 

where $H_0, H_1 > 0$ are finite constants. From the assumptions on $a$ it follows that there is a finite constant $B$ such that $b = sc + a(c) \leq B \forall s, c \in [0, 1]$, and that the total variation of $b$ is bounded by the total variation of $s$ and $c$. Consequently, the initial function $b^0(x) = s^0(x)c^0(x) + a(c^0(x))$ satisfies

$$0 \leq b^0(x) \leq B \quad \forall x \in \mathbb{R}$$

$$b^0 \in BV.$$ 

### 3. The Finite Difference Scheme

In this section we define a finite difference approximation to the system (2.1). Let $\Delta x$ and $\Delta t$ be the mesh sizes in space and time, respectively, and define $x_n = n \Delta x$ $\forall n \in \mathbb{Z}$ and $t_k = k \Delta t$ $\forall k \in \mathbb{Z}_+$. Let $s_n^k, b_n^k, c_n^k$ denote approximations to $s(x_n, t_k), b(x_n, t_k), c(x_n, t_k)$, respectively, and let $f_n^k = f(s_n^k, c_n^k)$. A standard explicit and space centered discretization of (2.1) then reads

$$\frac{s_n^k - s_n^{k-1}}{\Delta t} + \frac{f_{n+1}^{k-1} - f_{n-1}^{k-1}}{2 \Delta x} = \varepsilon \frac{s_{n+1}^{k-1} - 2s_n^{k-1} + s_{n-1}^{k-1}}{(\Delta x)^2}$$

$$\frac{b_n^k - b_n^{k-1}}{\Delta t} + \frac{c_{n+1}^{k-1}f_{n+1}^{k-1} - c_{n-1}^{k-1}f_{n-1}^{k-1}}{2 \Delta x} = \varepsilon \frac{b_{n+1}^{k-1} - 2b_n^{k-1} + b_{n-1}^{k-1}}{(\Delta x)^2}$$

$c_n^k$ solves the equation $s_n^kc_n^k + a(c_n^k) = b_n^k$. 

$\varepsilon$
The initial conditions are inserted by evaluating the initial functions in the mesh points, \( s_n^0 = s^0(n \Delta x), \ c_n^0 = c^0(n \Delta x), \ b_n^0 = b^0(n \Delta x) \).

Let the mesh parameters be such that

\[
(\Delta x)^2/2 \Delta t = \varepsilon, \tag{3.4}
\]

and recall that \( \varepsilon > 0 \) is fixed. Define

\[
\tau = \Delta t/\Delta x,
\]

which by (3.4) equals \( \Delta x/2\varepsilon \). Then, by the conditions on \( f \) and \( a \), the CFL inequality

\[
\tau \max \left( \left| \frac{\partial f}{\partial s}(s, c) \right|, \left| \frac{f(s, c)}{s + H_0} \right| \right) \leq 1 \quad \forall s, c \in [0, 1] \tag{3.5}
\]

is satisfied for a sufficiently small \( \Delta x \). We assume throughout the paper that \( \Delta x \) is chosen small enough to satisfy (3.5).

Since \( \Delta x, \Delta t \) satisfies (3.4), the finite difference scheme (3.1), (3.2) can be written in the form

\[
s_n^k = \frac{1}{2} (s_{n+1}^{k-1} + s_{n-1}^{k-1}) - \frac{1}{2}\tau (f_{n+1}^{k-1} - f_{n-1}^{k-1}) \tag{3.6}
\]

\[
b_n^k = \frac{1}{2} (b_{n+1}^{k-1} + b_{n-1}^{k-1}) - \frac{1}{2}\tau (c_{n+1}^{k-1}f_{n+1}^{k-1} - c_{n-1}^{k-1}f_{n-1}^{k-1}), \tag{3.7}
\]

which is the Lax–Friedrichs scheme.

Throughout the paper we denote by \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \) the usual \( L^1 \)- and \( L^\infty \)-norms. Observe that for some mesh function \( w \) the \( L^1 \)-norm is given by

\[
\|w\|_1 = \sum_{n \in \mathbb{Z}} |w_n| \Delta x,
\]

and the \( L^\infty \)-norm is given by

\[
\|w\|_\infty = \sup_{n \in \mathbb{Z}} |w_n|.
\]

4. Convergence of the Finite Difference Scheme

In this section we prove convergence of the finite difference solutions defined by (3.6) and (3.7) to a unique classical solution \((s, b)\) of Eq. (2.1). We start by proving that the approximate solutions stay bounded for all \( x \) and \( t > 0 \).

**Lemma 1.** Assume that \( 0 \leq s_n^0, c_n^0 \leq 1 \ \forall n \in \mathbb{Z} \). Then there exist numbers \( \beta_n^k \in [0, 1] \ \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}_+ \) such that

\[
c_n^k = \beta_n^k c_{n+1}^{k-1} + (1 - \beta_n^k) c_{n-1}^{k-1}. \tag{4.1}
\]
Furthermore, 

\[ s_n^k, c_n^k \in [0, 1] \quad \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}_+. \]  
\hspace{1cm} \text{(4.2)}

and 

\[ b_n^k \in [0, B] \quad \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}_+. \]  
\hspace{1cm} \text{(4.3)}

**Proof.** Recall that \( b_n^k = s_n^k c_n^k + a(c_n^k) \), and rewrite (3.7) in the form 

\[ s_n^k c_n^k + \frac{1}{2} (a(c_n^k) - a(c_n^{k-1})) + \frac{1}{2} (a(c_n^k) - a(c_n^{k-1})) = g_{n,k} + \tau f_{n+1} - \tau f_{n+1}^{k-1} c_{n+1}^{k-1}. \]  
\hspace{1cm} \text{(4.4)}

By the mean value theorem there exist bounded values \( \tilde{h}_n^{k-1} \) and \( \tilde{h}_n^{k-1} \) such that 

\[ a(c_n^k) - a(c_n^{k-1}) = \tilde{h}_n^{k-1}(c_n^k + c_n^{k-1}) \]  
\hspace{1cm} \text{(4.5)}

\[ a(c_n^k) - a(c_n^{k-1}) = \tilde{h}_n^{k-1}(c_n^k - c_n^{k-1}). \]  
\hspace{1cm} \text{(4.6)}

Since we have not yet shown that \( c_n^k \in [0, 1] \), the equalities (4.5) and (4.6) may be accomplished by extending the function \( a \) to be a smooth function satisfying (2.5) for all real arguments.

By using (4.4), (4.5), and (4.6), we obtain 

\[ (s_n^k + \frac{1}{2} \tilde{h}_n^{k-1} + \frac{1}{2} \tilde{h}_n^{k-1}) c_n^k = \frac{1}{2} (s_n^{k-1} + \tilde{h}_n^{k-1} - \tau f_{n+1}^{k-1}) c_n^{k-1} + \frac{1}{2} (s_n^{k-1} + \tilde{h}_n^{k-1} + \tau f_{n+1}^{k-1}) c_n^{k-1}. \]  
\hspace{1cm} \text{(4.7)}

Define 

\[ \beta_n^{k-1} = \frac{1}{2} \frac{s_n^{k-1} + \tilde{h}_n^{k-1} - \tau f_{n+1}^{k-1}}{s_n^{k-1} + \frac{1}{2} \tilde{h}_n^{k-1} + \frac{1}{2} \tilde{h}_n^{k-1}}, \]  
\hspace{1cm} \text{(4.8)}

then, by using (3.6), we obtain that 

\[ 1 - \beta_n^{k-1} = \frac{1}{2} \frac{s_n^{k-1} + \tilde{h}_n^{k-1} + \tau f_{n+1}^{k-1}}{s_n^{k-1} + \frac{1}{2} \tilde{h}_n^{k-1} + \frac{1}{2} \tilde{h}_n^{k-1}}, \]  
\hspace{1cm} \text{(4.9)}

and consequently 

\[ c_n^k = \beta_n^{k-1} c_n^{k-1} + (1 - \beta_n^{k-1}) c_n^{k-1}. \]  
\hspace{1cm} \text{(4.10)}

The CFL condition (3.5) implies that 

\[ s_n^{k-1} + \tilde{h}_n^{k-1} - \tau f_{n+1}^{k-1} \geq 0 \]  
\hspace{1cm} \text{(4.11)}

and that 

\[ s_n^{k-1} + \tilde{h}_n^{k-1} + \tau f_{n+1}^{k-1} \geq 0. \]  
\hspace{1cm} \text{(4.12)}
From (4.8), (4.9), (4.11), and (4.12) it follows that

$$0 \leq \beta_n^{k-1} \leq 1,$$

which completes the proof of (4.1).

Define the function

$$S(s_{-1}, s_1, c_{-1}, c_1) = \frac{1}{2}(s_1 + s_{-1}) - \frac{1}{2}\tau(f(s_1, c_1) - f(s_{-1}, c_{-1})),$$

for $$s_{-1}, s_1, c_{-1}, c_1 \in [0, 1]$$, and observe that

$$\frac{\partial S}{\partial s_{-1}} = \frac{1}{2} + \frac{1}{2} \frac{\partial f}{\partial s}(s_{-1}, c_{-1}) \geq 0$$

by the CFL condition (3.5). Similarly we obtain that

$$\frac{\partial S}{\partial s_1} = \frac{1}{2} - \frac{1}{2} \frac{\partial f}{\partial s}(s_1, c_1) \geq 0.$$ 

Consequently,

$$S(s_{-1}, s_1, c_{-1}, c_1) \leq S(1, 1, c_{-1}, c_1) = 1$$

and

$$S(s_{-1}, s_1, c_{-1}, c_1) \geq S(0, 0, c_{-1}, c_1) = 0.$$ 

Since $$s^k_n = S(s^{k-1}_{n-1}, s^{k-1}_{n+1}, c^{k-1}_{n-1}, c^{k-1}_{n+1})$$, we have established that

$$s^{k-1}_{n-1}, s^{k-1}_{n+1}, c^{k-1}_{n-1}, c^{k-1}_{n+1} \in [0, 1] \Rightarrow s^k_n \in [0, 1]. \quad (4.13)$$

The proofs of (4.2) and (4.3) in the lemma are now completed by induction on (4.1) and (4.13). \(\square\)

We remark that the lemma above also is valid in the case \(\Delta x = O(\Delta t)\), i.e., for the hyperbolic problem.

Define the total variation of some mesh function \(w\) by

$$TV(w) = \sum_{n \in \mathbb{Z}} |w_{n+1} - w_{n-1}|.$$

Since the initial functions are in \(BV\), the total variations of \(s^0, c^0,\) and \(b^0\) are bounded for all mesh sizes \(\Delta x\). The next lemma shows that the total variations of the approximate solutions remain bounded for all finite time, independent of the mesh size. The total variation of \(c\) is also bounded independently of the parameter \(\varepsilon\), but the estimates for \(s\) and \(b\) go to infinity for finite time as \(\varepsilon\) goes to zero. Therefore the results obtained in
this paper cannot be used to pass to the limit in $\varepsilon$ and to obtain results for
the hyperbolic problem.

**Lemma 2.** Assume that

$$TV(s^0) \leq S_0, \quad TV(c^0) \leq C_0, \quad TV(b^0) \leq B_0$$

(4.14)

for finite constants $S_0, C_0, B_0$, and that $0 \leq s_n^0, c_n^0 \leq 1 \forall n \in \mathbb{Z}$; then

(i) $TV(c^k) \leq TV(c^0) \leq C_0 \quad \forall k \in \mathbb{Z}_+$.

Furthermore, for a given $T < \infty$ and $\varepsilon > 0$ there are finite constants $K_s$ and
$K_b$, independent of the mesh size, such that

(ii) $TV(s^k) \leq K_s, \quad k \Delta t \leq T$

(iii) $TV(b^k) \leq K_b, \quad k \Delta t \leq T$.

**Proof.** We start by noting that (iii) is easily obtained from (i), (ii), and
Lemma 1. So it remains to prove (i) and (ii), and we start by proving (i).

In Lemma 1 we proved that $c_n^k$ is a convex combination of $c_{n-1}^{k-1}$ and
$c_{n+1}^{k-1}$, cf. (4.1); consequently

$$|c_n^k - c_n^{k-1}| + |c_n^k - c_n^{k-1}| = |c_n^{k-1} - c_n^{k-1}|.$$  (4.15)

By applying (4.15) and the triangle inequality, we get

$$TV(c^k) = \sum_n |c_n^k - c_n^{k-2}| \leq \sum_n (|c_n^k - c_n^{k-1}| + |c_n^{k-1} - c_n^{k-2}|)
= \sum_n (|c_n^{k-1} - c_n^{k-1}| - |c_n^k - c_n^{k-1}| + |c_n^{k-1} - c_n^{k-1}|)
= TV(c^{k-1}).$$

Since $TV(c^k) \leq TV(c^{k-1})$, (i) follows by induction. We remark again that
this result is independent of the size of $\varepsilon$.

The rest of this proof is devoted to (ii). The total variation bound in $s$ is
proved by applying some results from the paper of Nishida and Smoller [10] and a discrete Gronwall inequality. Define

$$u_n^k = s_n^k - s_n^{k-2}$$
$$v_n^k = c_n^k - c_n^{k-2}$$
$$p_n^k = f_n^k - f_n^{k-2}.$$

Then, by (3.6), we obtain

$$u_n^k = \frac{1}{2}(u_{n+1}^{k-1} + u_{n-1}^{k-1}) - \frac{1}{2}\tau(p_{n+1}^{k-1} - p_{n-1}^{k-1}).$$  (4.16)
By applying Lemma 2.1 in [10], we have that

\[ u^k_n = \sum_{|m| \leq k} \alpha^k_m u_{n+m}^0 - \tau \sum_{l=1}^{k} \sum_{m=0}^{l} A^l_m (p_{n+m}^{k-l} - p_{n-m}^{k-l}), \quad (4.17) \]

where

\[ \alpha^l_m = \frac{1}{2^l} \binom{l}{\frac{1}{2}(l - m)} \quad (4.18) \]

and

\[ A^l_m = \frac{1}{2^l} \frac{m}{l} \binom{l}{\frac{1}{2}(l - m)}. \quad (4.19) \]

Here

\[ \binom{l}{\frac{1}{2}(l - m)} \]

denotes the binomial coefficient which by definition is zero whenever \( \frac{1}{2}(l - m) \) is not an integer between 0 and \( l \). The following properties of \( \alpha^l_m \) and \( A^l_m \) were proved in [10],

\[ \sum_{|m| \leq l} \alpha^l_m = 1 \quad (4.20) \]

\[ \sum_{m=0}^{l} A^l_m \leq \frac{K_2}{\sqrt{l}}, \quad (4.21) \]

where \( K_2 \) is independent of \( l \).

Using the representation (4.17), we get

\[
\begin{align*}
\sum_{n \in \mathbb{Z}} |u^k_n| & \leq \sum_{n \in \mathbb{Z}} \sum_{|m| \leq k} \alpha^k_m |u_{n+m}^0| \\
& \quad + \tau \sum_{n \in \mathbb{Z}} \sum_{l=1}^{k} \sum_{m=0}^{l} A^l_m |p_{n+m}^{k-l}| \\
& \quad + \tau \sum_{n \in \mathbb{Z}} \sum_{l=1}^{k} \sum_{m=0}^{l} A^l_m |p_{n-m}^{k-l}| \\
& = I + II + III. \quad (4.22)
\end{align*}
\]

The first term is easily bounded by applying (4.20),

\[ I = \sum_{n \in \mathbb{Z}} \sum_{|m| \leq k} \alpha^k_m |u_{n+m}^0| \leq \sum_{n \in \mathbb{Z}} |u^k_n| \leq S_0. \quad (4.23) \]
By applying the estimate (4.21) in the second term of (4.22), we obtain

$$ II = \tau \sum_{n \in \mathbb{Z}} \sum_{l=1}^{k} \sum_{m=0}^{l} A_{m}^{l} |p_{n+m}^{k-l}| $$

$$ = \tau \sum_{l=1}^{k} \sum_{m=0}^{l} A_{m}^{l} \sum_{n \in \mathbb{Z}} |p_{n+m}^{k-l}| $$

$$ = \tau \sum_{l=1}^{k} \sum_{n \in \mathbb{Z}} |p_{n}^{k-l}| \sum_{m=0}^{l} A_{m}^{l} $$

$$ \leq K_{2}\tau \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \sum_{n \in \mathbb{Z}} |p_{n}^{k-l}|. \quad (4.24) $$

The third term of (4.22) is treated in the same way as the second. Recall that by (3.4)

$$ \tau = \frac{\Delta t}{\Delta x} = \sqrt{\frac{\Delta t}{2\varepsilon}}. \quad (4.25) $$

By inserting the bounds for I, II, and III into (4.22), we obtain, using (4.25), that

$$ \sum_{n \in \mathbb{Z}} |\mu_{n}^{k}| \leq S_{0} + \sqrt{\frac{2}{\varepsilon}} K_{2}\sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \sum_{n \in \mathbb{Z}} |p_{n}^{k-l}|. \quad (4.26) $$

Here the term

$$ \sum_{n \in \mathbb{Z}} |p_{n}^{k-l}| = \sum_{n \in \mathbb{Z}} |f_{n}^{k-l} - f_{n-2}^{k-l}| $$

can be bounded by the total variation of $s^{k-l}$ and $c^{k-l}$ by using the Lipschitz continuity of $f$, cf. (2.4),

$$ \sum_{n \in \mathbb{Z}} |p_{n}^{k-l}| \leq K_{1}(TV(s^{k-l}) + TV(c^{k-l})) \leq K_{1}(TV(s^{k-l}) + C_{0}), \quad (4.27) $$

where we have used (i). From (4.26) and (4.27) we obtain

$$ TV(s^{k}) \leq S_{0} + \sqrt{\frac{2}{\varepsilon}} K_{1}K_{2}C_{0}\sqrt{\Delta t} \sum_{l=1}^{l} + \sqrt{\frac{2}{\varepsilon}} K_{1}K_{2} \sum_{l=1}^{k} \frac{\Delta t}{l} TV(s^{k-l}). \quad (4.28) $$

Using the simple inequality

$$ \sum_{l=1}^{k} \frac{1}{\sqrt{l}} < 2\sqrt{k}. \quad (4.29) $$
we obtain

\[ TV(s^k) \leq S_0 + 2\sqrt{\frac{2}{\varepsilon}} K_1 K_2 C_0 \sqrt{k \Delta t} + \sqrt{\frac{2}{\varepsilon}} K_1 K_2 \sum_{j=0}^{k-1} \sqrt{\frac{\Delta t}{k-j}} TV(s^j), \]

where we have changed the order of the summation in the last term. By applying a discrete Gronwall inequality, cf. Sugiyama [13], and (4.29) we finally obtain

\[ TV(s^k) \leq \left( S_0 + 2\sqrt{\frac{2}{\varepsilon}} K_1 K_2 C_0 \sqrt{T/\varepsilon} \right)^{\frac{k-1}{\Delta t}} \left( 1 + \frac{2}{\varepsilon} K_1 K_2 \sqrt{\frac{\Delta t}{k-j}} \right) \]

\[ \leq \left( S_0 + 2\sqrt{2} K_1 K_2 C_0 \sqrt{T/\varepsilon} \right) e^{2\sqrt{2} K_1 K_2 \sqrt{T/\varepsilon}} \]

which completes the proof (ii) in the lemma. \( \square \)

We next show that the approximate solutions generated by the scheme (3.6) and (3.7) are \( L^1 \)-continuous in time.

**Lemma 3.** Assume that the total variations of \( s^0, c^0, \) and \( b^0 \) are bounded as in Lemma 2 and that \( 0 \leq s_n, c_n \leq 1 \) \( \forall n \in \mathbb{Z} \). Then there is a finite constant \( K \) independent of \( \Delta x \) and \( \Delta t \) such that

\[ \|s^k - s^n\|_1 + \|b^k - b^n\|_1 \leq K \sqrt{(k-p) \Delta t} \] (4.31)

for \( 0 \leq p \leq k, k \Delta t \leq T < \infty. \)

**Proof.** We start by observing that iterating on the scheme (3.6) we obtain

\[ s_n^k = \sum_{|m| \leq l} \alpha_m^l s_{n+m}^{k-l} - \frac{1}{2} \tau \sum_{|m| \leq j-1} \alpha_m^{j-1} \left( f_{n+m+1} - f_{n+m} \right) \] (4.32)

for \( 1 \leq l \leq k, \) where

\[ \alpha_m^l = \frac{1}{2^l} \left( \frac{l}{\frac{1}{2}(l-m)} \right) \]
as in the proof of Lemma 2. Let \( p < k \); then

\[
\|s^k - s^p\|_1 = \sum_{n \in \mathbb{Z}} \left| \sum_{|m| \leq k-p} \alpha_m^{k-p} s^p_{n+m} \right| \Delta x
\]

\[
- \frac{1}{2} \tau \sum_{j=1}^{k-p} \sum_{|m| \leq j-1} \alpha_m^{j-1} (f_{n+j} - f_{n+m-1}) - s^p_n \left| \Delta x \right.
\]

\[
\leq \sum_{n \in \mathbb{Z}} \sum_{|m| \leq k-p} \alpha_m^{k-p} |s^p_{n+m} - s^p_n| \Delta x
\]

\[
+ \frac{1}{2} \tau \sum_{j=1}^{k-p} \sum_{n \in \mathbb{Z}} \sum_{|m| \leq j-1} \alpha_m^{j-1} |f_{n+m+1} - f_{n+m-1}| \Delta x
\]

\[
= I + II,
\]

(4.33)

where we have used the fact that \( \sum_{|m| \leq k-p} \alpha_m^{k-p} = 1 \), cf. (4.20).

Let \( l = k - p \) be even and consider the first term of (4.33),

\[
I = \sum_{n \in \mathbb{Z}} \sum_{|m| \leq l} \alpha_m^l |s^p_{n+m} - s^p_n| \Delta x
\]

\[
= \sum_{m=-l}^{-1} \alpha_m^l \sum_{n \in \mathbb{Z}} |s^p_{n+m} - s^p_n| \Delta x + \sum_{m=1}^{l} \alpha_m^l \sum_{n \in \mathbb{Z}} |s^p_{n+m} - s^p_n| \Delta x
\]

\[
= \sum_{m=1}^{l} \alpha_m^l \sum_{n \in \mathbb{Z}} |s^p_{n+m} - s^p_n| \Delta x + \sum_{m=1}^{l} \alpha_m^l \sum_{n \in \mathbb{Z}} |s^p_{n+m} - s^p_n| \Delta x
\]

\[
= 2 \sum_{m=1}^{l} \alpha_m^l |s^p_{n+m} - s^p_n| \Delta x,
\]

(4.34)

where we have used the fact that \( \alpha_m^l = \alpha_{-m}^l \). Since \( l \) is even, we have that \( \alpha_m^l \neq 0 \) only for \( m \) even. For \( m \) even we have that

\[
s^p_{n+m} - s^p_n = \sum_{j=1}^{m/2} (s^p_{n+2j} - s^p_{n+2j-2}),
\]

and then by (4.34) we obtain

\[
I \leq 2 \sum_{n \in \mathbb{Z}} \sum_{m=1}^{l} \alpha_m^l \sum_{j=1}^{m/2} |s^p_{n+2j} - s^p_{n+2j-2}| \Delta x
\]

\[
= 2 \sum_{m=1}^{l} \alpha_m^l \sum_{j=1}^{m/2} TV(s^p) \Delta x = TV(s^p) \Delta x \sum_{m=1}^{l} m\alpha_m^l.
\]

(4.35)
Observe that \( m\alpha_m^l = lA_m^l \), cf. (4.18) and (4.19), such that by using the estimate (4.21), we obtain

\[
I \leq TV(s^p) \Delta x \sum_{m=1}^{l} m\alpha_m^l = TV(s^p) l \Delta x \sum_{m=1}^{l} A_m^l
\]

\[
\leq K_2 TV(s^p) \sqrt{2\varepsilon} \Delta t = K_2 TV(s^p) \sqrt{2\varepsilon (k - p)} \Delta t,
\]

where we have used the relation \( \Delta x = \sqrt{2\varepsilon} \Delta t \). Recall that \( K_2 \) is a finite constant; cf. (4.21). A bound similar to (4.36) can be obtained when \( k - p \) is odd.

It remains to estimate the second term of (4.33). By using the Lipschitz continuity of \( f \), we get

\[
\| s^k - s^p \|_1 \leq K \sqrt{(k - p) \Delta t}
\]

\[
\| b^k - b^p \|_1 \leq K \sqrt{(k - p) \Delta t}
\]

for \( p < k \).

Following Oleinik [11], we can now prove the existence and uniqueness of a smooth solution to (2.1). Let \( s^0(x) \) and \( c^0(x) \) be initial functions for the system (2.1) such that

\[
0 \leq s^0(x), c^0(x) \leq 1 \quad \forall x \in \mathbb{R}
\]

\[
s^0, c^0 \in BV;
\]
then \( b^0(x) = s^0(x)c^0(x) + a(c^0(x)) \) satisfies
\[
0 \leq b^0(x) \leq B \quad \forall x \in \mathbb{R}
\]
\[
b^0 \in BV.
\]

**Theorem 1.** Assume that the initial functions \( s^0, c^0, b^0 \) satisfy (4.40), (4.41), (4.42), (4.43). Then the approximate solutions generated by the finite difference scheme (3.6) and (3.7) converge to a unique solution \((s, c, b)\) of (2.1) as \( \Delta x = \sqrt{2\varepsilon \Delta t} \) goes to zero. Moreover, the solution has the following properties for \( 0 < t \leq T < \infty \):

1. \( 0 \leq s(x, t), c(x, t) \leq 1, 0 \leq b(x, t) \leq B, \forall x \).
2. \( TV(c(\cdot, t)) \leq TV(c^0), TV(s(\cdot, t)), TV(b(\cdot, t)) < \infty \).
3. \( s, c, \) and \( b \) are twice continuously differentiable in \( x \) and once in \( t \).
4. The initial data is assumed in the sense that
\[
\int_{-\infty}^{\infty} \left( \phi(x, t)s(x, t) - \phi(x, 0)s^0(x) \right) dx \to 0 \quad \text{as } t \to 0
\]
for any continuous function \( \phi \) with compact support in \( x \). If \( s^0(x) \) is continuous for \( x = \bar{x} \), then \( \lim_{t \to 0^+} x \to \bar{x} s(x, t) = s^0(\bar{x}) \). The initial data is assumed in the same way for \( b \).

**Proof.** The theorem is proved by a technique introduced by Oleinik [11]. After Lemmas 1, 2, and 3 are established, it follows from Theorem 3 in [11] that there is a sequence of mesh parameters \( \Delta x_i = \sqrt{2\varepsilon \Delta t_i} \) such that the family of approximate solutions converges boundedly a.e. to a pair of measurable functions \((s, b)\) as \( \Delta x_i \to 0 \). By Lemma 7 in [11], the limit \((s, b)\) is a weak solution of (2.1) in the sense that
\[
\int_{-\infty}^{\infty} \int_0^\infty (s\phi_t + f\phi_x + \varepsilon s\phi_{xx}) \, dx \, dt + \int_{-\infty}^\infty \phi(x, 0)s^0(x) \, dx = 0
\]
\[
\int_{-\infty}^{\infty} \int_0^\infty (b\phi_t + c\phi_x + \varepsilon b\phi_{xx}) \, dx \, dt + \int_{-\infty}^\infty \phi(x, 0)b^0(x) \, dx = 0
\]
for any test function \( \phi \) with compact support in \( 0 \leq t \leq T < \infty \).

By Theorem 6 in [11], the solution is twice continuously differentiable in \( x \) and once in \( t \), and the initial data is assumed as indicated in (4). The uniqueness follows from Theorem 7 in [11]. Since the solution is unique, the entire family of approximate solutions converges to \((s, b)\) as mesh the size goes to zero. The properties (1) and (2) of the solution follow from their discrete analogs in Lemmas 1 and 2. \( \Box \)
5. An Error Estimate

We are now in the position to apply an error estimate for the finite difference scheme (3.6), (3.7) proved by Hoff and Smoller [1]. They considered the parabolic problem

\[ v_t + g(v)_x = \varepsilon v_{xx} \]
\[ v(x, 0) = v^0(x). \]  

(5.1)

Here \( v \in \mathbb{R}^n \), \( g \in C^2 \), and \( \varepsilon > 0 \) is constant. In addition the initial total variation is assumed to be bounded, i.e., \( TV(v^0) = V < \infty \). Let \( v^k_n \ \forall n \in \mathbb{Z} \), \( \forall k \in \mathbb{Z}_+ \) be generated by the Lax–Friedrichs scheme for (5.1) with the initial conditions \( v^0_n = v^0(n \Delta x) \), and let \( v^k \) denote an approximation to the unique solution \( v \) of (5.2) defined by

\[ v^k(x, t) = \begin{cases} v^k_n & \text{for } (x, t) \in [n \Delta x, (n + 1) \Delta x) \times [k \Delta t, (k + 1) \Delta t). \end{cases} \]

Hoff and Smoller [1] prove that if there is a convex set in the \( v \)-space, where the approximate solutions are invariant and where \( g'(v) \) is bounded, then there is a finite constant \( K \) depending only on \( T, \varepsilon, \) and \( V \) such that

\[ \| v(\cdot, t) - v^k(\cdot, t) \|_{\infty} \leq \frac{K}{\sqrt{t}} \Delta x |\ln \Delta x| \]  

(5.2)

for \( \Delta x \) sufficiently small. Having assumed that the Lax–Friedrichs scheme is invariant in some convex set where \( g'(v) \) is bounded, Hoff and Smoller argue in [1] that the bounds on the sup norm and the total variation of the approximate solutions obtained by Nishida and Smoller [10] apply. Hence the existence and uniqueness are consequences of the convex invariant region for the Lax–Friedrichs scheme. We could therefore, by the same argument, state Theorem 1 after having proved the existence of the invariant region in Lemma 1. But since the work [10] deals only with the parabolic \( p \)-system, we have here, for the sake of completeness, given a proof of the total variation bound and the proof of the \( L^1 \)-continuity in time for our system.

By using the bounds of the approximate solutions established in the previous section, we can apply the error estimate (5.2) to our finite difference scheme. Let \((s, b)\) be the unique solution of (2.1) with \( \varepsilon > 0 \) fixed and with initial data satisfying (2.2), (2.3), (2.6), and (2.7), and let \((s^k_n, b^k_n)\) denote an extension of the finite difference solution generated by (3.6) and (3.7) to a function on \( \mathbb{R} \times [0, T] \) defined by

\[ (s^k_n, b^k_n)(x, t) = (s^k_n, b^k_n) \quad \text{for } (x, t) \in [n \Delta x, (n + 1) \Delta x) \times [k \Delta t, (k + 1) \Delta t). \]
Then it follows from (5.2) that there is a finite constant \( K \) depending only on \( T, \epsilon, TV(s^0), \) and \( TV(b^0) \) such that

\[
\| s(\cdot, t) - s_\Delta(\cdot, t) \|_\infty + \| b(\cdot, t) - b_\Delta(\cdot, t) \|_\infty \leq \frac{K}{\sqrt{t}} \Delta x |\ln \Delta x| \ 
\]

for \( 0 < t \leq T < \infty \).

6. Stability

In this section we consider the question of stability of the system (2.1). First we prove that the system is stable with respect to perturbations of the initial data in both \( L^1 \) and \( L^\infty \). Thereafter we prove stability with respect to perturbations in the fractional flow functions. Both results are shown by using the finite difference scheme (3.6), (3.7) to show their discrete analogs.

We start by proving that the finite difference scheme is stable in the sense that it depends continuously on the initial data in \( L^1 \) and \( L^\infty \). Let \((s^0, c^0)\) and \((\bar{s}^0, \bar{c}^0)\) be two initial functions satisfying the conditions

\[
0 \leq s_n^0, c_n^0, \bar{s}_n^0, \bar{c}_n^0 \leq 1 \quad \forall n \in \mathbb{Z} \\
TV(s^0), TV(c^0), TV(\bar{s}^0), TV(\bar{c}^0) < \infty, 
\]

where \( b = sc + a(c) \) as before. From these conditions it follows immediately that

\[
0 \leq b_n^0, \bar{b}_n^0 \leq B \quad \forall n \in \mathbb{Z} \\
TV(b^0), TV(\bar{b}^0) < \infty. 
\]

We generate two finite difference solutions \((s_n^k, c_n^k, b_n^k)\) and \((\bar{s}_n^k, \bar{c}_n^k, \bar{b}_n^k)\) with initial conditions \((s_n^0, c_n^0, b_n^0)\) and \((\bar{s}_n^0, \bar{c}_n^0, \bar{b}_n^0)\), respectively, by the schemes

\[
s_n^k = \frac{1}{2}(s_{n+1}^{k-1} + s_{n-1}^{k-1}) - \frac{1}{2}\tau(f_{n+1}^{k-1} - f_{n-1}^{k-1}) \\
b_n^k = \frac{1}{2}(b_{n+1}^{k-1} + b_{n-1}^{k-1}) - \frac{1}{2}\tau(c_{n+1}^{k-1}f_{n+1}^{k-1} - c_{n-1}^{k-1}f_{n-1}^{k-1}) 
\]

and

\[
\bar{s}_n^k = \frac{1}{2}(\bar{s}_{n+1}^{k-1} + \bar{s}_{n-1}^{k-1}) - \frac{1}{2}\tau(\bar{f}_{n+1}^{k-1} - \bar{f}_{n-1}^{k-1}) \\
\bar{b}_n^k = \frac{1}{2}(\bar{b}_{n+1}^{k-1} + \bar{b}_{n-1}^{k-1}) - \frac{1}{2}\tau(\bar{c}_{n+1}^{k-1}\bar{f}_{n+1}^{k-1} - \bar{c}_{n-1}^{k-1}\bar{f}_{n-1}^{k-1}), 
\]

where \( \bar{f}_n^k = f(\bar{s}_n^k, \bar{c}_n^k) \).
Then the following lemma assures the stability of the discrete process.

**Lemma 4.** Let \((s^k_n, c^k_n, b^k_n)\) and \((\tilde{s}^k_n, \tilde{c}^k_n, \tilde{b}^k_n)\) be finite difference solutions generated by (6.5), (6.6) and (6.7), (6.8), respectively, and with initial data satisfying (6.1) and (6.2). Then there is a finite constant \(K\) independent of \(\Delta x\) and \(\Delta t\) such that

\[
\|s^k - \tilde{s}^k\|_\infty + \|b^k - \tilde{b}^k\|_\infty \leq K(\|s^0 - \tilde{s}^0\|_\infty + \|b^0 - \tilde{b}^0\|_\infty) \tag{6.9}
\]

for \(k \Delta t \leq T < \infty\). If the initial data in addition satisfies

\[
\|s^0 - \tilde{s}^0\|_1 + \|b^0 - \tilde{b}^0\|_1 < \infty,
\]

then there is a finite constant \(K\) independent of \(\Delta x\) and \(\Delta t\) such that

\[
\|s^k - \tilde{s}^k\|_1 + \|b^k - \tilde{b}^k\|_1 \leq K(\|s^0 - \tilde{s}^0\|_1 + \|b^0 - \tilde{b}^0\|_1) \tag{6.10}
\]

for \(k \Delta t \leq T < \infty\).

**Proof.** Observe that by Lemma 1, the processes (6.5), (6.6) and (6.7), (6.8) are well defined in the sense that \(s^k_n, \tilde{s}^k_n, c^k_n, \tilde{c}^k_n \in [0, 1]\) and \(b^k_n, \tilde{b}^k_n \in [0, B] \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}_+\). Define

\[
u^k_n = s^k_n - \tilde{s}^k_n, \quad v^k_n = b^k_n - \tilde{b}^k_n
\]

Then, by subtracting (6.7) from (6.5), we obtain

\[
u^k_n = \frac{1}{2}(u^k_{n+1} + u^k_{n-1}) - \frac{1}{2}\tau(p^k_{n+1} - p^k_{n-1})\tag{6.11}
\]

and similarly

\[
u^k_n = \frac{1}{2}(v^k_{n+1} + v^k_{n-1}) - \frac{1}{2}\tau(q^k_{n+1} - q^k_{n-1}).\tag{6.12}
\]

Observe that (6.11) and (6.12) are of the form (4.16), and that we thereby have the representations

\[
u^k_n = \sum_{|m| \leq k} \alpha^k_m u^0_{n+m} - \tau \sum_{l=1}^k \sum_{m=0}^l A^l_m(p^k_{n+m} - p^k_{n-m}),\tag{6.13}
\]

\[
u^k_n = \sum_{|m| \leq k} \alpha^k_m v^0_{n+m} - \tau \sum_{l=1}^k \sum_{m=0}^l A^l_m(q^k_{n+m} - q^k_{n-m}),\tag{6.14}
\]

where \(\alpha^k_m\) and \(A^l_m\) are as defined in (4.18) and (4.19).
We start by proving (6.10). Using the representations (6.13) and (6.14), we obtain, in exactly the same way as in the proof of Lemma 2, that

\[ \|u^k\|_1 \leq \|u^0\|_1 + \sqrt{\frac{2}{\epsilon}} K_2 \sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \|p^{k-l}\|_1 \]

(6.15)

\[ \|v^k\|_1 \leq \|v^0\|_1 + \sqrt{\frac{2}{\epsilon}} K_2 \sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \|q^{k-l}\|_1, \]

(6.16)

where \( K_2 \) is a finite constant independent of \( \Delta x, \Delta t \), cf. (4.21).

Next, we want to express \( \|p^{k-l}\|_1 \) and \( \|q^{k-l}\|_1 \) in terms of \( \|u^{k-l}\|_1 \) and \( \|v^{k-l}\|_1 \). Let \( s, \bar{s}, c, \bar{c} \in [0, 1] \) and let \( b = sc + c(\bar{c}) \) and \( \bar{b} = \bar{s}c + a(\bar{c}) \); then \( b - \bar{b} = (s - \bar{s})c + (c - \bar{c})\bar{s} + h(\bar{c})(c - \bar{c}) \) for some \( \bar{c} \) between \( c \) and \( \bar{c} \), and consequently

\[ |c - \bar{c}| \leq \frac{1}{\bar{s} + h(\bar{c})} \{|b - \bar{b}| + c|s - \bar{s}|\} \leq \frac{1}{H_0} \{|b - \bar{b}| + |s - \bar{s}|\}, \]

cf. (2.5). Also, by using the Lipschitz continuity of \( f \), there is a finite constant \( \bar{K} \) such that

\[ |p| = |f(s, c) - f(s, \bar{c})| \leq \bar{K}\{|b - \bar{b}| + |s - \bar{s}|\} \]

(6.17)

and

\[ |q| = |cf(s, c) - \bar{c}f(s, \bar{c})| \leq \bar{K}\{|b - \bar{b}| + |s - \bar{s}|\}. \]

(6.18)

By applying (6.17) and (6.18) in (6.15) and (6.16), respectively, we have that

\[ \|u^k\|_1 \leq \|u^0\|_1 + \sqrt{\frac{2}{\epsilon}} \bar{K} K_2 \sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} (\|u^{k-l}\|_1 + \|v^{k-l}\|_1) \]

\[ \|v^k\|_1 \leq \|v^0\|_1 + \sqrt{\frac{2}{\epsilon}} \bar{K} K_2 \sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} (\|u^{k-l}\|_1 + \|v^{k-l}\|_1). \]

By adding these equations, we obtain

\[ \|u^k\|_1 + \|v^k\|_1 \leq \|u^0\|_1 + \|v^0\|_1 + 2\sqrt{\frac{2}{\epsilon}} \bar{K} K_2 \sqrt{\Delta t} \sum_{j=0}^{k-1} \frac{1}{\sqrt{k-j}} (\|u^j\|_1 + \|v^j\|_1), \]

where the order of the summation is changed. Then, by using a discrete
Gronwall inequality, cf. [13], we obtain

\[ \|u^k\|_1 + \|v^k\|_1 \leq (\|u^0\|_1 + \|v^0\|_1)e^{\sqrt{2} \Delta t/\varepsilon}, \]

in the same way as in the proof of Lemma 2. This concludes the proof of (6.10).

To prove (6.9), we start again by the representations (6.13) and (6.14). Using the triangle inequality and the estimate (4.20), we obtain

\[
\|u^k\|_m \leq \|u^0\|_m + 2\sum_{l=1}^{k} \sum_{m=0}^{l} A_m^l (p_{n+m+j}^k - p_{n-m}^k) \leq \|u^0\|_m + 2\sum_{l=1}^{k} \sum_{m=0}^{l} A_m^l \|p^{k-l}\|_\infty \sum_{m=0}^{l} \|p^{k-l}\|_\infty. \tag{6.19}\]

Since (6.19) holds \( \forall n \in \mathbb{Z} \), we have that

\[ \|u^k\|_\infty \leq \|u^0\|_\infty + \sqrt{2/\varepsilon} K_2 \Delta t \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \|p^{k-l}\|_\infty, \tag{6.20} \]

and in the same way we obtain

\[ \|v^k\|_\infty \leq \|v^0\|_\infty + \sqrt{2/\varepsilon} K_2 \Delta t \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \|q^{k-l}\|_\infty. \tag{6.21} \]

By using (6.17), (6.18) in (6.18), (6.21) and adding the inequalities, we obtain

\[
\|u^k\|_\infty + \|v^k\|_\infty \leq \|u^0\|_\infty + \|v^0\|_\infty + 2\sqrt{2/\varepsilon} K_2 \Delta t \sum_{j=1}^{k-1} \frac{1}{\sqrt{k-j}} (\|u^j\|_\infty + \|v^j\|_\infty),
\]

and then the discrete Gronwall inequality implies

\[ \|u^k\|_\infty + \|v^k\|_\infty \leq \left(\|u^0\|_\infty + \|v^0\|_\infty\right)e^{\sqrt{2} \Delta t/\varepsilon}. \tag*{\Box} \]

On the basis of the stability and convergence of the approximate solutions, we can now prove the following theorem.

**Theorem 2.** Let \((s, b)\) and \((\bar{s}, \bar{b})\) be the unique solutions of (2.1) for \( \varepsilon > 0 \) with initial data \((s^0, b^0)\) and \((\bar{s}^0, \bar{b}^0)\), respectively, where both initial
conditions are assumed to satisfy (2.2) and (2.3). Then there is a finite constant $K$ depending only on $T$ and $\varepsilon$ such that

$$
\|s(\cdot, t) - \bar{s}(\cdot, t)\|_\infty + \|b(\cdot, t) - \bar{b}(\cdot, t)\|_\infty \\
\leq K\left(\|s^0 - \bar{s}^0\|_\infty + \|b^0 - \bar{b}^0\|_\infty\right)
$$

(6.22)

for $0 < t \leq T < \infty$.

If the initial data in addition satisfies

$$
\|s^0 - \bar{s}^0\|_1 + \|b^0 - \bar{b}^0\|_1 < \infty,
$$

then there is a finite constant $K$ depending only on $T$ and $\varepsilon$ such that

$$
\|s(\cdot, t) - \bar{s}(\cdot, t)\|_1 + \|b(\cdot, t) - \bar{b}(\cdot, t)\|_1 \leq K\left(\|s^0 - \bar{s}^0\|_1 + \|b^0 - \bar{b}^0\|_1\right)
$$

(6.23)

for $0 < t \leq T < \infty$.

**Proof.** Let $(s_\Delta, b_\Delta)$ and $(\bar{s}_\Delta, \bar{b}_\Delta)$ denote the finite difference approximations to $(s, b)$ and $(\bar{s}, \bar{b})$, respectively, for some mesh size $\Delta x = \sqrt{2\varepsilon} \Delta t$ satisfying the CFL condition (3.5). Then, for some finite time $t > 0$, we obtain, by using Lemma 4 and the error estimate (5.3), that

$$
\|s(\cdot, t) - \bar{s}(\cdot, t)\|_\infty + \|b(\cdot, t) - \bar{b}(\cdot, t)\|_\infty \\
\leq \|s(\cdot, t) - s_\Delta(\cdot, t)\|_\infty + \|s_\Delta(\cdot, t) - \bar{s}_\Delta(\cdot, t)\|_\infty \\
+ \|\bar{s}_\Delta(\cdot, t) - \bar{s}(\cdot, t)\|_\infty + \|b(\cdot, t) - b_\Delta(\cdot, t)\|_\infty \\
+ \|b_\Delta(\cdot, t) - \bar{b}_\Delta(\cdot, t)\|_\omega + \|\bar{b}_\Delta(\cdot, t) - \bar{b}(\cdot, t)\|_\omega \\
\leq K\left(\frac{\Delta x|\ln \Delta x|}{\sqrt{t}} + \|s^0 - \bar{s}^0\|_\infty + \|b^0 - \bar{b}^0\|_\infty\right)
$$

for some finite constant $K$ depending only on $T$ and $\varepsilon$. By letting $\Delta x \to 0$, the stability result (6.22) follows.

By using the $L^1$-convergence of the finite difference scheme to the exact solution, cf. Oleinik [11, Theorem 3], we can prove the $L^1$-stability (6.23) in the same way in which we proved the $L^\infty$-stability. \qed

Next we consider the stability of the system (2.1) with respect to perturbations in the fractional flow functions. The fractional flow function for a polymer model is usually assumed to be a function of the relative permeability and the viscosity of the aqueous and oleic phase; sometimes also gravitational effects are taken into account. Since the relative permeability functions are given by data from laboratory experiments, cf. [9], it is
quite important to prove stability of the system (2.1) with respect to perturbations in the fractional flow functions. Our technique is again to start by proving the result for the approximate solutions.

Let \( f \) and \( f' \) be two fractional flow functions satisfying the requirement (2.4) and let \( \Delta x \) be small enough to satisfy the CFL condition (3.5) for both \( f \) and \( f' \). We generate two finite difference solutions \( \{(s_n^k), (c_n^k), (b_n^k)\} \) and \( \{((\tilde{s}_n^k)), (\tilde{c}_n^k), (\tilde{b}_n^k)\} \) with coinciding initial conditions \( \{(s_0^0), (c_0^0), (b_0^0)\} \) by the schemes

\[
\begin{align*}
    s_{n+1}^k &= \frac{1}{2} (s_{n+1}^{k-1} + s_{n-1}^{k-1}) - \frac{1}{2} \tau(f_{n+1}^{k-1} - f_{n-1}^{k-1}) \quad (6.24) \\
    b_{n+1}^k &= \frac{1}{2} (b_{n+1}^{k-1} + b_{n-1}^{k-1}) - \frac{1}{2} \tau(c_{n+1}^{k-1}f_{n+1}^{k-1} - c_{n-1}^{k-1}f_{n-1}^{k-1}) \quad (6.25) \\
\end{align*}
\]

and

\[
\begin{align*}
    \tilde{s}_{n+1}^k &= \frac{1}{2} (\tilde{s}_{n+1}^{k-1} + \tilde{s}_{n-1}^{k-1}) - \frac{1}{2} \tau(\tilde{f}_{n+1}^{k-1} - \tilde{f}_{n-1}^{k-1}) \quad (6.26) \\
    \tilde{b}_{n+1}^k &= \frac{1}{2} (\tilde{b}_{n+1}^{k-1} + \tilde{b}_{n-1}^{k-1}) - \frac{1}{2} \tau(\tilde{c}_{n+1}^{k-1}\tilde{f}_{n+1}^{k-1} - \tilde{c}_{n-1}^{k-1}\tilde{f}_{n-1}^{k-1}) \quad (6.27)
\end{align*}
\]

where \( \tilde{f}_n^k = \tilde{f}(\tilde{s}_n^k, \tilde{c}_n^k) \). The initial conditions \( (s_0^0, c_0^0) \) are assumed to satisfy the requirements

\[
\begin{align*}
    0 \leq s_n^0, c_n^0 \leq 1 & \quad \forall n \in \mathbb{Z} \quad (6.28) \\
    TV(s_0^0), TV(c_0^0) < \infty & \quad (6.29)
\end{align*}
\]

Then the following lemma assures stability with respect to perturbations in the fractional flow function.

**Lemma 5.** Let \( \{(s_n^k), (c_n^k), (b_n^k)\} \) and \( \{((\tilde{s}_n^k)), (\tilde{c}_n^k), (\tilde{b}_n^k)\} \) be finite difference solutions generated by (6.24), (6.25) and (6.26), (6.27), respectively, and with coinciding initial data satisfying (6.28) and (6.29). Then there is a finite constant \( K \) independent of \( \Delta x \) and \( \Delta t \) such that

\[
\|s^k - \tilde{s}^k\|_\infty + \|b^k - \tilde{b}^k\|_\infty \leq K\|f - \tilde{f}\|_\infty \quad (6.30)
\]

for \( k \Delta t \leq T < \infty \).

**Proof.** Since \( \tilde{f} \) satisfy the same conditions as \( f \), Lemma 1 shows that \( s_n^k, s_n^k, c_n^k, \tilde{c}_n^k \in [0, 1] \) and \( b_n^k, \tilde{b}_n^k \in [0, B] \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}_+ \). Define

\[
\begin{align*}
    u_n^k &= s_n^k - \tilde{s}_n^k, & v_n^k &= b_n^k - \tilde{b}_n^k \\
    p_n^k &= f_n^k - \tilde{f}_n^k, & q_n^k &= (c_n^k f_n^k - \tilde{c}_n^k \tilde{f}_n^k).
\end{align*}
\]
Then, in the same way as in Lemma 4, we obtain the representations

\[ u^k_n = \sum_{|m| \leq k} c_m u^0_{n+m} - \tau \sum_{l=1}^k \sum_{m=0}^l A^l_m (p^k_{n+m} - p^k_{n-m}) \]

\[ v^k_n = \sum_{|m| \leq k} c_m v^0_{n+m} - \tau \sum_{l=1}^k \sum_{m=0}^l A^l_m (q^k_{n+m} - q^k_{n-m}) \]

where \( a^l_m \) and \( A^l_m \) are as defined in (4.18) and (4.19). Since \( s^0_n = \bar{s}^0_n \), \( c^0_n = \bar{c}^0_n \) \( \forall n \in \mathbb{Z} \), we have \( u^0_n = v^0_n = 0 \) \( \forall n \in \mathbb{Z} \), and consequently

\[ |u^k_n| \leq \tau \sum_{l=1}^k \sum_{m=0}^l A^l_m |p^k_{n+m} - p^k_{n-m}| \]

\[ \leq 2 \tau \sum_{l=1}^k \|p^k_l\|_\infty \sum_{m=0}^l A^l_m \]

\[ \leq \sqrt{2/\epsilon} K_2 \sqrt{\Delta t} \sum_{l=1}^k \frac{1}{\sqrt{l}} \|p^k_l\|_\infty, \]

where we have used the estimate (4.21). Since this holds \( \forall n \in \mathbb{Z} \), we get

\[ \|u^k\|_\infty \leq \sqrt{2/\epsilon} K_2 \sqrt{\Delta t} \sum_{l=1}^k \frac{1}{\sqrt{l}} \|p^k_l\|_\infty \] (6.31)

and similarly

\[ \|v^k\|_\infty \leq \sqrt{2/\epsilon} K_2 \sqrt{\Delta t} \sum_{l=1}^k \frac{1}{\sqrt{l}} \|q^k_l\|_\infty. \] (6.32)

Let \( s, \bar{s}, c, \bar{c} \in [0, 1] \) and let \( b = sc + a(c) \) and \( \bar{b} = \bar{s} \bar{c} + a(\bar{c}) \); then

\[ |p| = |f(s, c) - \hat{f}(\bar{s}, \bar{c})| \leq |f(s, c) - f(\bar{s}, \bar{c})| + |f(\bar{s}, \bar{c}) - \hat{f}(\bar{s}, \bar{c})|. \]

Here the first term on the right-hand side can be bounded by \(|s - \bar{s}| \) and \(|b - \bar{b}| \) in exactly the same way as in Lemma 4, hence we have the bound

\[ |p| \leq \bar{K} \{ |s - \bar{s}| + |b - \bar{b}| \} + \|f - \hat{f}\|_\infty, \] (6.33)

where

\[ \|f - \hat{f}\|_\infty = \sup_{s, c \in [0, 1]} |f(s, c) - \hat{f}(s, c)|. \]

In the same way we obtain the bound

\[ |q| \leq |cf(s, s) - \bar{c}\hat{f}(\bar{s}, \bar{c})| \leq \bar{K} \{ |s - \bar{s}| + |b - \bar{b}| \} + \|f - \hat{f}\|_\infty. \] (6.34)
By using (6.33) and (6.34) in (6.31) and (6.32) and adding the inequalities, we obtain

\[\|u^k\|_\infty + \|v^k\|_\infty \leq 2\sqrt{2/\varepsilon} K_2 \sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \left\{ \tilde{K} \left( \|u^{k-l}\|_\infty + \|v^{k-l}\|_\infty \right) + \|f - \tilde{f}\|_\infty \right\} \]

\[\leq 4\sqrt{2/\varepsilon} K_2 \sqrt{k \Delta t} \|f - \tilde{f}\|_\infty \]

\[+ 2\sqrt{2/\varepsilon} \tilde{K} K_2 \sqrt{\Delta t} \sum_{l=1}^{k} \frac{1}{\sqrt{l}} \left( \|u^{k-l}\|_\infty + \|v^{k-l}\|_\infty \right).\]

And then, by the discrete Gronwall inequality, we get

\[\|u^k\|_\infty + \|v^k\|_\infty \leq 4 \sqrt{2T} \varepsilon^{4}\tilde{K} K_2 \sqrt{\Delta t} \varepsilon^{f - \tilde{f}} \|f - \tilde{f}\|_\infty \]

for all \(k \Delta t \leq T\), which concludes the proof of the lemma. \(\Box\)

By using the fact that the approximate solutions are stable with respect to perturbations in the fractional flow function, we can now prove a corresponding theorem for the solution of the system (2.1).

**Theorem 3.** Assume that \(f\) and \(\tilde{f}\) are two fractional flow functions satisfying (2.4), and let \((s^0, b^0)\) be an initial condition obeying (2.2) and (2.3). Let \((s, b)\) be the unique solution of

\[s_t + f(s, c) = \varepsilon s_{xx}, \]

\[b_t + (cf(s, c)) = \varepsilon b_{xx}, \]

\[s(x, 0) = s^0(x), \quad b(x, 0) = b^0(x),\]

and let \((\tilde{s}, \tilde{b})\) be the unique solution of

\[\tilde{s}_t + \tilde{f}(\tilde{s}, \tilde{c}) = \varepsilon \tilde{s}_{xx}, \]

\[\tilde{b}_t + (\tilde{c}\tilde{f}(\tilde{s}, \tilde{c})) = \varepsilon \tilde{b}_{xx}, \]

\[\tilde{s}(x, 0) = s^0(x), \quad \tilde{b}(x, 0) = b^0(x)\]

for \(\varepsilon > 0\). Then there is a finite constant \(K\) depending only on \(T\) and \(\varepsilon\) such that

\[\|s(\cdot, t) - \tilde{s}(\cdot, t)\|_\infty + \|b(\cdot, t) - \tilde{b}(\cdot, t)\|_\infty \leq K\|f - \tilde{f}\|_\infty \]

for \(0 < t \leq T < \infty\).

**Proof.** This is proved in the same way as Theorem 2, by using the discrete analog and the \(L^\infty\)-convergence of the finite difference scheme. \(\Box\)
7. A Numerical Example

Since most of the estimates obtained in the previous sections approach infinity as $T/\varepsilon$ goes to infinity, we are unable to prove that the solution of the parabolic problem (1.1) converges to the solution of the hyperbolic problem (1.2) as $\varepsilon$ tends to zero. However, since the solution of the Riemann problem of (1.2) is known, we can illustrate the convergence numerically.

Let the fractional flow function be given by

$$f(s, c) = \frac{s^2}{s^2 + (1 - s)^2 \left( \frac{1}{2} + c \right)},$$

and let the adsorption function be given by

$$a(c) = \frac{c}{5(1 + c)}.$$

We consider the Cauchy problem of (1.1) and (1.2) with initial data

$$\left(s^0(x), c^0(x)\right) = \begin{cases} (0.95, 1), & x < 0 \\ (0.8, 0), & x > 0. \end{cases} \quad (7.1)$$

For the hyperbolic problem (1.2), the solution of this problem is given by the construction presented by Johansen and Winther [6]. To study the

FIG. 1. The $s$ component of the solution for $\varepsilon = 0.01$. The mesh sizes for the parabolic problem are $\Delta x = 0.00067$ and $\Delta t = 0.00002$. 

...
convergence of the solution of the parabolic problem (1.1) to the solution of the hyperbolic problem (1.2), we compare the solution of (1.2) with the approximate solutions of (1.1) generated by the Lax–Friedrichs scheme for small values of $\varepsilon$. We have used two parameter sets, both of them satisfying the CFL condition (3.5). For both sets of parameters we plot the $s$ and $c$ components of the finite difference solution of (1.1) and the exact solution of Riemann problem (7.1) for the hyperbolic problem as a function of $x$ for $t = 1$ (Figs. 1–4). We observe from the figures that both the $s$ and the $c$ component of the solution of the parabolic problem seem to converge to the solution of the hyperbolic problem as $\varepsilon$ tends to zero, at least in $L^1$. 

**Fig. 2.** The $c$ component of the solution for $\varepsilon = 0.01$. The mesh sizes for the parabolic problem are $\Delta x = 0.00067$ and $\Delta t = 0.00002$.

**Fig. 3.** The $s$ component of the solution for $\varepsilon = 0.001$. The mesh sizes for the parabolic problem are $\Delta x = 0.00067$ and $\Delta t = 0.00002$. 
8. SOME REMARKS

As we mentioned in the Introduction, the system (2.1) models a situation where a polymer is added to the water in order to enhance an oil recovery process. In practical flooding situations one is often interested in using several different polymer components. Let \( c_i = c_i(x, t) \) denote the concentration of the \( i \)th polymer component in the aqueous phase. Then the parabolic system

\[
\begin{align*}
{s_i} + & f(s, c_1, \ldots, c_n) x = \varepsilon s_{xx} \\
(s c_i + a_i(c_i))_t + (c_i f(s, c_1, \ldots, c_n))_x &= \varepsilon (s c_i + a_i(c_i))_{xx},
\end{align*}
\]

models the displacement of the oleic phase by the aqueous phase. Here the term \( a_i = a_i(c_i) \) models the adsorption of the \( i \)th polymer component on the rock. Each adsorption function is assumed to fulfill the requirements (2.5). The fractional flow function is now a function of the concentration of all the polymer components; it is assumed to be smooth and to satisfy generalisations of (2.4).

The Riemann problem of (8.1) with \( \varepsilon = 0 \) was solved by Johansen and Winther [7], and an algorithmic version of their construction was given in [5], together with some examples of solutions.

We remark here that the Cauchy problem of (8.1), with \( \varepsilon > 0 \) and with initial data \( (s^0, c_1^0, \ldots, c_n^0) \) in the unit interval and with bounded total variation, can be handled exactly as in the case of \( n = 1 \). We also remark that the system (8.1) with \( a_i(c_i) = 0 \) can, with minor modifications, be handled in the same way as adsorption functions which satisfy the requirement (2.5).
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