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On a graph theoretical model for cyclic register allocation

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Abstract

In the process of compiling a computer programme, we consider the problem of allocating variables to registers within a loop. It can be formulated as a coloring problem in a circular arc graph (intersection graph of a family $\mathscr{F}$ of intervals on a circle). We consider the meeting graph of $\mathscr{F}$ introduced by Eisenbeis, Lelait and Marmol. Proceedings of the Fifth Workshop on Compilers for Parallel Computers, Malaga, June 1995, pp. 502–515. Characterizations of meeting graphs are developed and their basic properties are derived with graph theoretical arguments.

Furthermore some properties of the chromatic number for periodic circular arc graphs are derived.

Keywords: Register allocation; Compilation; Circular arc graphs; Coloring; Microprocessor design; Cyclic scheduling

1. Introduction

In the process of compiling a computer programme, register allocation is an important problem; basically it amounts to trying to keep as many variables as possible in registers, thereby avoiding the need to introduce spill code; we shall concentrate on loops which offer instances where the process needs to be optimized.

For the allocation of variables to registers, graph coloring models provide a fundamental tool. Basically we may associate a node of a graph G to each variable occurring in a programme; two nodes are linked if they correspond to variables which are simultaneously alive. Finding a minimum coloring of G corresponds to finding the smallest number of registers needed to store the variables. Various approaches have been described in the literature. Our purpose in this note is to start from the representation in terms of “meeting graph” introduced in [3] and to derive directly its properties from the

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structure of the graph. In this process we shall also generalize and hopefully simplify some of the results given in [6]. Our work lies in the scope of periodic assignment in periodic scheduling problems [5]. Assignment of periodic jobs to processors is formulated in terms of graph coloring in [5]; collections of jobs with different periods are considered. Worst-case performance of some heuristics is also studied.

For a comprehensive bibliography on register allocation problems we refer the reader to [6] which contains an extensive list of contributions in the area; all graph-theoretical terms not defined here can be found in [1].

In this note, we concentrate on a graph-theoretical model designed for loop cyclic register allocation. The motivations for dealing with this model are extensively discussed in [3,6]. Notice that we will exclusively consider the case of loops in a programme; furthermore, in order to exploit the instruction-level parallelism of the programmes and the performances of modern processors, we will have to consider that some of the lifetimes of the variables may span more than one iteration; this is due to loop software pipelining that make iterations overlap in time (see [6]).

As in [6] we will not consider the problem of loop scheduling so that in our model the basic data will consist of a family $\mathscr{F}$ of circular intervals (intervals on a circle which represents one iteration of the loop); as mentioned above these intervals may be longer than the circumference of the circle ("they intersect themselves") in case they are associated to a variable whose lifetime exceeds the length of one iteration.

A classical model consists in taking the intersection graph $G=(V, E)$ of $\mathscr{F}$: each interval in $\mathscr{F}$ is a node of G and we link two nodes in G if the corresponding intervals have a nonempty intersection. (Notice that for intervals longer than one iteration, a loop will be introduced on the corresponding node in G).

Such graphs are circular arc graphs; they have been extensively studied (see [4,8]). Observe that finding the chromatic number of general circular arc graphs is NP-complete, see [4].

Assigning the variables to registers in such a way that no two variables simultaneously alive are assigned to the same register amounts to coloring the nodes of the graph G representing $\mathscr{F}$. The presence of loops in the graph however makes a coloring impossible for the corresponding nodes. This is one reason why loop unrolling is introduced.

Given a family $\mathscr{F}$ of cyclic intervals and a positive integer k, we arrange k copies of the family $\mathscr{F}$ along a circle whose circumference is k times the one of the original circle. The intersection graph of this new family will be called a *k-unrolling* of G (or of $G(\mathscr{F})$); it will be denoted by $G^k(\mathscr{F})$ or simply G^k. Fig. 1 shows an example of family $\mathscr{F}$ with a 2-unrolling of G; the underlying circle has been cut for simplifying the representation.

Examples show that the chromatic number $\chi(G^k(\mathscr{F}))$ does change when k increases; the effect of loop unrolling on $G^k(\mathscr{F})$ has thus been studied by various authors using different tools (see [6]).

If the nodes of G are $a, b, \dots, z$, the nodes of G^k will be $a_1, \dots, a_k$, $b_1, \dots, b_k, \dots$, $z_1, \dots, z_k$. We shall say that a q-coloring of G^k is *cyclic* if there exists a permutation

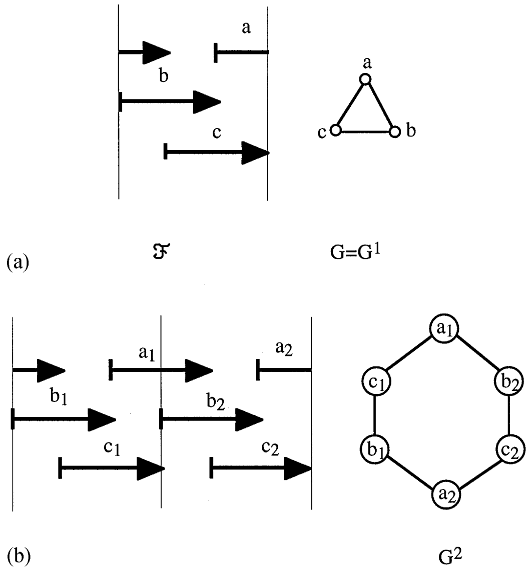

Fig. 1. A 2-unrolling G^2 of G. (a) The family $\mathscr{F}$ and the intersection graph $G = G^1$. (b) Two copies of $\mathscr{F}$ and the intersection graph G^2.

φ of the colors in $\{1,\dots,q\}$ such that for each node u_i the following holds:

if u_i has color c, then u_{i+1} has color $\varphi(c)$.

(Clearly the indices are taken modulo k between 1 and k).

The smallest q such that a graph G^k has a cyclic q-coloring is the *cyclic chromatic number* of G^k; it is denoted by $\chi_{\text{cyclic}}(G^k)$ while the usual chromatic number is $\chi(G^k)$. Clearly $\chi(G^k) \leqslant \chi_{\text{cyclic}}(G^k)$.

In the next sections we shall present the meeting graph introduced by Eisenbeis et al. [3]; we will characterize meeting graphs, derive their properties by graph-theoretical arguments and use them to generalize and simplify some results in loop cyclic register allocation.

2. The meeting graph

We are given a family $\mathscr{F}$ of cyclic intervals (located on a circle of circumference p). We may assume that p is integral and that all intervals in $\mathscr{F}$ are of the form $[i, j[$ where i, j are integers; notice that we may have in $\mathscr{F}$ intervals of length $|j - i| > p$; the endpoints i, j should thus be integers; they are taken modulo p between 1 and p.

The thickness of $\mathscr{F}$ at i, denoted by $r(\mathscr{F};\ i)$ is the number of intervals in $\mathscr{F}$ which contain a point i of the circle; an interval covering α times point i should be counted α times.

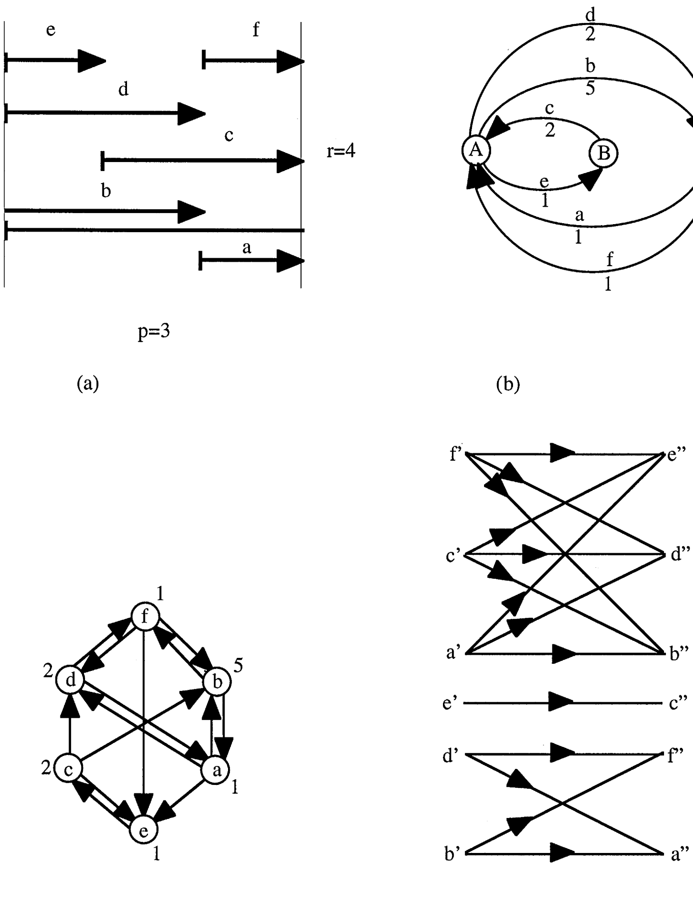

Fig. 2. A family $\mathscr{F}$ with the various graphs. (a) The family $\mathscr{F}$. (b) The graph $\mathscr{G}$. (c) The graph $\hat{G}$. (d) The split $\mathscr{S}(\hat{G})$.

We denote by $r(\mathscr{F})$ the maximum of $r(\mathscr{F};\ i)$ over all points i of the circle. It is the *thickness* of $\mathscr{F}$. Fig. 2a shows a family $\mathscr{F}$ of cyclic intervals with $r(\mathscr{F}) = 4$.

As in [6] we will assume without loss of generality that the thickness at all points i is constant (equal to $r(\mathscr{F})$): we may introduce unit intervals in $\mathscr{F}$ if needed.

Basically for the cyclic register allocation problem, the circle should be oriented according to the time axis.

This suggests that a representation by an oriented graph would be more appropriate than one using an unoriented graph, as for instance the circular arc graph.

We therefore define a graph $\mathscr{G}$ representing $\mathscr{F}$ as follows: the nodes of $\mathscr{G}$ are the integral points $1, 2, \dots, p$ of the circle; each interval $[i,\ j[$ of $\mathscr{F}$ with length r is associated with an arc $(i,\ j)$ of length r.

For the family $\mathscr{F}$ of Fig. 2a, we have in Fig. 2b the graph $\mathscr{G}$ representing $\mathscr{F}$.

Proposition 2.1. *For every node x of $\mathcal{G}$, we have*

$$\mathrm{d}^+_{\mathcal{G}}(x) = \mathrm{d}^-_{\mathcal{G}}(x)$$

(*the number of arcs entering x is equal to the number of arcs going out of x*).

This is an immediate consequence of the assumption of constant thickness of $\mathcal{F}$.

We recall that a circuit in a graph G is Eulerian if it uses each arc of G exactly once. It is a basic result of graph theory that such a circuit exists in G if and only if G is connected and satisfies $\mathrm{d}^+_G(x) = \mathrm{d}^-_G(x)$ for each node x (see [1]). Hence we can state:

Proposition 2.2. *Every connected component of $\mathcal{G}$ has a Eulerian circuit.*

Proposition 2.3. *In $\mathcal{G}$ every circuit has a length which is a multiple of p.*

Again this follows directly from the construction of $\mathcal{G}$ from $\mathcal{F}$: a circuit in $\mathcal{G}$ corresponds to a sequence of intervals in $\mathcal{F}$ such that the endpoint of any interval coincides with the initial point of the next one. Since all intervals are oriented according to time, we may come back to the starting point only after having gone around the circle (once or more).

Remark 2.1. Besides satisfying $\mathrm{d}^+_{\mathcal{G}}(x) = \mathrm{d}^-_{\mathcal{G}}(x)$ for each node x, graph $\mathcal{G}$ does not have any special structure. In fact for any graph G (with $\mathrm{d}^+_G(x) = \mathrm{d}^-_G(x)$ for each node x) we can associate a length $l(x, y)$ to each arc (x, y) so that G corresponds to a family of intervals on a circle.

This can be seen as follows: let $x_1, x_2, \ldots, x_n$ be the nodes of G; we may consider that they are placed regularly in this order around a circle of circumference n. Examine each arc (x_i, x_j): if $i < j$ we give a length $l(x_i, x_j) = j - i$, otherwise $(i \geqslant j)$ we give a length $l(x_i, x_j) = n - j + i$. Then G represents a family $\mathcal{F}$ of intervals obtained by associating to each arc (x_i, x_j) of G an interval $[i, j[$ starting at x_i.

A consequence of the above remark is that for constructing a Eulerian circuit in $\mathcal{G}$ we cannot hope to have a special algorithm based on a specific structure of the graph. We will have to apply any general algorithm for constructing a Eulerian circuit.

Now let us introduce the *adjoint* $H(G)$ of a graph G as a graph obtained by associating a node $\bar{u}$ to every arc $u = (x, y)$ of G. In $H(G)$ nodes $\bar{u}$ and $\bar{v}$ are linked by an arc $(\bar{u}, \bar{v})$ if they correspond to two arcs $u = (x, y)$, $v = (y, z)$ in G. $H(G)$ is sometimes called the *directed line-graph* of G. It appears in various applications (see for instance [2]).

Define in a graph $G = (V, U)$ the *neighborhoods* $N^+(x) = \{y \in V \mid (x, y) \in U\}$ and $N^-(x) = \{y \in V \mid (y, x) \in U\}$. It is known that a graph G (without parallel arcs) is an *adjoint* of some graph if and only if for any two nodes x, y $N^+(x) \cap N^+(y) \neq \emptyset$ implies $N^+(x) = N^+(y)$ (see [1]).

This defines an equivalence relation $\sim$ on the node set V of an adjoint G: for $x, y \in V$ we have $x \sim y$ if and only if $N^+(x) = N^+(y)$.

Now for any (oriented) graph $G=(V, U)$ we define the *split* $\mathscr{S}(G)$ as follows: each node x of G is replaced by two nodes x', x'' and each arc $u=(x, y)$ of G is replaced by an arc $u^* = (x', y'')$. So $\mathscr{S}(G) = (V', V'', U^*)$ is a bipartite graph.

If H is the adjoint of a graph G, then $\mathscr{S}(H)$ is a union of node disjoint complete bipartite graphs: for each class $C(x)$ of the equivalence relation $\sim$ defined above, we will have a complete bipartite graph on subsets $\{z' \in V' \mid z \in C(x)\}$ and $\{y'' \in V'' \mid y \in N^+(x)\}$.

One observes that the left sets (resp. right sets) of those bipartite graphs are pairwise disjoint.

Having stated these preliminary definitions and observations, we may now return to the family $\mathscr{F}$ of cyclic intervals and to the "meeting graphs".

For $\mathscr{F}$ the *meeting graph* $\hat{G}(\mathscr{F})$ or simply $\hat{G}$ is obtained by introducing a node $\hat{v}$ for each interval $v = [i, j[$ of $\mathscr{F}$ and we link nodes $\hat{u}$ and $\hat{v}$ by an arc $(\hat{u}, \hat{v})$ if they correspond to intervals $u = [i, j[$ and $v = [j, k[$ of $\mathscr{F}$. The graph $\hat{G}$ associated to the family $\mathscr{F}$ in Fig. 2a is given in Fig. 2c. Furthermore the split $\mathscr{S}(\hat{G})$ is shown in Fig. 2d.

We observe that $\hat{G}(\mathscr{F}) = H(\mathscr{G})$, i.e., the meeting graph of $\mathscr{F}$ is the adjoint of the graph $\mathscr{G}$ representing $\mathscr{F}$.

As consequences of this observation, we can mention the following.

Proposition 2.4 (Lelait [6]). *Every connected component of $\hat{G}$ has a Hamiltonian circuit.*

This follows directly from Proposition 2.2 and from the fact that $\hat{G}$ is the adjoint of $\mathscr{G}$: a Eulerian circuit in $\mathscr{G}$ corresponds to a Hamiltonian circuit in $\hat{G}$.

So for constructing a Hamiltonian circuit in a connected component of $\hat{G}$ we simply have to construct a Eulerian circuit in a connected component I of $\mathscr{G}$.

A simple such technique consists of choosing in I a node x as root and constructing a spanning oriented tree T (arborescence) directed towards x. Then, starting from x, one follows a path by choosing at each node an unused arc and by using the unique arc of T leaving a node only when there is no other unused arc.

We shall now denote by $K_{p,\,q}$ a complete bipartite graph with p (resp. q) nodes in the left (resp. right) set.

Proposition 2.5. *A graph S (without isolated nodes) is the split $\mathscr{S}(\hat{G})$ of a meeting graph $\hat{G}$ if and only if it is a collection of node disjoint complete graphs $K_{n_a,\,n_a}$, $K_{n_b,\,n_b}, \ldots, K_{n_t,\,n_t}$.*

Proof. Since $\hat{G}$ is the adjoint of some graph $\mathscr{G}$, we know from the above remarks that the split $\mathscr{S}(\hat{G})$ of $\hat{G}$ consists of node disjoint complete bipartite graphs.

Now, according to Proposition 2.1, $\mathscr{G}$ satisfies $d^+_{\mathscr{G}}(x) = d^-_{\mathscr{G}}(x)$ for each node x.

For each pair of arcs $u=(z, x)$, $v=(x, y)$ in $\mathscr{G}$ there will be in $\hat{G}=H(\mathscr{G})$ a pair of nodes u, v with an arc (u, v) and in $\mathscr{S}(\hat{G})$ a pair of nodes u', v'' with an arc (u', v''). So if $d^+_{\mathscr{G}}(x)=d^-_{\mathscr{G}}(x)=s$, then in $\mathscr{S}(\hat{G})$ every one of the s nodes u' corresponding to some arc u entering node x in $\mathscr{G}$ will be linked to the s nodes v'' corresponding to arcs v going out of x in $\mathscr{G}$. This will give a complete graphs $K_{s,\ s}$.

Conversely, assume that we are given a collection $\mathscr{K}=(K_{n_1,\ n_1},\dots,K_{n_d,\ n_d})$ of complete (oriented from left to right) bipartite graphs. We can construct a graph $\mathscr{G}$ such that $\mathscr{K}=\mathscr{S}(H(\mathscr{G}))$ as follows: we assume that the left nodes are labelled $x_1,\dots,x_n$ in $\mathscr{K}$ and that the labels of the right nodes form an arbitrary permutation of $x_1,\dots,x_n$.

To each $K_{i,\ i}$ associate a node I in the graph $\mathscr{G}$ to be constructed. Examine consecutively each symbol x_s: it occurs once in the left set of some graph, say, $K_{j,\ j}$, and once in the right set of some graph, say, $K_{r,\ r}$

We associate to x_s an arc $x_s=(L, J)$. Since all bipartite graphs in $\mathscr{K}$ are of the form $K_{i,\ i}$ there will be the same number of arcs leaving and entering a node in $\mathscr{G}$. Clearly $\mathscr{G}$ has an adjoint $H(\mathscr{G})$ whose split is $\mathscr{K}$. □

We have seen that every connected component of $\hat{G}$ has a Hamiltonian circuit. In applications one needs simply to have a collection of node disjoint circuits covering all nodes of $\hat{G}$. It corresponds to cyclic r-colorings of some k-unrolling G^k. More precisely each circuit C in $\hat{G}$ corresponds to a sequence of intervals which make $\rho(C)=\sum(w(x): x\in C)/p$ tours around the circle of circumference p, where $w(x)$ is the length of the interval of $\mathscr{F}$ represented by node x. This circuit defines a $\rho(C)$-coloring of the corresponding nodes in a $\rho(C)$-unrolling of G.

For instance the circuit $C_2=(e, c, b, a)$ in the decomposition given in Fig. 3a has $\rho(C_2)=\frac{9}{3}=3$; going along C_2 we give color i to the ith occurrence of each interval. We do the same for C_1 (with new colors) and we get the coloring of intervals shown in Fig. 3b. Since $\rho(C_1)+\rho(C_2)=4=r$, this will define an r-coloring of a k-unrolling of G with $k=l\mathrm{cm}\ (\rho(C_1),\rho(C_2))$. Here $l$cm denotes the smallest common multiple.

The permutation φ of colors associated to this cyclic 4-coloring is $\varphi=(1)\ (2\ 3\ 4)$: the intervals d and f get always color 1, while e, c, b, a get consecutively colors 2, 3 and 4.

Such a collection of circuits can be easily constructed by observing that it is simply a perfect matching (i.e., a collection of node disjoint arcs meeting all nodes) in $\mathscr{S}(\hat{G})$. Such a matching does exist since all bipartite graphs in $\mathscr{S}(\hat{G})$ are of the form $K_{s,\ s}$. Fig. 3c shows a perfect matching in $\mathscr{S}(\hat{G})$; one can verify that it defines a collection of three circuits in $\hat{G}$.

Hence, we can state the following.

Proposition 2.6. *There is a one-to-one correspondence between the perfect matchings in $\mathscr{S}(\hat{G})$ and the collection of node disjoint circuits covering all nodes of $\hat{G}$.*

As a consequence, we may easily examine what happens when one wants to introduce a new arc in a family $\mathscr{C}$ of node disjoint circuits covering all nodes of $\hat{G}$; this will be

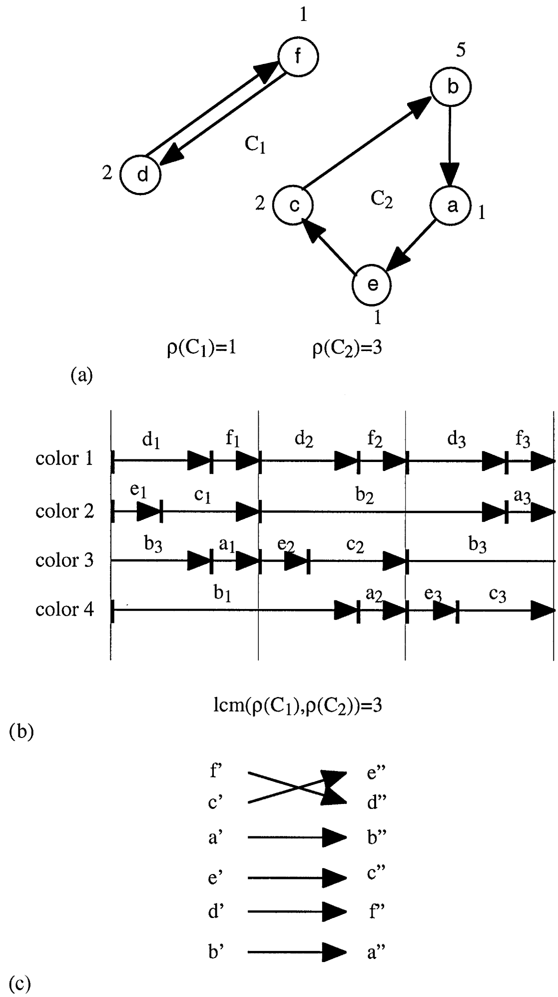

Fig. 3. Circuits and cyclic colorings. (a) A decomposition of $\hat{G}$ of Fig. 2 into two circuits. (b) A cyclic 4-coloring of G^3 associated to the decomposition in (a). (c) A perfect matching in $\mathscr{S}(\hat{G})$ corresponding to another decomposition into circuits.

needed in exploration procedures for finding a suitable cyclic r-coloring of G^k. Let M be the perfect matching in $\mathscr{S}(\hat{G})$ corresponding to $\mathscr{C}$. Suppose we want to introduce an arc (a, b) of $\hat{G}$ into $\mathscr{C}$. It corresponds to arc (a', b'') in $\mathscr{S}(\hat{G})$; let (a', c'') and (d', b'') be the arcs of M adjacent to a' and b'' in $\mathscr{S}(\hat{G})$. These arcs are in the same connected component $K_{i,\,i}$; so there is an arc (d', c'') in $K_{a,\,a}$; it corresponds to arc (d, c) in $\hat{G}$: so we may introduce (a, b) and (d, c) in $\mathscr{C}$ provided we remove (a, c) and (d, b). This corresponds to the "dual chords" defined in [6].

In fact $\mathscr{S}(\hat{G})$ allows us to see directly what are the arcs which can be introduced simultaneously into $\mathscr{C}$; we shall say that they are *compatible*. So two arcs of $\hat{G}$ are compatible if and only if they correspond to nonadjacent arcs in $\mathscr{S}(\hat{G})$.

We can formalize this as follows:

Proposition 2.7. *A collection of arcs of $\hat{G}$ are compatible if the corresponding arcs in $\mathscr{S}(\hat{G})$ are nonadjacent.*

In $\mathscr{S}(\hat{G})$ any set of nonadjacent arcs can be extended to a perfect matching.

This follows from the structure of $\mathscr{S}(\hat{G})$ described in Proposition 2.5. Any greedy algorithm will thus provide a perfect matching in $\mathscr{S}(\hat{G})$.

We can take advantage of this observation when we will have to explore the set of partitions of the node set of $\hat{G}$ into circuits $C_1,\dots,C_n$ in order to find one with a reasonably small $D = \mathrm{lcm}(\rho(C_1),\dots,\rho(C_n))$. Finding the smallest D is NP-complete (see [6]) so heuristics will have to be defined; Proposition 2.7 can thus be reformulated as follows: any collection of node disjoint paths in $\hat{G}$ can be extended to a family of node disjoint circuits covering all nodes of $\hat{G}$.

We shall end this section with a simple observation related to connectivity of $\hat{G}$ and a reformulation of the problem of finding an r-coloring in a circular arc graph G.

Proposition 2.8. *$\hat{G}(\mathscr{F})$ is connected if and only if for every nontrivial partition of the nodes of $\mathscr{G}$ into A, B, there is at least one arc of $\mathscr{G}$ between A and B.*

If $\mathscr{F}$ contains a collection of p consecutive unit intervals, then $\hat{G}(\mathscr{F})$ is connected and hence Hamiltonian.

Proof. The first part follows directly from the fact that $\hat{G}$ is connected if and only if $\mathscr{G}$ is connected.

Now consider any nontrivial partition A, B of the nodes of $\mathscr{G}(A,\ B \neq \emptyset)$. There must be one of the p unit intervals which corresponds to an arc $(x,\ y)$ with $x \in A,\ y \in B$, so $\hat{G}$ is connected. $\square$

Proposition 2.9. *For any family $\mathscr{F}$ of cyclic intervals with thickness $r(\mathscr{F}) = r$, the circular arc graph G will satisfy $\chi(G) = \chi_{\text{cyclic}}(G) = r$ if and only if in $\hat{G}$ there is a collection of r node disjoint circuits.*

Proof. (A) Assume G has an r-coloring (i.e. $\chi(G) = r$); this excludes the presence of loops in G. So, there are no intervals with length larger than p. From the assumption on $\mathscr{F}$ (constant thickness), each collection of nodes of the same colour i corresponds to intervals forming exactly one circuit C_i; we have $\rho(C_i) = 1$ since we are in $G = G^1$ $(1 \geqslant l\mathrm{cm}(\rho(C_1),\dots,\rho(C_r))$. So $\hat{G}$ has a collection of r node disjoint circuits.

(B) Assume that $\hat{G}$ contains a collection of r node disjoint circuits $C_1,\dots,C_r$; since for each circuit C we have $\rho(C) \geqslant 1$ and since $\sum_{i=1}^{r} \rho(C_i) \leqslant r$ for the node disjoint circuits $C_1,\dots,C_r$, we must have $\rho(C_i) = 1$ for $i = 1,\dots,r$. This defines an r-coloring of G. For $G = G^1$, any r-coloring is trivially a cyclic r-coloring. $\square$

This property can be used for checking whether $G(\mathscr{F})$ is $(r+i)$-colorable: one introduces i sequences of p consecutive unit intervals into $\mathscr{F}$. Let $\hat{G}(\mathscr{F}')$ be the new graph corresponding to this extended family $\mathscr{F}'$, we will have $r(\mathscr{F}') = r + i$. According to Proposition 2.9, $G(\mathscr{F}')$ will be $(r+i)$-colorable if and only if $\hat{G}(\mathscr{F}')$ has a collection of $r+i$ node disjoint circuits. From an $(r+i)$-coloring of $G(\mathscr{F}')$ we get an $(r+i)$-coloring of $G(\mathscr{F})$ by removing the nodes associated to the unit intervals in $\mathscr{F}' - \mathscr{F}$.

3. Unrolling loops and chromatic number

In this section we intend to examine the behavior of the chromatic number of the k-unrolling G^k of the circular arc graph $G = G^1$ associated to a family $\mathscr{F}$ of cyclic intervals. Related results on periodic scheduling and graph coloring are given in [5].

As mentioned, the graph G (circular arc graph) may have loops, so we shall have to choose values k such that the k-unrolling G^k has no more loops; it will certainly be the case if $k \geqslant \lceil \max_{1 \leqslant i \leqslant n} \ell_i / p \rceil$ where ℓ_i is the length of interval i and p the length of the circle.

In [3,6] it was shown that a value D always exists such that $\chi_{\text{cyclic}}(G^D) = r(\mathscr{F})$. In applications one wishes to find a D which is as small as possible; such a D can be obtained by finding a collection $\mathscr{C}$ of circuits $C_1, \ldots, C_n$ covering all nodes of $\hat{G}$ such that the $\text{lcm}(\rho(C_1), \ldots, \rho(C_n))$ is minimum. This is an NP-complete problem (see [6]) as mentioned.

We shall now first examine the case of usual q-colorings of G^k and extend some results of [6] while providing alternate derivations. Instead of coloring the nodes of G^k we shall sometimes consider that we are coloring the intervals directly.

Proposition 3.1. *Let $\mathscr{F}$ be a family consisting of a single interval making exactly r tours around the circle; then*

$$\chi(G^D) = \lceil D / \lfloor D/r \rfloor \rceil$$

Proof. Clearly if $D < r, G^D$ has a loop and no coloring is possible ($\chi(G^D) = \infty$). Let us assume that $D \geqslant r$ and let $u = \lceil D / \lfloor D/r \rfloor \rceil$.

(A) $\chi(G^D) \geqslant u$.

We have D intervals $I_1,\ I_2, \ldots, I_D$ of length $r \cdot p$ to color; with each colour, we can color at most $\lfloor D \cdot p / r \cdot p \rfloor$ intervals. So we need at least u colors.

(B) $\chi(G^D) \leqslant u$.

We shall construct a coloring of the D intervals with u colors.

$$\text{Let } D = \lambda \cdot r + \mu \quad (\lambda,\ \mu \text{ integers, } 0 \leqslant \mu < r)$$

$$\text{and } \mu = \alpha \cdot \lambda + \beta \quad (\alpha,\ \beta \text{ integers, } 0 \leqslant \beta < \lambda)$$

The coloring rules are the following:

with each color i ($i = 1, \ldots, r + \alpha$) color the intervals

(a) $I_{(j-1)(r+\alpha+1)+i}$ for $j = 1,\ 2, \ldots, \beta$,

(b) $I_{\beta(r+\alpha+1)+(k-1)(r+\alpha)+i}$ for $k = 1,\ 2, \ldots, \lambda - \beta$.

With color $r+\alpha+1$ color the intervals

(c) $I_{j(r+\alpha+1)}$ for $j=1,\ldots,\beta$.

Here we assume that if β (resp. $\lambda-\beta$) is zero, then (a), (c) (resp. (b)) do not exist. In particular if $D=\lambda\cdot r$, then $\mu=\alpha=\beta=0$ and the algorithm gives an r-coloring. It is immediate to verify that the intervals with the same color do not overlap (each interval $I_i=[a_i,\ b_i[$ satisfies $a_i=i\cdot p+1,\ b_i=i\cdot p+r\cdot p$).

For instance the last interval colored with $i\leqslant r+\alpha$ is $I_{\beta(r+\alpha+1)+(\lambda-\beta-1)(r+\alpha)+i}=I_{D-r-\alpha+i}$ which ends at $c=D\cdot p-\alpha\cdot p+i\cdot p\equiv i\cdot p-\alpha\cdot p$ and the first interval colored with $i\leqslant r+\alpha$ is I_i which starts at $d=i\cdot p+1$ so $c<d$. $\square$

For the general case, we consider that in $\mathscr{F}$ we have for $i=1,\ldots,p$ an interval I_i starting at i and making exactly r_i tours around the circle.

Let $\gamma=p(r_1+\cdots+r_p)+p-1$.

Proposition 3.2.

$$\chi(G^D)\leqslant\lceil D/\lfloor D\cdot p/\gamma\rfloor\rceil.$$

Proof. A simple way of deriving this result is to consider that we have indeed a single interval of length γ (by concatenating all p intervals and by including the unit spaces between the starting points of two consecutive intervals in $\mathscr{F}$).

Let $D\cdot p=\delta\cdot\gamma+\varepsilon$ ($\delta,\ \varepsilon$ integrals, $0\leqslant\varepsilon<\gamma$).

From Proposition 3.1 there exists a coloring with $\lceil D/\lfloor D\cdot p/\gamma\rfloor\rceil$ colors. $\square$

Corollary 3.3 (Lelait [6]). *There is a D such that for any $k\geqslant D$ $\chi(G^D)\leqslant r_1+\cdots+r_{p+1}$.*

Proof. We have $\lfloor D\cdot p/\gamma\rfloor=\delta$ and $\lceil D/\delta\rceil$

$$=\lceil(\delta\gamma+\varepsilon)/p\delta\rceil$$
$$=\lceil(\gamma+\varepsilon/\delta)(1/p)\rceil=r_1+\cdots+r_p+\lceil(\delta(p-1)+\varepsilon)/(p\cdot\delta)\rceil.$$

Choose D so that $\delta>\gamma$ (for instance $D\cdot p\geqslant\gamma^2$, i.e. $D\geqslant p(r_1+\cdots+r_p+1)^2$), so $\varepsilon<\gamma\leqslant\delta$ and $\lceil(\delta(p-1)+\varepsilon)/(p\cdot\delta)\rceil\leqslant 1$. $\square$

Remark 3.1. The value of D can be strongly improved by refining the proof technique; our purpose was simply to derive the result as simply as possible.

Remark 3.2. For the general case of Proposition 3.2 we are not able to derive an explicit formula giving $\chi(G^D)$ since the problem is NP-complete.

A q-coloring of G^k is *strongly cyclic* if it is a cyclic coloring associated to a permutation φ which is a cyclic permutation.

Let us now examine when a graph G^k has a strongly cyclic q-coloring. We shall assume that $k\geqslant r$.

Proposition 3.4. *Given a family $\mathscr{F}$ of cyclic intervals with constant thickness $r(\mathscr{F})= r$. The following statements are equivalent*:

(a) *for k large enough, G^k has a strongly cyclic r-coloring,*

(b) *$\hat{G}$ is connected.*

Proof.

(b) $\Rightarrow$ (a): It follows from Proposition 2.4 that $\hat{G}$ has a Hamiltonian cycle C; it corresponds to a sequence of cyclic intervals making exactly r tours around the circle. C defines a cyclic r-coloring of G^r: by going along the cycle r times, color with color i the ith occurrence of each interval. Clearly this r-coloring is strongly cyclic since it corresponds to the cyclic permutation $\varphi = (1,\ 2,\dots,r)$.

(a) $\Rightarrow$ (b): Assume $\hat{G}$ is not connected. Every cyclic r-coloring corresponds to a partition of the node set of $\hat{G}$ into circuits $C_1,\dots,C_t$; if $D = l\mathrm{cm}(\rho(C_1),\dots,\rho(C_t))$, it is a cyclic r-coloring of G^D.

In $\mathscr{F}$ there are at least two subsets $\mathscr{F}_1$, $\mathscr{F}_2$ of intervals such that no interval in $\mathscr{F}_1$ has an endpoint in common with some interval in $\mathscr{F}_2$. This means that in a cyclic r-coloring of some G^k no interval in $\mathscr{F}_1$ can have the same color as some interval in $\mathscr{F}_2$ (because in an r-coloring the intervals of the same color are consecutive without empty space between them since r is the thickness of $\mathscr{F}$). So there cannot be any strongly cyclic r-coloring (because in such a coloring the consecutive occurrences of each interval get successively the different r colors). So for no $k(\geqslant r), G^k$ will have a strongly cyclic r-coloring. □

Remark 3.3 (*Lelait* [6]). Notice that for any $\mathscr{F}$ with $r(\mathscr{F}) = r$ it always holds that G^{r+1} has a strongly cyclic $(r+1)$-coloring according to Proposition 2.7.

Strongly cyclic q-colorings are interesting when one uses a file of rotating registers; in such a system a variable stored during iteration i in register R_i is automatically transferred for iteration $i+1$ to register R_{i+1}. So we have a finite number s of such registers (which have to be considered cyclically, i.e R_1 follows R_s) and for a system an assignment will be possible if and only if one has a strongly cyclic q-coloring with $q\leqslant s$. The above remark tells us that if $r(\mathscr{F}) = r$, then we will always have either a strongly cyclic r-coloring or a strongly cyclic $(r+1)$-coloring. So an assignment will be possible if $s\geqslant r+1$.

4. Conclusions

Connections between (strongly) cyclic q-colorings and acyclic q-colorings are not completely understood yet. More research is needed along this line. We may however raise some questions and bring some answers: is there a value K such that for any circular arc graph $G = G^1$, $\chi_{\text{cyclic}}(G^k) - \chi(G^k)\leqslant K$? The answer is negative. Take a family $\mathscr{F}$ consisting of one interval of length $r = 2$ (here $p = 1$); then G^k is a cycle of length k, so $\chi(G^k)=2$ or 3 (depending on the parity of k), while $\chi_{\text{cyclic}}(G^k)=k$ if k is prime. So there is no constant K.

Here $K=K(k)$; given k and the size $|\mathscr{F}|$ of $\mathscr{F}$ what is the smallest $K(k,|\mathscr{F}|)$ such that for any circular arc graph G associated to a family $\mathscr{F}$ with given size $|\mathscr{F}|$ we have

$$\chi_{\text{cyclic}}(G^k) - \chi(G^k) \leqslant K(k,|\mathscr{F}|)?$$

Can one characterize the cases where $\chi(G^k)=\chi_{\text{cyclic}}(G^k)$?

There are many more questions to be examined; the applications to register allocation will undoubtedly suggest quite a few new ones.

At this stage we may simply conclude by emphasizing that the introduction of the meeting graph has provided a clear insight into the concept of cyclic q-coloring of k-unrollings G^k of a circular arc graph G.

In fact, the meeting graph $\hat{G}$ plays the same role with respect to $\mathscr{G}$ as the one which representation "potential-nodes" of precedence constraints plays with respect to "potential-arcs" representations in scheduling (see [7]).

In the case of $\hat{G}$ the tasks (intervals) are the nodes and sequencing constraints are represented by arcs, while in $\mathscr{G}$ the tasks (intervals) are arcs and sequencing constraints are represented by concatenation of arcs.

Here the sequencing problem is particular in the sense that we have a cyclic scheduling problem.

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