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Qualitative properties in nonlinear Volterra integro-differential equations with delay

Cemil Tunç *

Department of Mathematics Faculty of Sciences, Yüzüncü Yıl University, 65080 Van, Turkey

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Abstract

This paper considers a class of scalar and vector nonlinear Volterra integro-differential equations of the first order with a constant delay. We demonstrate the stability, uniform stability, boundedness, convergence and square integrability of the solutions. The technique of proof involves defining appropriate Lyapunov functionals. Our results improve the results obtained in the literature.

Keywords: Volterra integro-differential equation; Stability; Boundedness; Convergence; Square integrability; Lyapunov functional

1. Introduction

During the last forty years, many results have been obtained on the qualitative behavior of solutions to Volterra integro-differential equations without delay. For instance, we can refer the reader to the papers of Becker [2], Burton [3–6], Burton and Haddock [7], Burton and Mahfoud [9,10], Miller [15], Staffans [18], Tunc [19], and Vanualailai and Nakagiri [21]; the books of Burton [5], Corduneanu [11], Gripenberg et al. [14]; and the references cited therein. It should be noted that an important ingredient in the qualitative theory for ordinary and functional differential equations and integro-differential equations is Lyapunov's second method. In particular, Burton et al. [8] developed a Lyapunov theory that primarily seems to apply to Volterra integro-differential equations. They use Lyapunov functionals, which are (most of the time) non-increasing or strictly decreasing along solutions. Theoretically, this method is very appealing, and there are numerous applications in which its use is natural. However, it is a quite difficult task to find a suitable Lyapunov function or functional for an ordinary differential equation or a functional Volterra integro-differential equation. The key requirement of the method is to find a positive definite function or functional that is non-increasing along solutions. The situation becomes more difficult when we replace the ordinary differential equation with an integro-differential equation or a functional integro-differential equation. Moreover, in the literature, there are a few papers on the qualitative behavior of Volterra integro-differential equations with delay. See, for example, the recent papers of Adıvar and Raffoul [1], Graef and Tunc [13], Raffoul [16], Raffoul and Unal [17] and Tunc [20].

* Tel.: +90 5375517167.
E-mail address: cemtunc@yahoo.com
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In 1982, Burton [4] first considered the following non-linear homogeneous scalar Volterra integro-differential equation without delay:

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds, \tag{1}$$

where $t \geq 0$, $x \in \Re$, $A(t) : \Re \to [0, \infty)$ and $f, g : \Re \to \Re$ are continuous functions, and $B(t, s)$ is a continuous function for all $0 \leq s \leq t < \infty$. The author studied the stability, boundedness, and convergence of the bounded solutions of Eq. (1) by using the Lyapunov functionals.

In this paper, instead of Eq. (1), we consider the non-linear and non-homogeneous scalar Volterra integro-differential equation with delay:

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^t B(t,s)g(x(s))ds + e(t, x(t)), \tag{2}$$

where $t \geq 0$, τ is a positive constant representing a fixed delay; $x \in \Re$, $a(t) : [0, \infty) \to (0, \infty)$, $f, g : \Re \to \Re$ and $e : [0, \infty) \times \Re \to \Re$ are continuous functions with $f(0) = g(0) = 0$, and $B(t, s)$ is a continuous function for $0 \leq s \leq t < \infty$. We investigate the stability, boundedness and convergence of the bounded solutions of Eq. (2) when $t \to \infty$ by defining a Lyapunov functional.

Next, in the same work, Burton [4] considered the following nonlinear homogeneous vector Volterra integro-differential equation of the form

$$x'(t) = Ax(t) + \int_0^t B(t,s)E(x(s))x(s)ds, \tag{3}$$

where $t \geq 0$, x is an n-vector, $n \geq 1$, A is an $n \times n$-constant matrix, $B(t, s)$ is an $n \times n$-continuous matrix function for $0 \leq s \leq t < \infty$, and $E(x)$ is an $n \times n$-matrix valued continuous function for $x \in \Re^n$. Burton [4] discussed the stability, boundedness, and convergence of bounded solutions and the square integrability of solutions of Eq. (3) by means of a Lyapunov functional.

In this paper, instead of Eq. (3), we consider the non-linear homogeneous vector Volterra integro-differential equation with delay:

$$x'(t) = -D(t)x(t) + \int_{t-\tau}^t B(t,s)E(x(s))x(s)ds + Q(t, x(t)), \tag{4}$$

where $t \geq 0$, τ is positive constant representing a fixed delay, x is an n-vector, $n \geq 1$, $D(t)$ is an $n \times n$-continuous symmetric matrix function for all t, $t \in [0, \infty)$, $B(t, s)$ is an $n \times n$-continuous symmetric matrix function for $0 \leq s \leq t < \infty$, $E(x)$ is an $n \times n$-symmetric matrix of continuous functions, and $Q : [0, \infty) \times \Re^n \to \Re^n$ is a continuous function. We discuss the stability, uniform stability, boundedness and integrability of solutions of Eq. (4) by the Lyapunov functional approach.

In view of the mentioned information, it follows that the Volterra integro-differential equations discussed by Burton [4] have no delay. However, in this paper, the Volterra integro-differential equations to be studied are have a delay. In addition, it is clear that Eqs. (2) and (4) include the equations discussed by Burton [4], Eqs. (1) and (3), when $\tau = 0$. Our results will also be different from those obtained in the literature (see Adıvar and Raffoul [1], Becker [2], Burton [3–6], Burton and Haddock [7], Burton et al. [8], Burton and Mahfoud [9,10]), Corduneanu [11], Graef and Tunc [13], Gripenberg et al. [14], Miller [15], Raffoul [16], Raffoul and Unal [17], Staffans [18], Tunc [19], Vanualailai and Nakagiri [21] and the references thereof). In this way, we mean that the Volterra integro-differential equations discussed and the assumptions to be established here are different from those in the abovementioned papers. This paper also makes a contribution to the topic for the literature, and the paper may be useful for researchers working on the qualitative behavior of solutions to Volterra integro-differential equations with and without delay. These cases show the novelty and originality of this paper.

We give some basic information related to Eq. (2) and use the following notation throughout this paper.

For any $t_0 \geq 0$ and initial function $\varphi \in [t_0 - \tau, t_0]$, let $x(t) = x(t, t_0, \varphi)$ denote the solution of Eq. (2) on $[t_0 - \tau, \infty)$ such that $x(t) = \varphi(t)$ on $\varphi \in [t_0 - \tau, t_0]$.

Let $C[t_0, t_1]$ and $C[t_0, \infty)$ denote the set of all continuous real-valued functions on $[t_0, t_1]$ and $[t_0, \infty)$, respectively.

For $\varphi \in C[0, t_0]$, $|\varphi|_{t_0} := \sup\{|\varphi(t)| : 0 \leq t \leq t_0\}$.

Definition 1. The zero solution of Eq. (2) is stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi(t)|_{t_0} < \delta$ implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$.

Definition 2. The zero solution of Eq. (2) is uniformly stable if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi(t)|_{t_0} < \delta$ (any $t_0 \geq 0$) implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$.

Definition 3. The solutions of Eq. (2) are bounded if for each $T > 0$, there exists $D > 0$ such that

$$t_0 \geq 0, \quad \varphi \in C[0, t_0], \quad |\varphi(t)|_{t_0} < T \quad \text{and}$$
$$t \geq t_0 \quad \text{imply} \quad |x(t)| < D.$$

The following theorem is need for the stability result of this theorem and is a basic tool for our results.

Theorem 1 ((Driver [12])). If there exists a functional $V(t, \phi(.))$, defined whenever $t \geq t_0 \geq 0$ and $\phi \in C([0, t], \Re^n)$, such that

(i) $V(t, 0) \equiv 0$, V is continuous in t and locally Lipschitz in ϕ,
(ii) $V(t, \phi(.)) \geq W(|\phi(t)|)$, $W: [0, \infty) \to [0, \infty)$ is a continuous function with $W(0) = 0$, $W(r) > 0$ if $r > 0$, and W is strictly increasing (positive definiteness), and
(iii) $V'(t, \phi(.)) \leq 0$,

then the zero solution of Eq. (4) is stable, and

$$V(t, \phi(.)) = V(t, \phi(s) : 0 \leq s \leq t)$$

is called a Lyapunov functional for system (4).

The following lemma plays a key role in proving our boundedness result.

Lemma 1. If Ω be a real symmetric $n \times n$-matrix, then for any $x \in \Re^n$,

$$a_1|x|^2 \geq \langle \Omega X, X\rangle \geq a_0|x|^2,$$

where a_0 and a_1 are the least and greatest eigenvalues of Ω, respectively.

2. Results and discussion

Let

$$e(t, x(t)) \equiv 0$$

and

$$\theta(t) = a(t) - \frac{1}{2}\int_{t-\tau}^{t} |B(t, s)|ds - \frac{1}{2}m^2 \int_{t-\tau}^{\infty} |B(u, t)|du.$$

Assume that:

$(A1)$ There exist positive constants a_0, m and M such that

$g^2(x) \leq m^2 f^2(x)$ if $|x| \leq M$, $xf(x) > 0$, when $x \neq 0$,

$(A2)$ $a(t) \geq a_0 > 0$ for $t \geq 0$, $B(t, s) \geq 0$ for $0 \leq s \leq t < \infty$,

$\int_0^x |B(u + \tau, s|\, du$ is defined and continuous for $0 \leq s - \tau \leq t < \infty$.

Theorem 2. Assume that conditions $(A1)$ and $(A2)$ hold.

(i) If $\theta(t) \geq 0$ holds for $t \geq t_0 - \tau \geq 0$, then the zero solution of Eq. (2) is stable.
(ii) If $\theta(t) \geq 0$ holds for $t \geq t_0 - \tau \geq 0$, $M = \infty$, and $\int_0^x f(s)ds \to \infty$ as $|x| \to \infty$, then all solutions of Eq. (2) are bounded.
(iii) If $\theta(t) \geq \alpha_0 > 0$ holds for $t \geq t_0 - \tau \geq 0$ and $a(t)$ is bounded, then $\int_0^x f^2(s)ds < \infty$.

In addition, if $\frac{df(x)}{dx}$ is continuous, then bounded solutions of Eq. (2) tend to zero.

Proof. (i) We introduce a functional $V_0 = V_0(t) = V_0(t, x(t))$ defined by

$$V_0 = \int_0^x f(s)ds + \mu \int_0^t \int_{t-\tau}^{\infty} |B(u + \tau, s)|\, du f^2(x(s))ds, \tag{5}$$

where μ is a positive constant to be determined later in the proof.

If the assumptions of Theorem 2 hold, then it is clear that the functional V_0 is positive definite.

Differentiating the functional V_0 with respect to t, we obtain

$$\begin{aligned} V_0' &= -a(t)f^2(x) + f(x)\int_{t-\tau}^{t} B(t, s)g(x(s))ds \\ &\quad + \mu \int_{t-\tau}^{\infty} |B(u + \tau, t)|\, du f^2(x) \\ &\quad - \mu \int_0^t |B(t, s)| f^2(x(s))\, ds. \end{aligned} \tag{6}$$

By the assumptions of Theorem 2 and the estimate $|\alpha\beta| \leq 2^{-1}(\alpha^2 + \beta^2)$, it follows that

$$\begin{aligned} V_0' &\leq -a(t)f^2(x) + \frac{1}{2}\int_{t-\tau}^{t} |B(t, s)|(f^2(x(t)) \\ &\quad + g^2(x(s)))ds + \mu \int_{t-\tau}^{\infty} |B(u + \tau, t)|\, du f^2(x) \\ &\quad - \mu \int_0^t |B(t, s)| f^2(x(s))\, ds \\ &= -\left[a(t) - \frac{1}{2}\int_{t-\tau}^{t} |B(t, s)|\, ds \right. \\ &\quad \left. -\mu \int_{t-\tau}^{\infty} |B(u + \tau, t)|\, du\right] f^2(x) \\ &\quad + \frac{1}{2}\int_{t-\tau}^{t} |B(t, s)| g^2(x(s))ds \\ &\quad - \mu \int_0^t |B(t, s)| f^2(x(s))\, ds \\ &\leq -\left[a(t) - \frac{1}{2}\int_{t-\tau}^{t} |B(t, s)|\, ds \right. \\ &\quad \left. -\mu \int_{t-\tau}^{t} |B(u, t)|\, du\right] f^2(x) \\ &\quad - \left(\mu - \frac{1}{2}m^2\right) \int_{t-\tau}^{t} |B(t, s)| f^2(x(s))ds. \end{aligned}$$

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Let $\mu = \frac{1}{2}m^2$. We then have

$$V_0' \le -\theta(t) f^2(x) \le 0.$$

Hence, we can conclude that the zero solution of Eq. (2) is stable.

(ii) Integrating the estimate $V_0'(t) \le 0$ from zero to t, we have

$$\int_0^x f(s)ds + \frac{1}{2}m^2 \int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du f^2(x(s))\, ds$$
$$= V_0(t) \le V_0(t_0) = K > 0.$$

The boundedness of solutions can then be readily followed.

(iii) By (iii), it is clear that $|x'(t)|$ is bounded whenever $x(t)$ is bounded, and

$$V_0' \le -\alpha_0 f^2(x). \quad (7)$$

Integrating (7) from t_0 to t, we obtain

$$0 \le V_0(t) + \alpha_0 \int_{t_0}^t f^2(x(s))\, ds \le V_0(t_0) = K, \quad K > 0.$$

Hence, we can conclude that

$$\int_{t_0}^{\infty} f^2(x(s))ds < \infty.$$

It is now obvious that $\left(f^2(x(t))\right)'$ is bounded, so we have $f^2(x(t)) \to 0$, which implies that $x(t) \to 0$ as $t \to \infty$.

Let $e(t, x) \ne 0$. □

Theorem 3. Assume that conditions $(A1)$ with $0 < f_0 \le ((f(x))/x) \le F_0$, when $x \ne 0$, and $(A2)$ hold, where f_0 and F_0 are positive constants.

If $|e(t, x)| \le (e_0 + |x|)|q(t)|$, $|q(t)| \in L^1(0, \infty)$, and $q(t) \to 0$ as $t \to \infty$, then all solutions of Eq. (2) are bounded, where e_0 is a positive constant.

Proof. We consider the functional $V_0 = V_0(t)$ given by (5). In view of the assumption $((f(x))/x) \ge f_0 > 0$, it follows that

$$V_0 = \int_0^x \frac{f(s)}{s} sds + \mu \int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du f^2(x(s))ds$$
$$\ge \int_0^x f_0 sds + \frac{1}{2}m^2 \int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du f^2(x(s))ds$$
$$\ge \frac{1}{2} f_0 x^2.$$

Subject to the assumption that Theorem 3 holds, following the procedure of the previous proof, the following can clearly be obtained:

$$\begin{aligned}
V_0' &\le -\theta(t) f^2(x) + f(x)e(t, x) \\
&\le -\theta(t) f^2(x) + |f(x)||e(t, x)| \le |f(x)||e(t, x)| \\
&\le F_0|x||e(t, x)| \le F_0|x|(e_0 + |x|)|q(t)| \\
&= e_0 F_0|x||q(t)| + F_0|q(t)|x^2 \\
&\le e_0 F_0|q(t)| + F_0(1 + e_0)|q(t)|x^2 \\
&\le e_0 F_0|q(t)| + 2 f_0^{-1} F_0(1 + e_0)|q(t)|V_0.
\end{aligned}$$

Integrating the last estimate from zero to t, we have

$$V_0(t) \le V_0(0) + e_0 F_0 \int_0^t |q(s)|\, ds$$
$$+ 2 f_0^{-1} F_0(1 + e_0) \int_0^t V_0(s)|q(s)|ds.$$

An application of the Gronwall's inequality then leads to

$$V_0(t) \le \left\{ V_0(0) + e_0 F_0 \int_0^{\infty} |q(s)|\, ds \right\}$$
$$\times \exp\left(2 f_0^{-1} F_0(1 + e_0) \int_0^{\infty} |q(s)|\, ds \right)$$

so that

$$\frac{1}{2} f_0 x^2 \le V_0(t) \le \left\{ V_0(0) + e_0 F_0 \int_0^{\infty} |q(s)|\, ds \right\}$$
$$\times \exp\left(2 f_0^{-1} F_0(1 + e_0) \int_0^{\infty} |q(s)|ds \right).$$

The last inequality implies upper bounds for the functional V_0. Thus; we can conclude that all solutions of Eq. (2) are bounded.

Concerning Eq. (4), we now assume the following:

$(H1)$ Let $C(t)$ and $D(t)$ be $n \times n$-real symmetric and positive definite matrix functions bounded and $C(t)$ be continuously differentiable for $0 \le t < \infty$. We assume that there exists a positive constant γ such that

$$x^T[D^T(t)C(t) + C(t)D(t) - C'(t)]x = x^T R(t)x \ge \gamma |x|^2$$

for all $x \in \Re^n$, and $|C(t)| \le \lambda$, $\lambda \in \Re$, $\lambda > 0$.Define

$$\bar{\theta}(t, \lambda) = \gamma - \int_{t-\tau}^t |C(t)B(t, s)E(x(s))|\, ds$$
$$- \lambda \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du|E(x)|.$$

$(H2)$ There exists a positive constant M such that $|x| \le M$ implies $\bar{\theta}(t,\lambda) \ge 0$ for all $t \ge t_0 - \tau \ge 0$.

$(H3)$ $\int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du$ is defined and continuous for $0 \le s \le t < \infty$.

Let $Q(t, x) \equiv 0$. □

Theorem 4. Assume that conditions $(H1)$–$(H3)$ hold.

(i) The zero solution of Eq. (4) is then stable.
(ii) If $M = \infty$, then all solutions of Eq. (4) are bounded.
(iii) If $\bar{\theta}(t, k) \ge \alpha_0 > 0$, then all bounded solutions of Eq. (4) tend to zero and $\int_0^{\infty} x^T(s)x(s)\, ds < \infty$, where α_0 is a positive constant.
(iv) If $\int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du ds$ is bounded for $t_0 - \tau \le t < \infty$, then the zero solution of Eq. (4) is uniformly stable.

Proof. (i) We define a functional $V_1 = V_1(t) = V_1(t, x(t))$ by

$$V_1 = x^T C(t) x + \lambda \int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)| du |E(x(s))| x^T(s)x(s) ds. \quad (8)$$

By the assumptions of Theorem 4, it follows that the functional V_1 is positive definite.

Differentiating the functional V_1 with respect to t, we obtain

$$\begin{aligned}
V_1' &= (x')^T C(t)x + x^T C(t)x' + x^T C'(t)x + \lambda \int_{t-\tau}^{\infty} |B(u+\tau, t)| du |E(x)| x^T x - \lambda \int_0^t |B(t, s)| E(x(s))| x^T(s)x(s) ds \\
&= \left[-x^T D^T(t) + \int_{t-\tau}^{t} x^T(s) E^T(x(s)) B^T(t, s) ds \right] C(t)x + x^T C'(t)x + x^T C(t) \left[-D(t)x + \int_{t-\tau}^{t} B(t, s) E(x(s)) x(s) ds \right] \\
&\quad + \lambda \int_{t-\tau}^{\infty} |B(u+\tau, t)| du |E(x(t))| x^T x - \lambda \int_0^t |B(t, s)|\, |E(x(s))|\, x^T(s)x(s) ds \\
&= -x^T [D^T(t)C(t) + C(t)D(t) - C'(t)]x + 2x^T C(t) \int_{t-\tau}^{t} B(t, s) E(x(s)) x(s) ds + \lambda \int_{t-\tau}^{\infty} |B(u+\tau, t)|\, du |E(x)| x^T x \\
&\quad - \lambda \int_0^t |B(t, s)||E(x(s))| x^T(s)x(s)\, ds \le -x^T R(t)x + \int_{t-\tau}^{t} |C(t)B(t, s)E(x(s))| (x^T(s)x(s) + x^T(t)x(t)) ds \\
&\quad + \lambda \int_{t-\tau}^{\infty} |B(u+\tau, t)| du |E(x)| x^T x - \lambda \int_0^t |B(t, s)||E(x(s))| x^T(s)x(s) ds \\
&\le - \left[\gamma - \int_{t-\tau}^{t} |C(t)B(t, s)E(x(s))|\, ds - \lambda \int_{t-\tau}^{\infty} |B(u+\tau, t)|\, du\, |E(x)| \right] x^T x \\
&\quad - (\lambda - |C(t)|) \int_0^t |B(t, s)||E(x(s))| x^T(s)x(s) ds - \int_0^{t-\tau} |C(t)B(t, s)E(x(s))|\, x^T(s)x(s) ds \\
&\le - \left[\gamma - \int_{t-\tau}^{t} |C(t)B(t, s)E(x(s))|\, ds - \lambda \int_{t-\tau}^{\infty} |B(u+\tau, s)| du |E(x)| \right] x^T x \\
&\quad - (\lambda - |C(t)|) \int_0^t |B(t, s)| du |E(x(s))| x^T(s)x(s)\, ds.
\end{aligned}$$

It follows that the assumptions of Theorem 4 imply that

$$V_1'(t) \le 0.$$

We can conclude that the zero solution of Eq. (4) is stable.

(ii) By integrating the estimate $V_1'(t) \le 0$ from zero to t, it can be easily concluded that all solutions of Eq. (4) are bounded.

(iii) It is obvious that the right-hand side of Eq. (4) is bounded for bounded x. We can then obtain that bounded solutions tend to zero and $\int_0^{\infty} x^T(s)x(s)ds < \infty$.

(iv) By the assumption that $\int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du ds$ is bounded for $t_0 - \tau \le t < \infty$, it can be written that $\int_0^t \int_{t-\tau}^{\infty} |B(u+\tau, s)|\, du ds \le K$ for some positive constant K. In addition, for a given $\varepsilon > 0$, let φ be a continuous initial functional on an interval $[0, t_0]$ with $|\varphi|_{t_0} < \delta$. Because $V_1(t, x(.))$ is a positive definite and decreasing functional, it is clear from (8) that

$$x^T C(t)x \le V_1(t, x(.)) \le V_1(t_0, \varphi) \le \varphi^T(t_0)|C(t)|\varphi(t_0) + K\delta^2 \le \varphi^T(t_0)\lambda\varphi(t_0) + K\delta^2 \le (\lambda + K)\delta^2.$$

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This estimates leads to uniform stability of the zero solution of Eq. (4).

Let $Q(t, x) \neq 0$. □

Theorem 5. Assume that assumptions (*H*1)–(*H*3) hold with $c_1 \geq \lambda_i(C(t)) \geq c_0$, where c_0 and c_1 are the least and greatest eigenvalues of $C(t)$, respectively. If the assumption $|Q(t, x)| \leq \theta(t)(|x|+1)$, $\theta : \Re^+ \to \Re^+$, $\Re^+ = [0, \infty)$, θ is a continuous function such that $\int_0^\infty \theta(s)ds < \infty$ and $\theta(t) \to 0$ as $t \to \infty$, then all solutions of Eq. (4) are bounded.

Proof. To prove this theorem, we study the stability of solutions by using a Lyapunov functional. By noting the assumption that $c_1 \geq \lambda_i(C(t)) \geq c_0$, it follows from (8) that

$$V_1 \geq x^T C(t)x \geq c_0|x|^2.$$

In view of the assumptions of Theorem 5 and following the procedure in the proof of stability result of Theorem 4, we can easily obtain

$$V_1' \leq 2|Q(t, x)||C(t)||x|.$$

The rest of the proof can be easily completed, so we omit the details. □

3. Conclusions

A class of scalar and vector nonlinear Volterra integro-differential equations of the first order is considered. The stability, uniform stability, boundedness, convergence and integrability of solutions are discussed by using Lyapunov's functional approach. The obtained results improve some results in the literature.

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