Uniqueness of positive solutions for semilinear elliptic systems

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Abstract
We prove uniqueness of positive solutions for the system
\[ \begin{align*}
\Delta u &= -\lambda f(v), \quad \Delta v = -\mu g(u) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{align*} \]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) and \( f(x) \sim x^p, \ g(x) \sim x^q \) at \( \infty \) for some positive numbers \( p, q \) with \( pq < 1 \).

Keywords: Uniqueness; Positive solutions; Semilinear systems

1. Introduction

Consider the boundary value problem
\[ \begin{align*}
\Delta u &= -\lambda f(v) \quad \text{in } \Omega, \\
\Delta v &= -\mu g(u) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{align*} \]  

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where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \), \( f, g : \mathbb{R}^+ \to \mathbb{R}^+ \), and \( \lambda, \mu \) are positive parameters.

In [2], Dalmasso proved existence and uniqueness of positive solutions to (I) when the composition \( f o (cg) \) is sublinear at \( \infty \) and superlinear at 0 for each \( c > 0 \). The uniqueness part in [2] required that \( f(x)/x^p \) and \( g(x)/x^q \) are nonincreasing on \( \mathbb{R}^+ \) for some \( p, q > 0 \) with \( pq < 1 \). In the case when \( f(x)/x^p \) and \( g(x)/x^q \) are nonincreasing only for large \( x \), it was established in [5] that the system (I) has a unique positive solution provided that \( \min(\lambda, \mu) < \frac{1}{\lambda^p} \) is sufficiently large, and \( \Omega \) is a ball. The arguments in [5] rely on the fact that positive solutions to (I) in a ball are radially symmetric and decreasing (see [4,8]). In this paper, we shall extend the results in [5] to the case of a bounded domain in \( \mathbb{R}^N \). In particular, our results give uniqueness for a class of sublinear bi-harmonic boundary value problems. Similar results for the single equation case were obtained in [3,6]. Our approach is based on sub- and super-solutions, and maximum principles.

2. Existence and uniqueness results

We make the following assumptions:

(H.1) \( f, g : \mathbb{R}^+ \to \mathbb{R}^+ \) are nondecreasing, continuous, \( C^1 \) on \((0, \infty)\), and

\[
\lim_{x \to 0^+} \sup_{x} xf'(x) < \infty, \quad \lim_{x \to 0^+} \sup_{x} x g'(x) < \infty.
\]

(H.2) There exist positive numbers \( \beta, \delta, p, q \) with \( pq < 1 \), such that

\[
\beta x^p \leq f(x) \leq \delta x^p, \quad \beta x^q \leq g(x) \leq \delta x^q
\]

for all \( x \geq 0 \), and for \( p_1 > p, q_1 > q \),

\[
\frac{f(x)}{x^{p_1}} \quad \text{and} \quad \frac{g(x)}{x^{q_1}}
\]

are nonincreasing for \( x \) large.

Our main result is

**Theorem 1.** Let (H.1)–(H.2) hold. Then the system (I) has a unique positive solution for \( \min(\lambda, \mu) \) large.

The next lemma provides sharp estimates for solutions to (I). When \( \Omega \) is a ball, it was established in [5].

We shall denote the norm in \( C^k(\bar{\Omega}) \) by \( \| \cdot \|_k \).

**Lemma 1.** Let \((u, v)\) be a positive solution of (I). Then there exist positive constants \( M_1 \) and \( M_i, 1 \leq i \leq 4 \), such that

\[
M_1(\lambda, \mu) \frac{1}{\lambda^p} d(x, \partial \Omega) \leq u(x) \leq M_2(\lambda, \mu) \frac{1}{\lambda^p} d(x, \partial \Omega),
\]

\[
M_3(\mu, \lambda) \frac{1}{\mu^q} d(x, \partial \Omega) \leq v(x) \leq M_4(\mu, \lambda) \frac{1}{\mu^q} d(x, \partial \Omega)
\]

for \( \min(\lambda, \mu) \) large. Here \( d(x, \partial \Omega) \) denotes the distance from \( x \) to \( \partial \Omega \).
Proof. Let \((u, v)\) be a positive solution of (I). We first establish the upper estimate for \(v\). In what follows, we shall denote by \(C_i\) positive constants independent of \(\lambda, \mu, u, v\). Using the equations for \(u, v\), we obtain
\[
 u(x) = \lambda \int_{\Omega} K(x, y) f(v(y)) \, dy, \quad v(x) = \mu \int_{\Omega} K(x, y) g(u(y)) \, dy,
\]
where \(K(x, y)\) denotes the Green’s function of \(-\Delta\) with Dirichlet boundary conditions. Thus, by (H.2),
\[
 |u|_0 \leq \lambda C f(|v|_0) \leq \lambda C_1 \delta |v|_0^p \tag{1}
\]
and
\[
 |v|_0 \leq \mu C g(|u|_0) \leq \mu C_1 \delta |u|_0^q \tag{2}
\]
From (1) and (2), it follows that
\[
 |u|_0 \leq C_2 \left( \lambda \mu^p \right)^{\frac{1}{1-pq}},
\]
which, together with (H.2) and regularity estimates, implies
\[
 |v|_1 \leq \mu C_3 |g(u)|_0 \leq \mu C_3 \delta |u|^q_0 \leq \mu C_3 \delta C_2 \left( \lambda \mu^p \right)^{\frac{q}{1-pq}} \equiv M_4 \left( \mu \lambda^q \right)^{\frac{1}{1-pq}},
\]
and
\[
 v(x) \leq M_4 \left( \mu \lambda^q \right)^{\frac{1}{1-pq}} d(x, \partial \Omega)
\]
follows from the mean value theorem. The upper estimate for \(u\) follows in the same manner.

Next, let \(x_0 \in \Omega\) and \(R > 0\) be such that \(B \equiv B(x_0, R) \subset \Omega\). Here \(B(x_0, R)\) denotes the open ball centered at \(x_0\) with radius \(R\). Then \((u, v)\) is a supersolution for
\[
 \begin{cases}
 \Delta \bar{u} = -\lambda f(\bar{v}) & \text{in } B, \\
 \Delta \bar{v} = -\mu g(\bar{u}) & \text{in } B, \\
 \bar{u} = \bar{v} = 0 & \text{on } \partial B. 
\end{cases}
\]
(I*)

We shall construct a positive subsolution \((u_0, v_0)\) for (I*) with \(u_0 \leq u\) and \(v_0 \leq v\) in \(B\). To this end, let \(\varepsilon > 0\) and let \(\tilde{u}, \tilde{v}\) be the solution of
\[
 \begin{cases}
 \Delta \tilde{u} = -\mu \beta \varepsilon^{1/p} \tilde{v}^p & \text{in } B, \\
 \Delta \tilde{v} = -\mu \beta \varepsilon^{1/p} \tilde{u}^q & \text{in } B, \\
 \tilde{u} = \tilde{v} = 0 & \text{on } \partial B, 
\end{cases}
\]
whose existence follows from [2,5]. Define \(u_0 = \varepsilon^{1/pq} \tilde{u}, v_0 = \mu \beta \varepsilon^{1/p} \tilde{v}\), where \(\beta\) is given by (H.2). A direct calculation gives
\[
 \Delta u_0 = -\varepsilon^{(1/pq)} \tilde{v}^p \geq -\lambda \beta \left( \mu \beta \varepsilon^{1/p} \tilde{v} \right)^p \geq -\lambda f \left( \mu \beta \varepsilon^{1/p} \tilde{v} \right) = -\lambda f(v_0)
\]
if \(\lambda \mu^p > 1\) and \(\varepsilon\) is sufficiently small, and
\[
 \Delta v_0 = -\mu \beta \varepsilon^{1/p} \tilde{u}^q \geq -\mu \beta \varepsilon^{1/pq} \tilde{u}^q \geq -\mu g \left( \varepsilon^{1/pq} \tilde{u} \right) = -\mu g(u_0),
\]
i.e., \((u_0, v_0)\) is a subsolution for \((I^*)\). Clearly \(u_0 \leq u\) and \(v_0 \leq v\) in \(B\) for small \(\varepsilon\). Hence there exists a solution \((\tilde{u}, \tilde{v})\) to \((I^*)\) with \(\tilde{u} \leq u, \tilde{v} \leq v\). Since \(\tilde{u}\) is radially symmetric, it follows from [5, Lemma 4] that

\[
    u(x) \geq \tilde{M}_1(\lambda \mu p)^{\frac{1}{1-pq}} \text{ for } |x - x_0| \leq \frac{R}{2}, \tag{3}
\]

for \(\min(\lambda \mu p, \mu \lambda q)\) large, where \(\tilde{M}_1\) is a positive constant independent of \(u, v, \lambda, \mu\).

Let \(\Omega = \Omega \setminus B(x_0, R/2)\) and let \(\phi\) be the solution of

\[
    \begin{cases}
    \Delta \phi = 0 & \text{in } \tilde{\Omega}, \\
    \phi = 0 & \text{on } \partial \Omega, \\
    \phi = 1 & \text{on } \partial B(x_0, R/2).
    \end{cases}
\]

Since \(\Delta u \leq 0\) in \(\Omega\), the maximum principle (see, e.g., [1,7]) implies

\[
    u(x) \geq \tilde{M}_1(\lambda \mu p)^{\frac{1}{1-pq}} \phi(x) \geq \tilde{M}_1(\lambda \mu p)^{\frac{1}{1-pq}} d(x, \partial \Omega) \text{ in } \tilde{\Omega},
\]

where \(\tilde{M}_1\) is a positive constant satisfying \(\tilde{M}_1 \phi(x) \geq M_1 d(x, \partial \Omega)\) for \(x \in \tilde{\Omega}\). Combine this and (3), we obtain the lower estimate for \(u\). This completes the proof of Lemma 1. \(\square\)

**Lemma 2.** Let \((u, v)\) be a solution to \((I)\) and let \(w_0\) satisfy

\[
    \begin{cases}
    \Delta w_0 = -g(u) & \text{in } \Omega, \\
    w_0 = 0 & \text{on } \partial \Omega.
    \end{cases}
\]

Then for \(\min(\lambda \mu p, \mu \lambda q)\) large, there exists a positive number \(c\) independent of \(u, v, \lambda, \mu\), such that

\[
    w_0(x) \geq cd(x, \partial \Omega) \text{ for } x \in \Omega.
\]

**Proof.** Let \(\varepsilon_0 > 0\). It follows from (H.2) and Lemma 1 that for \(\min(\lambda \mu p, \mu \lambda q)\) large,

\[
    g(u(x)) \geq \beta(u(x))^q \geq \beta\left[M_1(\lambda \mu p)^{\frac{1}{1-pq}} d(x, \partial \Omega)\right]^q > 1
\]

if \(d(x, \partial \Omega) > \varepsilon_0\). Thus

\[
    \Delta w_0 \leq \begin{cases}
    -1 & \text{if } d(x, \partial \Omega) > \varepsilon_0, \\
    0 & \text{if } d(x, \partial \Omega) \leq \varepsilon_0,
    \end{cases}
\]

and the lemma follows from the maximum principle.

We are in a position to give the

**Proof of Theorem 1.** The existence part follows from [2]. Let \((u, v)\) and \((u_1, v_1)\) be solutions to \((I)\) and suppose that \(\min(\lambda \mu p, \mu \lambda q)\) is large enough so Lemmas 1 and 2 apply. By Lemma 1,

\[
    \frac{M_1}{M_2} u_1 \leq u \leq \frac{M_2}{M_1} u_1 \text{ in } \Omega.
\]
Let \( \alpha = \sup \{ c > 0 : u \geq cu_1 \text{ in } \Omega \} \). Then \( \alpha_0 \leq \alpha \leq \alpha_0^{-1} \), where \( \alpha_0 = M_1/M_2 \). We claim that \( \alpha \geq 1 \). Suppose to the contrary that \( \alpha < 1 \). Let \( q_1, q_2, p_1, p_2 > 0 \) be such that \( q_2 > q_1 > q, p_2 > p \) and \( p_2 q_2 < 1 \). Let \( A > 0 \) be such that

\[
\frac{g(x)}{x^{q_1}} \text{ is nonincreasing for } x > A,
\]

and define \( \Omega_1 = \{ x \in \Omega : u_1(x) > A/\alpha_0 \} \). Then

\[
g\left(\alpha u_1(x)\right) \geq \alpha^{q_1} g\left(u_1(x)\right) \text{ for } x \in \Omega_1,
\]

while if \( x \in \Omega \setminus \Omega_1 \),

\[
\left| g\left(u_1(x)\right) - g\left(\alpha u_1(x)\right) \right| \leq K (1 - \alpha),
\]

where \( K = \frac{1}{\alpha_0} \sup \{|xg'(x)| : 0 < x \leq A/\alpha_0\} \), which implies

\[
g\left(\alpha u_1(x)\right) \geq g\left(u_1(x)\right) - K (1 - \alpha) \text{ for } x \in \Omega \setminus \Omega_1.
\]

Define the operator \( T : C(\bar{\Omega}) \to C(\bar{\Omega}) \) by \( Tz = w \) if

\[
\Delta w = -z \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega.
\]

Let \( w = Tg(\alpha u_1) \). Then it follows from (4), (5) and the maximum principle that \( w \geq \bar{w} \), where \( \bar{w} \) satisfies

\[
\Delta \bar{w} = \begin{cases} -\alpha^{q_1} g(u_1) & \text{in } \Omega_1, \\ -g(u_1(x)) + K (1 - \alpha) & \text{in } \Omega \setminus \Omega_1, \end{cases} \quad \bar{w} = 0 \text{ on } \partial \Omega.
\]

Let \( w_0 = Tg(u_1) \). Then \( \Delta w_0 = -g(u_1) \) and therefore

\[
\Delta \left( \bar{w} - \alpha^{q_1} w_0 \right) = \begin{cases} 0 & \text{in } \Omega_1, \\ (\alpha^{q_1} - 1) g(u_1) + K (1 - \alpha) & \text{in } \Omega \setminus \Omega_1. \end{cases}
\]

Note that there exists a positive constant \( K_1 \) depending only on \( A, \alpha_0, K, q_1 \) such that

\[
\left| (\alpha^{q_1} - 1) g(u_1) + K (1 - \alpha) \right| \leq K_1 (1 - \alpha) \text{ in } \Omega \setminus \Omega_1.
\]

Using regularity estimates, we obtain for \( r > N \),

\[
\left| \bar{w} - \alpha^{q_1} w_0 \right|_1 \leq C K_1 (1 - \alpha) \left( \int_{\Omega \setminus \Omega_1} dx \right)^{1/r}.
\]

Since

\[
M_1 \left( \lambda \mu^p \right)^{1 - \frac{1}{nq}} d(x, \partial \Omega) \leq u_1(x) \leq \frac{A}{\alpha_0} \text{ on } \Omega \setminus \Omega_1,
\]

it follows that

\[
d(x, \partial \Omega) \leq \frac{A}{\alpha_0 M_1 \left( \lambda \mu^p \right)^{1 - \frac{1}{nq}}} \text{ for } x \in \Omega \setminus \Omega_1,
\]

and therefore the right-hand side of (6) goes to 0 as \( \lambda \mu^p \to \infty \). Let \( \varepsilon > 0 \), then it follows from (6) and the mean value theorem that

\[
\bar{w}(x) - \alpha^{q_1} w_0(x) \geq -\varepsilon (1 - \alpha) d(x, \partial \Omega), \quad x \in \Omega,
\]
for $\min(\lambda, \mu^p, \mu^q)$ large, which implies by Lemma 2 that
\[
\bar{w}(x) - \alpha_q^2 w_0(x) \geq (\alpha_q^1 - \alpha_q^2) w_0(x) - \epsilon(1 - \alpha) d(x, \partial \Omega) \\
\geq c \alpha_q^1 (1 - \alpha_q^{q_2 - q_1}) d(x, \partial \Omega) - \epsilon(1 - \alpha) d(x, \partial \Omega) \\
\geq \left[ \min(1, q - q_1) c \alpha_q^{q_1} - \epsilon \right] (1 - \alpha) d(x, \partial \Omega) > 0
\]
if $\epsilon$ is sufficiently small. Consequently, $w(x) \geq \bar{w}(x) \geq \alpha_q^2 w_0(x)$, or
\[
T g(\alpha u_1) \geq \alpha^q_2 T g(u_1).
\] (7)

Since
\[
\Delta v = -\mu g(u) \leq -\mu g(\alpha u_1),
\]
it follows from (7) that
\[
v \geq \mu T g(\alpha u_1) \geq \mu \alpha^q_2 T g(u_1) = \alpha^q_2 v_1.
\]
This implies
\[
\Delta u = -\lambda f(v) \leq -\lambda f(\alpha^q_2 v_1).
\] (8)

Now similarly
\[
T f(\alpha^q_2 v_1) \geq \alpha^{p_2 q_2} T f(v_1).
\] (9)

(8), (9) and the maximum principle imply
\[
u \geq \lambda T f(\alpha^q_2 v_1) \geq \lambda \alpha^{p_2 q_2} T f(v_1) = \alpha^{p_2 q_2} u_1,
\]
which is a contradiction since $\alpha^{p_2 q_2} > \alpha$. Thus $\alpha \geq 1$, i.e., $u \geq u_1$ and therefore $u = u_1$. Similarly, $v = v_1$, completing the proof of Theorem 1.

As a consequence of Theorem 1, consider the fourth order boundary value problem
\[
\begin{align*}
\Delta^4 u &= -\mu g(u) \quad \text{in } \Omega, \\
u &= \Delta u = 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (II)

where $\mu$ is a positive parameter. It is assumed that

(H.3) $g : R^+ \rightarrow R^+$ is continuous, nondecreasing, $C^1$ on $(0, \infty)$, and
\[
\lim_{x \to 0^+} \sup_{x g'(x)} < \infty.
\]

(H.4) There exist positive numbers $q, \beta, \delta$ with $q < 1$, such that
\[
\beta x^q \leq g(x) \leq \delta x^q
\]
for all $x \geq 0$, and for $q_1 > q$,
\[
\frac{g(x)}{x^{q_1}} \text{ is nonincreasing for } x \text{ large.}
\]

Then we have
Theorem 2. Let (H.3)–(H.4) hold. Then the problem (II) has a unique positive solution for $\mu$ large.

Proof. Let $v = \Delta u$, then $\Delta v = -\mu g(u)$, and uniqueness follows from Theorem 1 if $\mu$ is sufficiently large. Here $\lambda = 1$. \qed

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References