The relationship between metrics and $\mathcal{F}$-equalities is investigated; the latter are a special case of $\mathcal{F}$-equivales, a natural generalization of the classical concept of an equivalence relation. It is shown that in the construction of metrics from $\mathcal{F}$-equalities triangular norms with an additive generator play a key role. Conversely, in the construction of $\mathcal{F}$-equalities from metrics this role is played by triangular norms with a continuous additive generator or, equivalently, by continuous Archimedean triangular norms. These results are then applied to the biresidual operator $\mathcal{E}$ of a triangular norm $\mathcal{F}$. It is shown that $\mathcal{E}$ is a $\mathcal{F}$-equality on $[0, 1]$ if and only if $\mathcal{F}$ is left-continuous. Furthermore, it is shown that to any left-continuous triangular norm $\mathcal{F}$ there correspond two particular $\mathcal{F}$-equalities on $\mathcal{F}(X)$, the class of fuzzy sets in a given universe $X$; one of these $\mathcal{F}$-equalities is obtained from the biresidual operator $\mathcal{E}$ by means of a natural extension procedure. These $\mathcal{F}$-equalities then give rise to interesting metrics on $\mathcal{F}(X)$.

**Key Words:** additive generator; Archimedean property; biresidual operator; metric; $\mathcal{F}$-equality; triangular norm.

1. INTRODUCTION

The concept of a similarity relation was introduced by Zadeh [17] as a generalization of the concept of an equivalence relation. Also, simi-
Larity relations have been generalized by replacing the min-transitivity with the more general \( \mathcal{T} \)-transitivity, with \( \mathcal{T} \) an arbitrary triangular norm (t-norm) [12].

**Definition 1** ([4]). Consider a t-norm \( \mathcal{T} \). A binary fuzzy relation \( E \) in a universe \( X \) is called a \( \mathcal{T} \)-equivalence on \( X \) if it is reflexive, symmetric, and \( \mathcal{T} \)-transitive, i.e., if for any \((x, y, z)\) in \( X^3 \),

\[
\begin{align*}
(E1) & \quad E(x, x) = 1; \\
(E2) & \quad E(x, y) = E(y, x); \quad \text{and} \\
(E3) & \quad \mathcal{T}(E(x, y), E(y, z)) \leq E(x, z).
\end{align*}
\]

\( \mathcal{T} \)-equivalences are also called indistinguishability operators [13], fuzzy equalities [8], and equality relations [9]. Clearly, \( M \)-equivalences (with \( M \) the minimum operator) are nothing but similarity relations. \( W \)-equivalences (with \( W \) the Łukasiewicz t-norm defined by \( W(x, y) = \max(x + y - 1, 0) \)) are called likeness relations. A one-to-one correspondence between \( \mathcal{T} \)-equivalences and \( \mathcal{T} \)-partitions, a generalization of the concept of a partition, was recently exposed in [4].

In this paper, we deal with \( \mathcal{T} \)-equalities, a special type of \( \mathcal{T} \)-equivalence.

**Definition 2**. Consider a t-norm \( \mathcal{T} \). A \( \mathcal{T} \)-equivalence \( E \) in a universe \( X \) is called a \( \mathcal{T} \)-equality on \( X \) if for any \((x, y)\) in \( X^2 \),

\[
(E1') \quad E(x, y) = 1 \iff x = y.
\]

Recall that a t-norm \( \mathcal{T}^* \) is called weaker than a t-norm \( \mathcal{T} \), denoted \( \mathcal{T}^* \leq \mathcal{T} \), if \( \forall (x, y) \in [0, 1]^2 \) \( \mathcal{T}^*(x, y) \leq \mathcal{T}(x, y) \). The following proposition then is immediate.

**Proposition 1**. Consider a binary fuzzy relation \( E \) in a universe \( X \) and a t-norm \( \mathcal{T} \). If \( E \) is a \( \mathcal{T} \)-equivalence (resp., \( \mathcal{T} \)-equality), then it is also a \( \mathcal{T}^* \)-equivalence (resp., \( \mathcal{T}^* \)-equality) for any t-norm \( \mathcal{T}^* \) that is weaker than \( \mathcal{T} \).

Bezdek and Harris [1] have discussed the relationship between likeness relations and pseudo-metrics. More general investigations into the relationship between pseudo-metrics and \( \mathcal{T} \)-equivalences were done by Wagenknecht [15]. A complete study was carried out by De Baets and Mesiar [5].

**Definition 3**. An \( X^2 \to [0, \infty] \) mapping \( d \) is called a pseudo-metric on \( X \) if for any \((x, y, z)\) in \( X^3 \),

\[
\begin{align*}
(P1) & \quad d(x, x) = 0; \\
(P2) & \quad d(x, y) = d(y, x); \quad \text{and} \\
(P3) & \quad d(x, z) \leq d(x, y) + d(y, z).
\end{align*}
\]
In this paper, we will show that $\mathcal{T}$-equalities are related to metrics as $\mathcal{T}$-equivalences are to pseudo-metrics.

**Definition 4.** A pseudo-metric $d$ on $X$ is called a metric if for any $(x, y)$ in $X^2$,

\[(P1') \quad d(x, y) = 0 \iff x = y.\]

2. ADDITIVE GENERATORS AND ARCHIMEDEAN t-NORMS

In this section, we recall some important results concerning additive generators of t-norms (see e.g. [10–12]) and the relationship to the Archimedean property.

**Definition 5.** A strictly decreasing $[0, 1] \to [0, \infty]$ mapping $f$ with $\text{Rng}(f)$ relatively closed under addition, i.e.,

\[(\forall (u, v) \in \text{Rng}(f)^2) (u + v \in \text{Rng}(f) \lor u + v > f(0)),\]

such that $f(1) = 0$, is called an additive generator.

**Definition 6.** Consider a $[0, 1] \to [0, \infty]$ mapping $f$; then the pseudo-inverse of $f$ is the $[0, \infty] \to [0, 1]$ mapping $f^{-1}$ defined by

\[f^{-1}(x) = \inf \{ t \mid t \in [0, 1] \land f(t) \leq x \}.\]

Note that this pseudo-inverse is always decreasing. The pseudo-inverse $f^{-1}$ of a continuous additive generator $f$ is given by

\[f^{-1}(x) = f^{-1}(\min(f(0), x)).\]

**Theorem 1.** Consider an additive generator $f$; then the $[0, 1]^2 \to [0, 1]$ mapping $\mathcal{T}$ defined by

\[\mathcal{T}(x, y) = g(f(x) + f(y)),\]

where $g$ is an arbitrary $[0, \infty] \to [0, 1]$ mapping such that

\[g(x) = \begin{cases} f^{-1}(x), & \text{if } x \in \text{Rng}(f), \\ 0, & \text{if } x > f(0), \end{cases}\]

is a t-norm.

A suitable candidate for the mapping $g$ in the foregoing theorem is the pseudo-inverse $f^{-1}$ of $f$.

The continuity of an additive generator $f$ is equivalent with its left-continuity in the point 1 and with the continuity of the generated t-norm $\mathcal{T}$. Note that if a continuous t-norm $\mathcal{T}$ has an additive generator $f$, then this additive generator is uniquely determined up to a nonzero positive multiplicative constant.
Example 1. (i) The mapping \( f \) defined by \( f(x) = -\log x \) is an additive generator of the algebraic product, i.e., of the t-norm \( P \) defined by \( P(x, y) = xy \).

(ii) The mapping \( f \) defined by \( f(x) = 1 - x \) is an additive generator of the Łukasiewicz t-norm \( W \).

(iii) The mapping \( f \) defined by
\[
 f(x) = \begin{cases} 
 2 - x, & \text{if } x \in [0, 1[, \\
 0, & \text{if } x = 1, 
\end{cases}
\]
is an additive generator of the weakest t-norm \( Z \) defined by
\[
 Z(x, y) = \begin{cases} 
 \min(x, y), & \text{if } \max(x, y) = 1, \\
 0, & \text{otherwise}. 
\end{cases}
\]

Not all t-norms have an additive generator. An example of such a t-norm is the minimum operator \( M \). The fact that a t-norm has an additive generator is closely related to the Archimedean property.

Definition 7. A t-norm \( T \) is called Archimedean if
\[
(\forall (x, y) \in ]0, 1[) (\exists n \in \mathbb{N}) (x^n < y),
\]
where \( x^n \) stands for \( T(x, \ldots, x) \) (\( n \) times).

Proposition 2. A continuous t-norm \( T \) is Archimedean if and only if
\[
(\forall x \in ]0, 1[) (T(x, x) < x).
\]

Each t-norm with an additive generator is Archimedean. The converse is not true in general, but holds for instance for continuous t-norms.

Theorem 2. A \([0, 1]^2 \to [0, 1]\) mapping \( T \) is a continuous Archimedean t-norm if and only if there exists a continuous additive generator \( f \) such that
\[
 T(x, y) = f^{-1}(f(x) + f(y)).
\]

3. PSEUDO-METRICS AND \( T \)-EQUIVALENCES

In this section, we briefly recall our previous results concerning the construction of pseudo-metrics from \( T \)-equivalences, and vice versa.

If the cardinality of the universe \( X \) is smaller than 3, then for any t-norm \( T \), any \( T \)-equivalence \( E \) on \( X \), and any additive generator \( f \) it holds that the mapping \( d = f \circ E \) is a pseudo-metric on \( X \); in fact, any \([0, 1] \to [0, \infty]\) mapping \( f \) such that \( f(1) = 0 \) will do here. Therefore, only universes with higher cardinality are of interest to us.

Theorem 3 ([5]). Consider a universe \( X \) with \( \#X > 2 \), a t-norm \( T^* \) with additive generator \( f \), and a t-norm \( T \). Then the following statements
are equivalent:

(i) $\mathcal{T}^*$ is weaker than $\mathcal{T}$; i.e., $\mathcal{T}^* \leq \mathcal{T}$.

(ii) For any $\mathcal{T}$-equality $E$ on $X$, the $X^2 \to [0, \infty]$ mapping $d = f \circ E$ is a pseudo-metric on $X$.

In the converse problem, namely the construction of $\mathcal{T}$-equivalences from pseudo-metrics, continuous additive generators play an important role. In a counterexample, we have shown that this continuity requirement cannot be dropped [5].

**Proposition 3 ([5]).** Consider a pseudo-metric $d$ on a universe $X$ and a continuous Archimedean $t$-norm $\mathcal{T}^*$ with additive generator $f$; then the binary fuzzy relation $E = f^{(-1)} \circ d$ in $X$ is a $\mathcal{T}^*$-equivalence on $X$.

4. METRICS AND $\mathcal{T}$-EQUALITIES

The results from the previous section can be made more specific for metrics and $\mathcal{T}$-equalities. We will show how to construct metrics from $\mathcal{T}$-equalities and vice versa.

**Theorem 4.** Consider a universe $X$ with $\#X > 2$, a $t$-norm $\mathcal{T}^*$ with additive generator $f$, and a $t$-norm $\mathcal{T}$. Then the following statements are equivalent:

(i) $\mathcal{T}^*$ is weaker than $\mathcal{T}$; i.e., $\mathcal{T}^* \leq \mathcal{T}$.

(ii) For any $\mathcal{T}$-equality $E$ on $X$, the $X^2 \to [0, \infty]$ mapping $d = f \circ E$ is a metric on $X$.

**Proof.** We will first prove the implication (i) $\Rightarrow$ (ii). Suppose that $\mathcal{T}^* \leq \mathcal{T}$. Since any $\mathcal{T}$-equality is a $\mathcal{T}$-equivalence, it follows from Theorem 3 that $d$ is a pseudo-metric on $X$. Now consider $x$ and $y$ in $X$ such that $d(x, y) = 0$; then we have to show that $x = y$. From $d(x, y) = 0$ it follows that $f(E(x, y)) = 0$. Since $f$ is strictly decreasing and $f(1) = 0$, it follows that $E(x, y) = 1$, whence $x = y$.

Next, we prove the implication (ii) $\Rightarrow$ (i). Consider $(a, b) \in [0, 1]^2$; then we have to show that $\mathcal{T}^*(a, b) \leq \mathcal{T}(a, b)$. If $a = 1$ or $b = 1$, then always $\mathcal{T}^*(a, b) = \mathcal{T}(a, b)$. We can therefore assume that $(a, b) \in [0, 1]^2$. We construct the following binary fuzzy relation $E$ in $X$: First, for all $u$ in $X$ we put $E(u, u) = 1$. Next, we consider three different elements $x$, $y$, and $z$ of $X$ and define

$$
E(x, y) = a,
$$
$$
E(y, z) = b,
$$
$$
E(x, z) = \mathcal{T}(a, b).
$$
Furthermore, for any $u$ and $v$ in $X\setminus\{x, y, z\}$, $u \neq v$, we put $E(u, v) = 0$. One easily verifies that $E$ is a $\mathcal{T}$-equality on $X$. It then holds that the mapping $d = f \circ E$ is a metric on $X$. This means in particular that

$$f(E(x, z)) \leq f(E(x, y)) + f(E(y, z)).$$

Since $f^{(-1)}$ is decreasing, it follows that

$$f^{(-1)}(f(E(x, z))) \geq f^{(-1)}(f(E(x, y)) + f(E(y, z))).$$

Since $f^{(-1)}(f(E(x, z))) = E(x, z)$, it then follows that $\mathcal{T}(a, b) \geq \mathcal{T}^*(a, b)$. 

**Corollary 1.** (i) Consider an $M$-equality $E$ on $X$; then for any additive generator $f$ it holds that the mapping $d = f \circ E$ is a metric on $X$.

(ii) Consider an arbitrary $t$-norm $\mathcal{T}$ and a $\mathcal{T}$-equality $E$ on $X$; then the mapping $d = f \circ E$, with $f$ an additive generator of the weakest $t$-norm $Z$, is a metric on $X$.

(iii) Consider a $t$-norm $\mathcal{T}$ such that $W \leq \mathcal{T}$ and a $\mathcal{T}$-equality $E$ on $X$; then the mapping $d = 1 - E$ is a metric on $X$.

**Proposition 4.** Consider a metric $d$ on a universe $X$ and a continuous Archimedean $t$-norm $\mathcal{T}^*$ with additive generator $f$; then the binary fuzzy relation $E = f^{(-1)} \circ d$ in $X$ is a $\mathcal{T}^*$-equality on $X$.

**Proof.** According to Proposition 3, $E$ is a $\mathcal{T}^*$-equivalence on $X$. Now consider $x$ and $y$ in $X$ such that $E(x, y) = 1$; then we have to show that $x = y$. From $E(x, y) = 1$ it follows that $f^{(-1)}(d(x, y)) = 1$; i.e., $f^{-1}(\min(f(0), d(x, y))) = 1$. Since $f$ is strictly decreasing and $f(1) = 0$, it follows that $\min(f(0), d(x, y)) = 0$. Since $f(0) > f(1) = 0$, we can conclude that $d(x, y) = 0$, whence $x = y$.

**Corollary 2.** Consider a metric $d$ on $X$, then the binary fuzzy relation $E = \max(1 - d, 0)$ is a $W$-equality on $X$.

5. THE BIRESIDUAL OPERATOR OF A $t$-NORM

5.1. Definition and Properties

In the following section we study two particular $\mathcal{T}$-equalities on $\mathcal{F}(X)$. One of them is based on the biresidual operator $\mathcal{E}_f$ of a $t$-norm $\mathcal{T}$ that is used for measuring the degree of equality of real numbers taken from the unit interval. In fact, we show that the biresidual operator $\mathcal{E}_f$ of a $t$-norm $\mathcal{T}$ is a $\mathcal{T}$-equality on $[0, 1]$ if and only if $\mathcal{T}$ is left-continuous. Note that by a left-continuous $t$-norm we mean a $t$-norm with left-continuous partial mappings.
Consider a t-norm $\mathcal{T}$. The residual implicator $\mathcal{I}_\mathcal{T}$ of $\mathcal{T}$ is the binary operator on $[0, 1]$ defined by

$$\mathcal{I}_\mathcal{T}(x, y) = \sup\{z \mid z \in [0, 1] \wedge \mathcal{T}(x, z) \leq y\}.$$ 

Note that the residual implicator $\mathcal{I}_\mathcal{T}$ is hybrid monotonous; i.e., it has decreasing first and increasing second partial mappings.

**Proposition 5 ([6]).** Consider a continuous Archimedean t-norm $\mathcal{T}$ with an additive generator $f$; then its residual implicator $\mathcal{I}_\mathcal{T}$ is given by

$$\mathcal{I}_\mathcal{T}(x, y) = f^{-1}(\max(0, f(y) - f(x))).$$

**Proposition 6 ([6]).** Consider a t-norm $\mathcal{T}$; then the following properties hold, for any $(x, y)$ in $[0, 1]^2$:

(i) $x \leq y \Rightarrow \mathcal{I}_\mathcal{T}(x, y) = 1$;
(ii) $\mathcal{I}_\mathcal{T}(1, y) = y$ (the neutrality principle); and
(iii) $\mathcal{I}_\mathcal{T}(x, \mathcal{T}(x, y)) \geq y$.

**Proposition 7 ([6]).** Consider a left-continuous t-norm $\mathcal{T}$; then the following equivalence holds, for any $(x, y)$ in $[0, 1]^2$:

$$x \leq y \Leftrightarrow \mathcal{I}_\mathcal{T}(x, y) = 1.$$ 

**Theorem 5.** Consider a t-norm $\mathcal{T}$; then the following statements are equivalent:

(i) $\mathcal{T}$ is left-continuous;
(ii) $(\forall (x, y) \in [0, 1]^2) (\mathcal{T}(x, \mathcal{I}_\mathcal{T}(x, y)) \leq y)$; and
(iii) $(\forall (x, y, z) \in [0, 1]^3) (\mathcal{T}(\mathcal{I}_\mathcal{T}(x, y), \mathcal{I}_\mathcal{T}(y, z)) \leq \mathcal{I}_\mathcal{T}(x, z)).$

**Proof.** The implication (i) $\Rightarrow$ (ii) is well-known (see e.g. [2]). We will prove the converse implication. For $\mathcal{T}$ to be left-continuous, it suffices to show that for any $x$ in $[0, 1]$ and for any nonempty family $(y_i)_{i \in I}$ in $[0, 1]$ the following equality holds:

$$\mathcal{T}\left(x, \sup_{i \in I} y_i\right) = \sup_{i \in I} \mathcal{T}(x, y_i).$$

For any $i \in I$ it holds that $\mathcal{T}(x, y_i) \leq \sup_{i \in I} \mathcal{T}(x, y_i)$, whence

$$y_i \leq \mathcal{I}_\mathcal{T}\left(x, \sup_{i \in I} \mathcal{T}(x, y_i)\right)$$

and also

$$\sup_{i \in I} y_i \leq \mathcal{I}_\mathcal{T}\left(x, \sup_{i \in I} \mathcal{T}(x, y_i)\right).$$
The monotonicity of $\mathcal{T}$ and (ii) then imply that

$$\mathcal{T}\left(x, \sup_{i \in I} y_i\right) \leq \mathcal{T}\left(x, \mathcal{I}_x(x, \sup_{i \in I} \mathcal{T}(x, y_i))\right) \leq \sup_{i \in I} \mathcal{T}(x, y_i).$$

The converse inequality,

$$\mathcal{T}\left(x, \sup_{i \in I} y_i\right) \geq \sup_{i \in I} \mathcal{T}(x, y_i),$$

follows immediately from the monotonicity of $\mathcal{T}$.

The implication (i) $\Rightarrow$ (iii) is also well-known (see e.g. [2, 7]). The implication (iii) $\Rightarrow$ (ii) follows easily by applying (iii) to the triplet $(1, x, y)$ and using the neutrality principle. This completes the proof.  

**Definition 9** ([4, 9]). Consider a t-norm $\mathcal{T}$. The biresidual operator $\mathcal{E}_{\mathcal{T}}$ of $\mathcal{T}$ is the binary operator on $[0, 1]$ defined by

$$\mathcal{E}_{\mathcal{T}}(x, y) = \min(\mathcal{I}_x(x, y), \mathcal{I}_y(y, x)).$$

In the foregoing definition, the minimum operator could, without effect, be replaced with the t-norm $\mathcal{T}$ (due to Proposition 6(i)). Note that the biresidual operator $\mathcal{E}_{\mathcal{T}}$ of a t-norm $\mathcal{T}$ can also be written as

$$\mathcal{E}_{\mathcal{T}}(x, y) = \mathcal{I}_x(x, \min(x, y)).$$

**Proposition 8.** Consider two t-norms $\mathcal{T}^*$ and $\mathcal{T}$; then the following implication holds:

$$\mathcal{T}^* \leq \mathcal{T} \Rightarrow \mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}}.$$

**Proof.** If $\mathcal{T}^* \leq \mathcal{T}$, then it easily follows that $\mathcal{I}_y \geq \mathcal{I}_x$, whence also that $\mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}}$.  

**Proposition 9.** Consider two t-norms $\mathcal{T}^*$ and $\mathcal{T}$. If $\mathcal{T}^*$ is left-continuous, then the following implication holds:

$$\mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}} \Rightarrow \mathcal{T}^* \leq \mathcal{T}.$$

**Proof.** Let $\mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}}$ and suppose there exists $(x, y) \in [0, 1]^2$ such that $\mathcal{T}^*(x, y) > \mathcal{T}(x, y)$. Due to the left-continuity of $\mathcal{T}^*$, it then follows that

$$\mathcal{I}_{\mathcal{T}^*}(x, \mathcal{T}(x, y)) = \sup\{z \mid z \in [0, 1] \land \mathcal{T}^*(x, z) \leq \mathcal{T}(x, y)\} < y.$$

On the other hand, we have that

$$\mathcal{I}_{\mathcal{T}}(x, \mathcal{T}(x, y)) = \sup\{z \mid z \in [0, 1] \land \mathcal{T}(x, z) \leq \mathcal{T}(x, y)\} \geq y.$$

It then easily follows, since $\mathcal{T}(x, y) \leq x$, that

$$\mathcal{E}_{\mathcal{T}}(x, \mathcal{T}(x, y)) = \mathcal{I}_{\mathcal{T}}(x, \mathcal{T}(x, y)) < y \leq \mathcal{I}_{\mathcal{T}^*}(x, \mathcal{T}(x, y)) = \mathcal{E}_{\mathcal{T}^*}(x, \mathcal{T}(x, y)),$$

a contradiction.  

5.2. The Biresidual Operator as a $\mathcal{F}$-Equality

**Lemma 1.** Consider a $t$-norm $\mathcal{F}$; then the following properties are equivalent:

(i) $(\forall (x, y, z) \in [0, 1]^3) \ (\mathcal{F}(\mathcal{I}_x(x, y), \mathcal{I}_x(y, z)) \leq \mathcal{I}_x(x, z))$.

(ii) $(\forall (x, y, z) \in [0, 1]^3) \ (z < y < x \Rightarrow \mathcal{F}(\mathcal{I}_x(x, y), \mathcal{I}_x(y, z)) \leq \mathcal{I}_x(x, z))$.

**Proof.** The implication (i) $\Rightarrow$ (ii) is trivial. For the implication (ii) $\Rightarrow$ (i) to hold, it suffices to show that for any $(x, y, z) \in [0, 1]^3$ such that $z < y < x$ the inequality

$$\mathcal{F}(\mathcal{I}_x(x, y), \mathcal{I}_x(y, z)) \leq \mathcal{I}_x(x, z) \quad (1)$$

always holds. We consider the following cases.

(i) The case $x \leq z$. Since $\mathcal{I}_x(x, z) = 1$, the inequality (1) is trivially fulfilled.

(ii) The case $z < x$ and $x \leq y$. Since the first partial mappings of $\mathcal{I}_x$ are decreasing, it follows that $\mathcal{I}_x(y, z) \leq \mathcal{I}_x(x, z)$. Together with $\mathcal{I}_x(x, y) = 1$, the inequality (1) follows.

(iii) The case $z < x$, $y < x$, and $y \leq z$. Since the second partial mappings of $\mathcal{I}_y$ are increasing, it follows that $\mathcal{I}_y(x, y) \leq \mathcal{I}_y(x, z)$. Together with $\mathcal{I}_y(y, z) = 1$, this means the inequality (1) again follows.

**Theorem 6.** Consider a $t$-norm $\mathcal{F}$; then its biresidual $\mathcal{E}_x$ is a $\mathcal{F}$-equality on $[0, 1]$ if and only if $\mathcal{F}$ is left-continuous.

**Proof.** We will first give the proof from right to left. From Proposition 7 it immediately follows that $\mathcal{E}_x(x, y) = 1$ if and only if $x = y$. The symmetry of $\mathcal{E}_x$ is trivially fulfilled. We will now show the $\mathcal{F}$-transitivity of $\mathcal{E}_x$. Consider $(x, y, z) \in [0, 1]^3$; then

$$\mathcal{F}(\mathcal{E}_x(x, y), \mathcal{E}_x(y, z)) = \mathcal{F}(\min(\mathcal{I}_x(x, y), \mathcal{I}_x(y, x)), \min(\mathcal{I}_x(y, z), \mathcal{I}_x(z, y))) \leq \min(\mathcal{F}(\mathcal{I}_x(x, y), \mathcal{I}_x(y, z)), \mathcal{F}(\mathcal{I}_x(z, y), \mathcal{I}_x(y, x))).$$

With Theorem 5 it then follows that

$$\mathcal{F}(\mathcal{E}_x(x, y), \mathcal{E}_x(y, z)) \leq \min(\mathcal{I}_x(x, z), \mathcal{I}_x(z, x)) = \mathcal{E}_x(x, z).$$

Next, we give the proof from left to right. Consider an arbitrary $(x, y, z) \in [0, 1]^3$ such that $z < y < x$. Then it holds that $\mathcal{E}_x(x, y) = \mathcal{I}_x(x, y)$, $\mathcal{E}_x(y, z) = \mathcal{I}_x(y, z)$, and $\mathcal{E}_x(x, z) = \mathcal{I}_x(x, z)$. Since $\mathcal{E}_x$ is $\mathcal{F}$-transitive, it follows that

$$\mathcal{F}(\mathcal{I}_x(x, y), \mathcal{I}_x(y, z)) \leq \mathcal{I}_x(x, z).$$

From this, with Lemma 1 and Theorem 5, the left-continuity of $\mathcal{F}$ follows.
In the following proposition we consider a left-continuous t-norm $\mathcal{T}$ with an additive generator $f$. This implies, however, the left-continuity and hence also the continuity of $f$. Consequently, $\mathcal{T}$ is a continuous Archimedean t-norm.

**Proposition 10.** Consider a continuous Archimedean t-norm $\mathcal{T}$ with additive generator $f$; then the $[0, 1]^2 \to [0, \infty]$ mapping $d = f \circ \mathcal{E}_\mathcal{T}$ is a metric on $[0, 1]$. Moreover, it holds that

$$d(x, y) = |f(x) - f(y)|.$$  

**Proof.** It follows immediately from Theorems 4 and 6 that $d$ is a metric on $[0, 1]$. Consider $(x, y) \in [0, 1]^2$; then it follows with Proposition 5 that

$$\mathcal{E}_\mathcal{T}(x, y) = \min(f^{-1}(\max(0, f(y) - f(x))), f^{-1}(\max(0, f(x) - f(y)))) .$$

Since $f$ is decreasing, it then follows that

$$d(x, y) = f(\mathcal{E}_\mathcal{T}(x, y)) = \max(f(y) - f(x), f(x) - f(y), 0)$$

$$= |f(x) - f(y)| .$$

\[\square\]

**Example 2.** Consider the Łukasiewicz t-norm $W$ with additive generator $f(x) = 1 - x$; then the metric $d = f \circ \mathcal{E}_W$ on $[0, 1]$ is given by $d(x, y) = |x - y|$.

The foregoing proposition can be generalized as follows.

**Proposition 11.** Consider a t-norm $\mathcal{T}^*$ with additive generator $f$ and a left-continuous t-norm $\mathcal{T}$. If $\mathcal{T}^* \leq \mathcal{T}$, then the $[0, 1]^2 \to [0, \infty]$ mapping $d = f \circ \mathcal{E}_\mathcal{T}$ is a metric on $[0, 1]$.

**Proof.** It follows immediately from Theorems 4 and 6.

The following “converse” proposition is quite remarkable, as it allows one to decide, considering one particular (potential) $\mathcal{T}$-equality, whether one t-norm is weaker than another. Note that for any additive generator $f$ and any t-norm $\mathcal{T}$, since $\mathcal{E}_\mathcal{T}$ is always reflexive and symmetric, the mapping $d = f \circ \mathcal{E}_\mathcal{T}$ satisfies (P1) and (P2).

**Proposition 12.** Consider a t-norm with an additive generator $f$ and a t-norm $\mathcal{T}$. If the $[0, 1]^2 \to [0, \infty]$ mapping $d = f \circ \mathcal{E}_\mathcal{T}$ is a pseudo-metric on $[0, 1]$, then $\mathcal{T}^* \leq \mathcal{T}$.
Proof. Suppose there exists \((a, b) \in [0, 1]^2\) such that \(\mathcal{T}^*(a, b) > \mathcal{T}(a, b)\). By assumption, it holds for any \((x, y, z) \in [0, 1]^3\) that
\[
 f\left(\mathcal{E}_\mathcal{T}(x, z)\right) \leq f\left(\mathcal{E}_\mathcal{T}(x, y)\right) + f\left(\mathcal{E}_\mathcal{T}(y, z)\right).
\]
Since \(f^{-1}\) is decreasing, it then follows that
\[
 f^{-1}\left(f\left(\mathcal{E}_\mathcal{T}(x, z)\right)\right) \geq f^{-1}\left(f\left(\mathcal{E}_\mathcal{T}(x, y)\right) + f\left(\mathcal{E}_\mathcal{T}(y, z)\right)\right)
\]
or, equivalently, that
\[
 \mathcal{E}_\mathcal{T}(x, z) \geq \mathcal{T}^*\left(\mathcal{E}_\mathcal{T}(x, y), \mathcal{E}_\mathcal{T}(y, z)\right). \tag{2}
\]
Now choose \(x = \mathcal{T}(a, b), y = b,\) and \(z = 1;\) then \(x \leq y \leq z\). One easily verifies that for this choice it holds that
\[
 \mathcal{E}_\mathcal{T}(x, z) = \mathcal{E}_\mathcal{T}(z, x) = \mathcal{E}_\mathcal{T}(1, \mathcal{T}(a, b)) = \mathcal{T}(a, b),
\]
\[
 \mathcal{E}_\mathcal{T}(x, y) = \mathcal{E}_\mathcal{T}(y, x) = \mathcal{E}_\mathcal{T}(b, \mathcal{T}(b, a)) \geq a,
\]
and
\[
 \mathcal{E}_\mathcal{T}(y, z) = \mathcal{E}_\mathcal{T}(z, y) = \mathcal{E}_\mathcal{T}(1, b) = b.
\]
Substituting the above results into (2), we obtain that \(\mathcal{T}(a, b) \geq \mathcal{T}^*(a, b),\) a contradiction. \(\blacksquare\)

Note that in the foregoing proposition it is not necessary to impose left-continuity on the t-norm \(\mathcal{T},\) since it is not required that \(\mathcal{E}_\mathcal{T}\) is a \(\mathcal{F}\)-equality, but only that \(d = f \circ \mathcal{E}_\mathcal{T}\) is a pseudo-metric. However, if we do impose left-continuity, then this proposition is a stronger version of the implication (ii) \(\Rightarrow\) (i) of Theorem 4, for the special case of \(X = [0, 1],\) since it allows one to conclude the comparability of t-norms by considering one particular \(\mathcal{F}\)-equality. We can then state the following corollary.

**Corollary 3.** Consider a t-norm \(\mathcal{T}^*\) with an additive generator \(f\) and a left-continuous t-norm \(\mathcal{T}.\) If for the \(\mathcal{F}\)-equality \(\mathcal{E}_\mathcal{T}\) on \([0, 1]\) it holds that the \([0, 1]^2 \rightarrow [0, \infty]\) mapping \(d = f \circ \mathcal{E}_\mathcal{T}\) is a metric on \([0, 1],\) then \(\mathcal{T}^* \leq \mathcal{T}\).

6. METRICS AND \(\mathcal{F}\)-EQUALITIES ON \(\mathcal{F}(X)\)

6.1. Two Particular \(\mathcal{F}\)-Equalities on \(\mathcal{F}(X)\)

In this subsection, we study two particular \(\mathcal{F}\)-equalities on \(\mathcal{F}(X),\) the class of fuzzy sets on a universe \(X.\)

**Definition 10** ([4]). Consider a t-norm \(\mathcal{T}.\) The binary fuzzy relation \(E_\mathcal{F}\) in \(\mathcal{F}(X)\) is defined, for any two fuzzy sets \(A\) and \(B\) in \(X,\) as
\[
 E_\mathcal{F}(A, B) = \inf_{x \in X} \mathcal{E}_\mathcal{T}(A(x), B(x)).
\]
DEFINITION 11 ([7, 14, 16]). Consider a t-norm \( \mathcal{T} \). The binary fuzzy relation \( E_\mathcal{T} \) in \( \mathcal{T}(X) \) is defined, for any two fuzzy sets \( A \) and \( B \) in \( X \), as

\[
E_\mathcal{T}(A, B) = \mathcal{T} \left( \inf_{x \in X} \mathcal{J}_\mathcal{T}(A(x), B(x)), \inf_{x \in X} \mathcal{J}_\mathcal{T}(B(x), A(x)) \right),
\]

PROPOSITION 13. Consider a t-norm \( \mathcal{T} \); then it holds that \( E_\mathcal{T} \subseteq E^\mathcal{T} \).

Proof. Consider two fuzzy sets \( A \) and \( B \) in \( X \); then it holds that

\[
E_\mathcal{T}(A, B) = \mathcal{T} \left( \inf_{x \in X} \mathcal{J}_\mathcal{T}(A(x), B(x)), \inf_{x \in X} \mathcal{J}_\mathcal{T}(B(x), A(x)) \right)
\leq \inf_{x \in X} \mathcal{T} \left( \mathcal{J}_\mathcal{T}(A(x), B(x)), \mathcal{J}_\mathcal{T}(B(x), A(x)) \right)
= \inf_{x \in X} \mathcal{E}_\mathcal{T}(A(x), B(x)) = E^\mathcal{T}(A, B).
\]

Note that when \( \mathcal{T} = M \), it obviously holds that \( E_M = E^M \). Also, if \( \#X = 1 \), say \( X = \{x\} \), then for any t-norm \( \mathcal{T} \) it holds that \( E_\mathcal{T}(A, B) = E^\mathcal{T}(A, B) = \mathcal{E}_\mathcal{T}(A(x), B(x)) \).

In the following theorem, we show that any \( \mathcal{T} \)-equality on \([0, 1]\) can be extended, by means of the infimum operator, to a \( \mathcal{T} \)-equality on \( \mathcal{T}(X) \).

THEOREM 7. Consider a t-norm \( \mathcal{T} \) and a binary fuzzy relation \( E \) in \([0, 1]\). Define the binary fuzzy relation \( \mathcal{E}' \) in \( \mathcal{T}(X) \) as follows, for any two fuzzy sets \( A \) and \( B \) in \( X \):

\[
\mathcal{E}'(A, B) = \inf_{x \in X} E(A(x), B(x)).
\]

Then the following statements are equivalent:

(i) \( E \) is a \( \mathcal{T} \)-equality on \([0, 1]\),
(ii) \( \mathcal{E}' \) is a \( \mathcal{T} \)-equality on \( \mathcal{T}(X) \).

Proof. We will first prove the implication (i) \( \Rightarrow \) (ii).

(a) Consider two fuzzy sets \( A \) and \( B \) in \( X \); then the following chain of equivalences holds:

\[
\mathcal{E}'(A, B) = 1 \iff \inf_{x \in X} E(A(x), B(x)) = 1
\iff (\forall x \in X)(E(A(x), B(x)) = 1)
\iff (\forall x \in X)(A(x) = B(x)) \iff A = B.
\]

(b) The symmetry of \( \mathcal{E}' \) is obvious.
(c) Consider three fuzzy sets $A$, $B$, and $C$ in $X$; then

$$\mathcal{T}(E'(A, B), E'(B, C)) = \mathcal{T}\left(\inf_{x \in X} E(A(x), B(x)), \inf_{x \in X} E(B(x), C(x))\right)$$

$$\leq \inf_{x \in X} \mathcal{T}(E(A(x), B(x)), E(B(x), C(x))).$$

Since $E$ is $\mathcal{T}$-transitive, it follows that

$$\mathcal{T}(E'(A, B), E'(B, C)) \leq \inf_{x \in X} E(A(x), C(x)) = E'(A, C).$$

Next, we prove the implication (ii) $\Rightarrow$ (i). Consider $(a, b, c) \in [0, 1]^3$ and the corresponding constant fuzzy sets $A(x) = a$, $B(x) = b$, and $C(x) = c$ in $X$. It then holds that $E'(A, B) = E(a, b)$, $E'(B, C) = E(b, c)$, and $E'(A, C) = E(a, c)$.

(a) Let $E(a, b) = 1$. Then also $E'(A, B) = 1$, which implies that $A = B$ and also that $a = b$.

(b) The symmetry of $E$ follows immediately from the symmetry of $E'$.

(c) The $\mathcal{T}$-transitivity of $E'$ implies that

$$\mathcal{T}(E'(A, B), E'(B, C)) \leq E'(A, C)$$

and hence also that

$$\mathcal{T}(E(a, b), E(b, c)) \leq E(a, c).$$


\textbf{Corollary 4.} Consider a $t$-norm $\mathcal{T}$. The binary fuzzy relation $E^\mathcal{T}$ is a $\mathcal{T}$-equality on $\mathcal{F}(X)$ if and only if $\mathcal{T}$ is left-continuous.

\textbf{Proof.} It follows immediately from Theorems 6 and 7.

\textbf{Theorem 8.} Consider a $t$-norm $\mathcal{T}$. The binary fuzzy relation $E_\mathcal{T}$ is a $\mathcal{T}$-equality on $\mathcal{F}(X)$ if and only if $\mathcal{T}$ is left-continuous.

\textbf{Proof.} The proof from right to left was given by Gottwald [7]. Indeed, he has shown that for a left-continuous $t$-norm $\mathcal{T}$, $E_\mathcal{T}$ is a $\mathcal{T}$-equivalence on $\mathcal{F}(X)$. He further demonstrated that in this case $E_\mathcal{T}(A, B) = 1$ if and only if $A = B$. This means that $E_\mathcal{T}$ is a $\mathcal{T}$-equality on $\mathcal{F}(X)$.

For the proof from left to right, consider $(a, b, c) \in [0, 1]^3$ such that $c < b < a$. Consider $x_0$ in $X$ and construct the fuzzy sets $A$, $B$, and $C$ in $X$ as follows:

$$A(x) = \begin{cases} 
    a, & \text{if } x = x_0, \\
    0, & \text{elsewhere.} 
\end{cases}$$

$$B(x) = \begin{cases} 
    b, & \text{if } x = x_0, \\
    0, & \text{elsewhere.} 
\end{cases}$$

$$C(x) = \begin{cases} 
    c, & \text{if } x = x_0, \\
    0, & \text{elsewhere.} 
\end{cases}$$
One easily verifies that in this case \( \mathcal{E}_{\overline{T}}(A, B) = \mathcal{R}_{\overline{T}}(a, b) \), \( \mathcal{E}_{\overline{T}}(B, C) = \mathcal{R}_{\overline{T}}(b, c) \), and \( \mathcal{E}_{\overline{T}}(A, C) = \mathcal{R}_{\overline{T}}(a, c) \). Since \( \mathcal{E}_{\overline{T}} \) is \( \overline{T} \)-transitive, it then follows that

\[
\mathcal{I}_{\overline{T}}(\mathcal{R}_{\overline{T}}(A, B), \mathcal{R}_{\overline{T}}(B, C)) \leq \mathcal{R}_{\overline{T}}(A, C)
\]

and hence also that

\[
\mathcal{I}_{\overline{T}}(\mathcal{R}_{\overline{T}}(a, b), \mathcal{R}_{\overline{T}}(b, c)) \leq \mathcal{R}_{\overline{T}}(a, c).
\]

With Lemma 1 and Theorem 5, it then follows that \( \mathcal{I} \) is left-continuous.

The fuzzy relation \( \mathcal{E}_{\overline{T}} \) is inspired by the following classical equivalence, for any two sets \( A \) and \( B \) in \( X \):

\[
A = B \iff A \subseteq B \land B \subseteq A.
\]

The inclusion of a fuzzy set \( A \) in \( X \) in a fuzzy set \( B \) in \( X \) is then measured by \( \inf_{x \in X} \mathcal{I}(\mathcal{R}_{\overline{T}}(A(x), B(x))) \), with \( \mathcal{I} \) a fuzzy implication operator, such as the residual implicator \( \mathcal{I} \) (see e.g. [3, 14]).

6.2. Metrics on \( \mathcal{F}(X) \) Based on \( \mathcal{E}_{\overline{T}} \)

**Proposition 14.** Consider a \( \overline{T} \)-norm \( \overline{\mathcal{T}}^* \) with an additive generator \( f \) and a left-continuous \( \overline{T} \)-norm \( \mathcal{T} \) such that

\[ \overline{\mathcal{T}}^* \leq \mathcal{T} \]; then the \( \mathcal{F}(X)^2 \rightarrow [0, \infty] \) mapping \( d = f \circ \mathcal{E}_{\overline{T}} \) is a metric on \( \mathcal{F}(X) \).

**Proof.** It follows immediately from Theorems 4 and 8.

Note that for any additive generator \( f \) and any \( \overline{T} \)-norm \( \mathcal{T} \), since \( \mathcal{E}_{\overline{T}} \) is always reflexive and symmetric, the mapping \( d = f \circ \mathcal{E}_{\overline{T}} \) satisfies (P1) and (P2).

**Theorem 9.** Consider a \( \overline{T} \)-norm \( \overline{\mathcal{T}}^* \) with additive generator \( f \) and a \( \overline{T} \)-norm \( \mathcal{T} \). If the \( \mathcal{F}(X)^2 \rightarrow [0, \infty] \) mapping \( d = f \circ \mathcal{E}_{\overline{T}} \) is a pseudo-metric on \( \mathcal{F}(X) \), then \( \overline{\mathcal{T}}^* \leq \mathcal{T} \).

**Proof.** Suppose there exists \( (a, b) \in [0, 1]^2 \) such that \( \overline{\mathcal{T}}^*(a, b) > \mathcal{T}(a, b) \). By assumption, it holds for any \( (A, B, C) \in \mathcal{F}(X)^3 \) that

\[
f(\mathcal{E}_{\overline{T}}(A, C)) \leq f(\mathcal{E}_{\overline{T}}(A, B)) + f(\mathcal{E}_{\overline{T}}(B, C)).
\]

As in the proof of Proposition 12, it then follows that

\[
\mathcal{E}_{\overline{T}}(A, C) \geq \overline{\mathcal{T}}^*(\mathcal{E}_{\overline{T}}(A, B), \mathcal{E}_{\overline{T}}(B, C)).
\]
Now consider two different elements \( x_0 \) and \( y_0 \) of \( X \) and construct the fuzzy sets \( A, B, \) and \( C \) in \( X \) as

\[
A(x) = \begin{cases} 
1, & \text{if } x = x_0, \\
0, & \text{elsewhere}.
\end{cases}
\]

\[
B(x) = \begin{cases} 
0, & \text{if } x = x_0, \\
1, & \text{if } x = y_0, \\
0, & \text{elsewhere}.
\end{cases}
\]

\[
C(x) = \begin{cases} 
0, & \text{if } x = y_0.
\end{cases}
\]

One easily verifies that for this choice it holds that \( E_T(A, B) = a \), \( E_T(B, C) = b \), and \( E_T(A, C) = f(a, b) \). Substituting these results in (3), we obtain that \( \mathcal{F}(a, b) \leq \mathcal{F}^*(a, b) \), a contradiction.

Corollary 5. Consider a \( T \)-norm \( \mathcal{T}^* \) with an additive generator \( f \) and a left-continuous \( T \)-norm \( \mathcal{T} \). If for the \( T \)-equality \( E_T \) on \( \mathcal{F}(X) \) it holds that the \( \mathcal{F}(X)^2 \to [0, \infty] \) mapping \( d = f \circ E_T \) is a metric on \( \mathcal{F}(X) \), then \( \mathcal{T}^* \leq \mathcal{T} \).

As a corollary of Proposition 14 and Theorem 9, we rediscover the main theorem of Gottwald in [7].

Corollary 6 ([7]). Consider a left-continuous \( T \)-norm \( \mathcal{T} \). The \( \mathcal{F}(X)^2 \to [0, \infty] \) mapping \( d = 1 - E_T \) is a metric on \( \mathcal{F}(X) \) if and only if \( W \leq \mathcal{T} \).

We cite Gottwald here [7]: “The intuition behind that relation comes from the interpretation of \( E_T \) as a graded measure of the equality of fuzzy sets or of their indistinguishability. The negation of such an indistinguishability relation \( E_T \) hence should be a kind of graded distinguishability and thus (perhaps) even a kind of ‘distance’.”

An important remark should be made here: In Corollary 6, the operation \( 1– \) should not be interpreted as the standard negation, but as an additive generator of the Łukasiewicz \( T \)-norm. Only this insight can lead to the more general results presented in this paper.

6.3. Metrics on \( \mathcal{F}(X) \) Based on \( E_T^2 \)

Propositions similar to those in the previous subsection can be written for the \( T \)-equality \( E_T^2 \). The first proposition is an extended version of Proposition 10.

Proposition 15. Consider a continuous Archimedean \( T \)-norm \( \mathcal{T} \) with an additive generator \( f \); then the \( \mathcal{F}(X)^2 \to [0, \infty] \) mapping \( d = f \circ E_T^2 \) is a metric on \( \mathcal{F}(X) \). Moreover, it holds, for any two fuzzy sets \( A \) and \( B \) in \( X \), that

\[
d(A, B) = \sup_{x \in X} |f(A(x)) - f(B(x))|.
\]
Proposition 12. Since $f$ is continuous and decreasing, it then follows that

$$d(A, B) = f(E^\tau (A, B)) = \left( \inf_{x \in X} E^\tau_x (A(x), B(x)) \right).$$


Proof. It follows immediately from Corollary 4 and Theorem 4 that $d$ is a metric on $\mathcal{F}(X)$. Consider $(A, B) \in \mathcal{F}(X)^2$; then

$$d(A, B) = \sup_{x \in X} f(\varepsilon^\tau_x (A(x), B(x))).$$

As in the proof of Proposition 10, it then follows that

$$d(A, B) = \sup_{x \in X} |f(A(x)) - f(B(x))|.$$

Example 3. Consider the Łukasiewicz t-norm $W$ with an additive generator $f(x) = 1 - x$; then the metric $d = f \circ E_W$ on $\mathcal{F}(X)$ is given by, for any two fuzzy sets $A$ and $B$ in $X$,

$$d(A, B) = \sup_{x \in X} |A(x) - B(x)|.$$

Proposition 16. Consider a t-norm $\mathcal{T}^*$ with an additive generator $f$ and a left-continuous t-norm $\mathcal{T}$ such that $\mathcal{T}^* \leq \mathcal{T}$; then the $\mathcal{T}(X)^2 \rightarrow [0, \infty]$ mapping $d = f \circ E^\mathcal{T}$ is a metric on $\mathcal{T}(X)$.

Proof. It follows immediately from Corollary 4 and Theorem 4.

The following theorem (also its proof) is an extended version of Proposition 12.

Theorem 10. Consider a t-norm $\mathcal{T}^*$ with an additive generator $f$ and a t-norm $\mathcal{T}$. If the $\mathcal{T}(X)^2 \rightarrow [0, \infty]$ mapping $d = f \circ E^\mathcal{T}$ is a pseudo-metric on $\mathcal{T}(X)$, then $\mathcal{T}^* \leq \mathcal{T}$.

Proof. Suppose there exists $(a, b) \in [0, 1]^2$ such that $\mathcal{T}^*(a, b) > \mathcal{T}(a, b)$. As in the proof of Proposition 12, it follows that for any $(A, B, C) \in \mathcal{T}(X)^3$ it holds that

$$E^\mathcal{T}(A, C) \geq \mathcal{T}^*(E^\mathcal{T}(A, B), E^\mathcal{T}(B, C)).$$

Now consider $x_0$ in $X$ and construct the fuzzy sets $A, B,$ and $C$ in $X$ as follows:

$$A(x) = \begin{cases} \mathcal{T}(a, b), & \text{if } x = x_0, \\ 0, & \text{elsewhere}. \end{cases}$$

$$B(x) = \begin{cases} b, & \text{if } x = x_0, \\ 0, & \text{elsewhere}. \end{cases}$$

$$C(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{elsewhere}. \end{cases}$$
One easily verifies that for this choice it holds that $E^\mathcal{F}(A, C) = \mathcal{F}(a, b)$, $E^\mathcal{F}(A, B) \geq a$, and $E^\mathcal{F}(B, C) = b$. Substituting these findings in (4) we obtain $\mathcal{F}(a, b) \geq \mathcal{F}^*(a, b)$, a contradiction.

**Corollary 7.** Consider a t-norm $\mathcal{F}^*$ with an additive generator $f$ and a left-continuous t-norm $\mathcal{F}$. If for the $\mathcal{F}$-equality $E^\mathcal{F}$ on $\mathcal{F}(X)$ the $\mathcal{F}(X)^2 \to [0, \infty]$ mapping $d = f \circ E^\mathcal{F}$ is a metric on $\mathcal{F}(X)$, then $\mathcal{F}^* \leq \mathcal{F}$.

**REFERENCES**