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Metrics and $\mathcal{T}$-Equalities

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The relationship between metrics and $\mathcal{T}$-equalities is investigated; the latter are a special case of $\mathcal{T}$-equivalences, a natural generalization of the classical concept of an equivalence relation. It is shown that in the construction of metrics from $\mathcal{T}$-equalities triangular norms with an additive generator play a key role. Conversely, in the construction of $\mathcal{T}$-equalities from metrics this role is played by triangular norms with a continuous additive generator or, equivalently, by continuous Archimedean triangular norms. These results are then applied to the biresidual operator $\mathcal{E}_{\mathcal{T}}$ of a triangular norm $\mathcal{T}$. It is shown that $\mathcal{E}_{\mathcal{T}}$ is a $\mathcal{T}$-equality on [0, 1] if and only if $\mathcal{T}$ is left-continuous. Furthermore, it is shown that to any left-continuous triangular norm $\mathcal{T}$ there correspond two particular $\mathcal{T}$-equalities on $\mathcal{F}(X)$, the class of fuzzy sets in a given universe X; one of these $\mathcal{T}$-equalities is obtained from the biresidual operator $\mathcal{E}_{\mathcal{T}}$ by means of a natural extension procedure. These $\mathcal{T}$-equalities then give rise to interesting metrics on $\mathcal{F}(X)$.

Key Words: additive generator; Archimedean property; biresidual operator; metric; $\mathcal{T}$-equality; triangular norm.

1. INTRODUCTION

The concept of a similarity relation was introduced by Zadeh [17] as a generalization of the concept of an equivalence relation. Also, simi-

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larity relations have been generalized by replacing the min-transitivity with the more general $\mathcal{T}$-transitivity, with $\mathcal{T}$ an arbitrary triangular norm (t-norm) [12].

DEFINITION 1 ([4]). Consider a t-norm $\mathcal{T}$. A binary fuzzy relation E in a universe X is called a $\mathcal{T}$-equivalence on X if it is reflexive, symmetric, and $\mathcal{T}$-transitive, i.e., if for any (x, y, z) in X^3,

(E1) $E(x, x) = 1$;

(E2) $E(x, y) = E(y, x)$; and

(E3) $\mathcal{T}(E(x, y), E(y, z)) \leq E(x, z)$.

$\mathcal{T}$-equivalences are also called indistinguishability operators [13], fuzzy equalities [8], and equality relations [9]. Clearly, M-equivalences (with M the minimum operator) are nothing but similarity relations. W-equivalences (with W the Łukasiewicz t-norm defined by $W(x, y) = \max(x + y - 1, 0)$) are called likeness relations. A one-to-one correspondence between $\mathcal{T}$-equivalences and $\mathcal{T}$-partitions, a generalization of the concept of a partition, was recently exposed in [4].

In this paper, we deal with $\mathcal{T}$-equalities, a special type of $\mathcal{T}$-equivalence.

DEFINITION 2. Consider a t-norm $\mathcal{T}$. A $\mathcal{T}$-equivalence E in a universe X is called a $\mathcal{T}$-equality on X if for any (x, y) in X^2,

(E1′) $E(x, y) = 1 \Leftrightarrow x = y$.

Recall that a t-norm $\mathcal{T}^*$ is called weaker than a t-norm $\mathcal{T}$, denoted $\mathcal{T}^* \leq \mathcal{T}$, if $(\forall (x, y) \in [0, 1]^2)$ $(\mathcal{T}^*(x, y) \leq \mathcal{T}(x, y))$. The following proposition then is immediate.

PROPOSITION 1. *Consider a binary fuzzy relation E in a universe X and a* t-*norm $\mathcal{T}$. If E is a $\mathcal{T}$-equivalence* (*resp*., *$\mathcal{T}$-equality*), *then it is also a $\mathcal{T}^*$-equivalence* (*resp*., *$\mathcal{T}^*$-equality*) *for any* t-*norm $\mathcal{T}^*$ that is weaker than $\mathcal{T}$.*

Bezdek and Harris [1] have discussed the relationship between likeness relations and pseudo-metrics. More general investigations into the relationship between pseudo-metrics and $\mathcal{T}$-equivalences were done by Wagenknecht [15]. A complete study was carried out by De Baets and Mesiar [5] (see Section 3).

DEFINITION 3. An $X^2 \to [0, \infty]$ mapping d is called a pseudo-metric on X if for any (x, y, z) in X^3,

(P1) $d(x, x) = 0$;

(P2) $d(x, y) = d(y, x)$; and

(P3) $d(x, z) \leq d(x, y) + d(y, z)$.

In this paper, we will show that $\mathcal{T}$-equalities are related to metrics as $\mathcal{T}$-equivalences are to pseudo-metrics.

DEFINITION 4. A pseudo-metric d on X is called a metric if for any (x, y) in X^2,

(P1′) $d(x, y) = 0 \Leftrightarrow x = y$.

2. ADDITIVE GENERATORS AND ARCHIMEDEAN t-NORMS

In this section, we recall some important results concerning additive generators of t-norms (see e.g. [10–12]) and the relationship to the Archimedean property.

DEFINITION 5. A strictly decreasing $[0, 1] \to [0, \infty]$ mapping f with $\mathrm{Rng}(f)$ relatively closed under addition, i.e.,

$$(\forall\,(u, v) \in \mathrm{Rng}(f)^2)(u + v \in \mathrm{Rng}(f) \vee u + v > f(0)),$$

such that $f(1) = 0$, is called an additive generator.

DEFINITION 6. Consider a $[0, 1] \to [0, \infty]$ mapping f; then the pseudo-inverse of f is the $[0, \infty] \to [0, 1]$ mapping $f^{(-1)}$ defined by

$$f^{(-1)}(x) = \inf\{t \mid t \in [0, 1] \wedge f(t) \leq x\}.$$

Note that this pseudo-inverse is always decreasing. The pseudo-inverse $f^{(-1)}$ of a continuous additive generator f is given by

$$f^{(-1)}(x) = f^{-1}(\min(f(0), x)).$$

THEOREM 1. *Consider an additive generator f; then the $[0, 1]^2 \to [0, 1]$ mapping $\mathcal{T}$ defined by*

$$\mathcal{T}(x, y) = g(f(x) + f(y)),$$

where g is an arbitrary $[0, \infty] \to [0, 1]$ mapping such that

$$g(x) = \begin{cases} f^{-1}(x), & \text{if } x \in \mathrm{Rng}(f), \\ 0, & \text{if } x > f(0), \end{cases}$$

is a t-*norm*.

A suitable candidate for the mapping g in the foregoing theorem is the pseudo-inverse $f^{(-1)}$ of f.

The continuity of an additive generator f is equivalent with its left-continuity in the point 1 and with the continuity of the generated t-norm $\mathcal{T}$. Note that if a continuous t-norm $\mathcal{T}$ has an additive generator f, then this additive generator is uniquely determined up to a nonzero positive multiplicative constant.

EXAMPLE 1. (i) The mapping f defined by $f(x) = -\log x$ is an additive generator of the algebraic product, i.e., of the t-norm P defined by $P(x, y) = xy$.

(ii) The mapping f defined by $f(x) = 1 - x$ is an additive generator of the Łukasiewicz t-norm W.

(iii) The mapping f defined by

$$f(x) = \begin{cases} 2 - x, & \text{if } x \in [0, 1[, \\ 0, & \text{if } x = 1, \end{cases}$$

is an additive generator of the weakest t-norm Z defined by

$$Z(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Not all t-norms have an additive generator. An example of such a t-norm is the minimum operator M. The fact that a t-norm has an additive generator is closely related to the Archimedean property.

DEFINITION 7. A t-norm $\mathcal{T}$ is called Archimedean if

$$(\forall (x, y) \in]0, 1[^2)\ (\exists n \in \mathbb{N})(x^{(n)} < y),$$

where $x^{(n)}$ stands for $\mathcal{T}(x, \ldots, x)$ (n times).

PROPOSITION 2. *A continuous* t-*norm* $\mathcal{T}$ *is Archimedean if and only if* $(\forall x \in]0, 1[)\ (\mathcal{T}(x, x) < x)$.

Each t-norm with an additive generator is Archimedean. The converse is not true in general, but holds for instance for continuous t-norms.

THEOREM 2. *A* $[0, 1]^2 \to [0, 1]$ *mapping* $\mathcal{T}$ *is a continuous Archimedean* t-*norm if and only if there exists a continuous additive generator* f *such that*

$$\mathcal{T}(x, y) = f^{(-1)}(f(x) + f(y)).$$

3. PSEUDO-METRICS AND $\mathcal{T}$-EQUIVALENCES

In this section, we briefly recall our previous results concerning the construction of pseudo-metrics from $\mathcal{T}$-equivalences, and *vice versa*.

If the cardinality of the universe X is smaller than 3, then for any t-norm $\mathcal{T}$, any $\mathcal{T}$-equivalence E on X, and any additive generator f it holds that the mapping $d = f \circ E$ is a pseudo-metric on X; in fact, any $[0, 1] \to [0, \infty]$ mapping f such that $f(1) = 0$ will do here. Therefore, only universes with higher cardinality are of interest to us.

THEOREM 3 ([5]). *Consider a universe* X *with* $\#X > 2$, *a* t-*norm* $\mathcal{T}^*$ *with additive generator* f, *and a* t-*norm* $\mathcal{T}$. *Then the following statements* *are equivalent*:

(i) $\mathcal{T}^*$ *is weaker than* $\mathcal{T}$; *i.e.*, $\mathcal{T}^* \leq \mathcal{T}$.

(ii) *For any* $\mathcal{T}$*-equivalence* E *on* X, *the* $X^2 \to [0, \infty]$ *mapping* $d = f \circ E$ *is a pseudo-metric on* X.

In the converse problem, namely the construction of $\mathcal{T}$-equivalences from pseudo-metrics, continuous additive generators play an important role. In a counterexample, we have shown that this continuity requirement cannot be dropped [5].

PROPOSITION 3 ([5]). *Consider a pseudo-metric* d *on a universe* X *and a continuous Archimedean* t*-norm* $\mathcal{T}^*$ *with additive generator* f; *then the binary fuzzy relation* $E = f^{(-1)} \circ d$ *in* X *is a* $\mathcal{T}^*$*-equivalence on* X.

4. METRICS AND $\mathcal{T}$-EQUALITIES

The results from the previous section can be made more specific for metrics and $\mathcal{T}$-equalities. We will show how to construct metrics from $\mathcal{T}$-equalities and *vice versa*.

THEOREM 4. *Consider a universe* X *with* $\#X > 2$, *a* t*-norm* $\mathcal{T}^*$ *with additive generator* f, *and a* t*-norm* $\mathcal{T}$. *Then the following statements are equivalent*:

(i) $\mathcal{T}^*$ *is weaker than* $\mathcal{T}$; *i.e.*, $\mathcal{T}^* \leq \mathcal{T}$.

(ii) *For any* $\mathcal{T}$*-equality* E *on* X, *the* $X^2 \to [0, \infty]$ *mapping* $d = f \circ E$ *is a metric on* X.

Proof. We will first prove the implication (i) $\Rightarrow$ (ii). Suppose that $\mathcal{T}^* \leq \mathcal{T}$. Since any $\mathcal{T}$-equality is a $\mathcal{T}$-equivalence, it follows from Theorem 3 that d is a pseudo-metric on X. Now consider x and y in X such that $d(x, y) = 0$; then we have to show that $x = y$. From $d(x, y) = 0$ it follows that $f(E(x, y)) = 0$. Since f is strictly decreasing and $f(1) = 0$, it follows that $E(x, y) = 1$, whence $x = y$.

Next, we prove the implication (ii) $\Rightarrow$ (i). Consider $(a, b) \in [0, 1]^2$; then we have to show that $\mathcal{T}^*(a, b) \leq \mathcal{T}(a, b)$. If $a = 1$ or $b = 1$, then always $\mathcal{T}^*(a, b) = \mathcal{T}(a, b)$. We can therefore assume that $(a, b) \in [0, 1[^2$. We construct the following binary fuzzy relation E in X: First, for all u in X we put $E(u, u) = 1$. Next, we consider three different elements x, y, and z of X and define

$$E(x, y) = a,$$
$$E(y, z) = b,$$
$$E(x, z) = \mathcal{T}(a, b).$$

Furthermore, for any u and v in $X \backslash \{x, y, z\}$, $u \neq v$, we put $E(u, v) = 0$. One easily verifies that E is a $\mathcal{T}$-equality on X. It then holds that the mapping $d = f \circ E$ is a metric on X. This means in particular that

$$f(E(x, z)) \leq f(E(x, y)) + f(E(y, z)).$$

Since $f^{(-1)}$ is decreasing, it follows that

$$f^{(-1)}(f(E(x, z))) \geq f^{(-1)}(f(E(x, y)) + f(E(y, z))).$$

Since $f^{(-1)}(f(E(x, z))) = E(x, z)$, it then follows that $\mathcal{T}(a, b) \geq \mathcal{T}^*(a, b)$. ∎

COROLLARY 1. (i) *Consider an M-equality E on X; then for any additive generator f it holds that the mapping $d = f \circ E$ is a metric on X.*

(ii) *Consider an arbitrary* t-*norm $\mathcal{T}$ and a $\mathcal{T}$-equality E on X; then the mapping $d = f \circ E$, with f an additive generator of the weakest* t-*norm Z, is a metric on X.*

(iii) *Consider a* t-*norm $\mathcal{T}$ such that $W \leq \mathcal{T}$ and a $\mathcal{T}$-equality E on X; then the mapping $d = 1 - E$ is a metric on X.*

PROPOSITION 4. *Consider a metric d on a universe X and a continuous Archimedean* t-*norm $\mathcal{T}^*$ with additive generator f; then the binary fuzzy relation $E = f^{(-1)} \circ d$ in X is a $\mathcal{T}^*$-equality on X.*

Proof. According to Proposition 3, E is a $\mathcal{T}^*$-equivalence on X. Now consider x and y in X such that $E(x, y) = 1$; then we have to show that $x = y$. From $E(x, y) = 1$ it follows that $f^{(-1)}(d(x, y)) = 1$; i.e., $f^{-1}(\min(f(0), d(x, y))) = 1$. Since f is strictly decreasing and $f(1) = 0$, it follows that $\min(f(0), d(x, y)) = 0$. Since $f(0) > f(1) = 0$, we can conclude that $d(x, y) = 0$, whence $x = y$. ∎

COROLLARY 2. *Consider a metric d on X, then the binary fuzzy relation $E = \max(1 - d, 0)$ is a W-equality on X.*

5. THE BIRESIDUAL OPERATOR OF A t-NORM

5.1. *Definition and Properties*

In the following section we study two particular $\mathcal{T}$-equalities on $\mathcal{F}(X)$. One of them is based on the biresidual operator $\mathcal{E}_{\mathcal{T}}$ of a t-norm $\mathcal{T}$ that is used for measuring the degree of equality of real numbers taken from the unit interval. In fact, we show that the biresidual operator $\mathcal{E}_{\mathcal{T}}$ of a t-norm $\mathcal{T}$ is a $\mathcal{T}$-equality on [0, 1] if and only if $\mathcal{T}$ is left-continuous. Note that by a left-continuous t-norm we mean a t-norm with left-continuous partial mappings.

DEFINITION 8 (see e.g. [6]). Consider a t-norm $\mathcal{T}$. The residual implicator $\mathcal{I}_{\mathcal{T}}$ of $\mathcal{T}$ is the binary operator on [0, 1] defined by

$$\mathcal{I}_{\mathcal{T}}(x, y) = \sup\{z \mid z \in [0, 1] \wedge \mathcal{T}(x, z) \leq y\}.$$

Note that the residual implicator $\mathcal{I}_{\mathcal{T}}$ is hybrid monotonous; i.e., it has decreasing first and increasing second partial mappings.

PROPOSITION 5 ([6]). *Consider a continuous Archimedean* t-*norm* $\mathcal{T}$ *with an additive generator* f; *then its residual implicator* $\mathcal{I}_{\mathcal{T}}$ *is given by*

$$\mathcal{I}_{\mathcal{T}}(x, y) = f^{-1}(\max(0, f(y) - f(x))).$$

PROPOSITION 6 ([6]). *Consider a* t-*norm* $\mathcal{T}$; *then the following properties hold, for any* (x, y) *in* $[0, 1]^2$:

(i) $x \leq y \Rightarrow \mathcal{I}_{\mathcal{T}}(x, y) = 1$;

(ii) $\mathcal{I}_{\mathcal{T}}(1, y) = y$ (*the neutrality principle*); *and*

(iii) $\mathcal{I}_{\mathcal{T}}(x, \mathcal{T}(x, y)) \geq y$.

PROPOSITION 7 ([6]). *Consider a left-continuous* t-*norm* $\mathcal{T}$; *then the following equivalence holds, for any* (x, y) *in* $[0, 1]^2$:

$$x \leq y \Leftrightarrow \mathcal{I}_{\mathcal{T}}(x, y) = 1.$$

THEOREM 5. *Consider a* t-*norm* $\mathcal{T}$; *then the following statements are equivalent*:

(i) $\mathcal{T}$ *is left-continuous*;

(ii) $(\forall (x, y) \in [0, 1]^2)\ (\mathcal{T}(x, \mathcal{I}_{\mathcal{T}}(x, y)) \leq y)$; *and*

(iii) $(\forall (x, y, z) \in [0, 1]^3)\ (\mathcal{T}(\mathcal{I}_{\mathcal{T}}(x, y), \mathcal{I}_{\mathcal{T}}(y, z)) \leq \mathcal{I}_{\mathcal{T}}(x, z))$.

Proof. The implication (i) $\Rightarrow$ (ii) is well-known (see e.g. [2]). We will prove the converse implication. For $\mathcal{T}$ to be left-continuous, it suffices to show that for any x in [0, 1] and for any nonempty family $(y_i)_{i \in I}$ in [0, 1] the following equality holds:

$$\mathcal{T}\left(x, \sup_{i \in I} y_i\right) = \sup_{i \in I} \mathcal{T}(x, y_i).$$

For any $i \in I$ it holds that $\mathcal{T}(x, y_i) \leq \sup_{i \in I} \mathcal{T}(x, y_i)$, whence

$$y_i \leq \mathcal{I}_{\mathcal{T}}\left(x, \sup_{i \in I} \mathcal{T}(x, y_i)\right)$$

and also

$$\sup_{i \in I} y_i \leq \mathcal{I}_{\mathcal{T}}\left(x, \sup_{i \in I} \mathcal{T}(x, y_i)\right).$$

The monotonicity of $\mathcal{T}$ and (ii) then imply that

$$\mathcal{T}\left(x, \sup_{i\in I} y_i\right) \leq \mathcal{T}\left(x, \mathcal{I}_{\mathcal{T}}(x, \sup_{i\in I} \mathcal{T}(x, y_i))\right) \leq \sup_{i\in I} \mathcal{T}(x, y_i).$$

The converse inequality,

$$\mathcal{T}\left(x, \sup_{i\in I} y_i\right) \geq \sup_{i\in I} \mathcal{T}(x, y_i),$$

follows immediately from the monotonicity of $\mathcal{T}$.

The implication (i) $\Rightarrow$ (iii) is also well-known (see e.g. [2, 7]). The implication (iii) $\Rightarrow$ (ii) follows easily by applying (iii) to the triplet $(1, x, y)$ and using the neutrality principle. This completes the proof. ∎

DEFINITION 9 ([4, 9]). Consider a t-norm $\mathcal{T}$. The biresidual operator $\mathcal{E}_{\mathcal{T}}$ of $\mathcal{T}$ is the binary operator on [0, 1] defined by

$$\mathcal{E}_{\mathcal{T}}(x, y) = \min(\mathcal{I}_{\mathcal{T}}(x, y), \mathcal{I}_{\mathcal{T}}(y, x)).$$

In the foregoing definition, the minimum operator could, without effect, be replaced with the t-norm $\mathcal{T}$ (due to Proposition 6(i)). Note that the biresidual operator $\mathcal{E}_{\mathcal{T}}$ of a t-norm $\mathcal{T}$ can also be written as

$$\mathcal{E}_{\mathcal{T}}(x, y) = \mathcal{I}_{\mathcal{T}}(\max(x, y), \min(x, y)).$$

PROPOSITION 8. *Consider two* t-*norms* $\mathcal{T}^*$ *and* $\mathcal{T}$; *then the following implication holds*:

$$\mathcal{T}^* \leq \mathcal{T} \Rightarrow \mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}}.$$

Proof. If $\mathcal{T}^* \leq \mathcal{T}$, then it easily follows that $\mathcal{I}_{\mathcal{T}^*} \geq \mathcal{I}_{\mathcal{T}}$, whence also that $\mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}}$. ∎

PROPOSITION 9. *Consider two* t-*norms* $\mathcal{T}^*$ *and* $\mathcal{T}$. *If* $\mathcal{T}^*$ *is left-continuous, then the following implication holds*:

$$\mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}} \Rightarrow \mathcal{T}^* \leq \mathcal{T}.$$

Proof. Let $\mathcal{E}_{\mathcal{T}^*} \geq \mathcal{E}_{\mathcal{T}}$ and suppose there exists $(x, y) \in [0, 1]^2$ such that $\mathcal{T}^*(x, y) > \mathcal{T}(x, y)$. Due to the left-continuity of $\mathcal{T}^*$, it then follows that

$$\mathcal{I}_{\mathcal{T}^*}(x, \mathcal{T}(x, y)) = \sup\{z \mid z \in [0, 1] \wedge \mathcal{T}^*(x, z) \leq \mathcal{T}(x, y)\} < y.$$

On the other hand, we have that

$$\mathcal{I}_{\mathcal{T}}(x, \mathcal{T}(x, y)) = \sup\{z \mid z \in [0, 1] \wedge \mathcal{T}(x, z) \leq \mathcal{T}(x, y)\} \geq y.$$

It then easily follows, since $\mathcal{T}(x, y) \leq x$, that

$$\mathcal{E}_{\mathcal{T}^*}(x, \mathcal{T}(x, y)) = \mathcal{I}_{\mathcal{T}^*}(x, \mathcal{T}(x, y)) < y \leq \mathcal{I}_{\mathcal{T}}(x, \mathcal{T}(x, y)) = \mathcal{E}_{\mathcal{T}}(x, \mathcal{T}(x, y)),$$

a contradiction. ∎

5.2. *The Biresidual Operator as a $\mathcal{T}$-Equality*

LEMMA 1. *Consider a* t-*norm* $\mathcal{T}$; *then the following properties are equivalent*:

(i) $(\forall (x, y, z) \in [0, 1]^3)\ (\mathcal{T}(\mathcal{I}_{\mathcal{T}}(x, y), \mathcal{I}_{\mathcal{T}}(y, z)) \leq \mathcal{I}_{\mathcal{T}}(x, z))$.

(ii) $(\forall (x, y, z) \in [0, 1]^3)\ (z < y < x \Rightarrow \mathcal{T}(\mathcal{I}_{\mathcal{T}}(x, y), \mathcal{I}_{\mathcal{T}}(y, z)) \leq \mathcal{I}_{\mathcal{T}}(x, z))$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. For the implication (ii) $\Rightarrow$ (i) to hold, it suffices to show that for any $(x, y, z) \in [0, 1]^3$ such that $\neg(z < y < x)$ the inequality

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}(x, y), \mathcal{I}_{\mathcal{T}}(y, z)) \leq \mathcal{I}_{\mathcal{T}}(x, z) \tag{1}$$

always holds. We consider the following cases.

(i) The case $x \leq z$. Since $\mathcal{I}_{\mathcal{T}}(x, z) = 1$, the inequality (1) is trivially fulfilled.

(ii) The case $z < x$ and $x \leq y$. Since the first partial mappings of $\mathcal{I}_{\mathcal{T}}$ are decreasing, it follows that $\mathcal{I}_{\mathcal{T}}(y, z) \leq \mathcal{I}_{\mathcal{T}}(x, z)$. Together with $\mathcal{I}_{\mathcal{T}}(x, y) = 1$, the inequality (1) follows.

(iii) The case $z < x$, $y < x$, and $y \leq z$. Since the second partial mappings of $\mathcal{I}_{\mathcal{T}}$ are increasing, it follows that $\mathcal{I}_{\mathcal{T}}(x, y) \leq \mathcal{I}_{\mathcal{T}}(x, z)$. Together with $\mathcal{I}_{\mathcal{T}}(y, z) = 1$, this means the inequality (1) again follows. ∎

THEOREM 6. *Consider a* t-*norm* $\mathcal{T}$; *then its biresidual operator* $\mathcal{E}_{\mathcal{T}}$ *is a* $\mathcal{T}$-*equality on* [0, 1] *if and only if* $\mathcal{T}$ *is left-continuous*.

Proof. We will first give the proof from right to left. From Proposition 7 it immediately follows that $\mathcal{E}_{\mathcal{T}}(x, y) = 1$ if and only if $x = y$. The symmetry of $\mathcal{E}_{\mathcal{T}}$ is trivially fulfilled. We will now show the $\mathcal{T}$-transitivity of $\mathcal{E}_{\mathcal{T}}$. Consider $(x, y, z) \in [0, 1]^3$; then

$$\begin{aligned}\mathcal{T}(\mathcal{E}_{\mathcal{T}}(x,y),\mathcal{E}_{\mathcal{T}}(y,z)) &= \mathcal{T}(\min(\mathcal{I}_{\mathcal{T}}(x,y),\mathcal{I}_{\mathcal{T}}(y,x)),\min(\mathcal{I}_{\mathcal{T}}(y,z),\mathcal{I}_{\mathcal{T}}(z,y)))\\ &\leq \min(\mathcal{T}(\mathcal{I}_{\mathcal{T}}(x,y),\mathcal{I}_{\mathcal{T}}(y,z)),\mathcal{T}(\mathcal{I}_{\mathcal{T}}(z,y),\mathcal{I}_{\mathcal{T}}(y,x))).\end{aligned}$$

With Theorem 5 it then follows that

$$\mathcal{T}(\mathcal{E}_{\mathcal{T}}(x, y), \mathcal{E}_{\mathcal{T}}(y, z)) \leq \min(\mathcal{I}_{\mathcal{T}}(x, z), \mathcal{I}_{\mathcal{T}}(z, x)) = \mathcal{E}_{\mathcal{T}}(x, z).$$

Next, we give the proof from left to right. Consider an arbitrary $(x, y, z) \in [0, 1]^3$ such that $z < y < x$. Then it holds that $\mathcal{E}_{\mathcal{T}}(x, y) = \mathcal{I}_{\mathcal{T}}(x, y)$, $\mathcal{E}_{\mathcal{T}}(y, z) = \mathcal{I}_{\mathcal{T}}(y, z)$, and $\mathcal{E}_{\mathcal{T}}(x, z) = \mathcal{I}_{\mathcal{T}}(x, z)$. Since $\mathcal{E}_{\mathcal{T}}$ is $\mathcal{T}$-transitive, it follows that

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}(x, y), \mathcal{I}_{\mathcal{T}}(y, z)) \leq \mathcal{I}_{\mathcal{T}}(x, z).$$

From this, with Lemma 1 and Theorem 5, the left-continuity of $\mathcal{T}$ follows. ∎

In the following proposition we consider a left-continuous t-norm $\mathcal{T}$ with an additive generator f. This implies, however, the left-continuity and hence also the continuity of f. Consequently, $\mathcal{T}$ is a continuous Archimedean t-norm.

PROPOSITION 10. *Consider a continuous Archimedean* t-*norm* $\mathcal{T}$ *with additive generator* f; *then the* $[0,1]^2 \to [0,\infty]$ *mapping* $d = f \circ \mathcal{E}_{\mathcal{T}}$ *is a metric on* [0, 1]. *Moreover, it holds that*

$$d(x, y) = |f(x) - f(y)|.$$

Proof. It follows immediately from Theorems 4 and 6 that d is a metric on [0, 1]. Consider $(x, y) \in [0,1]^2$; then it follows with Proposition 5 that

$$\mathcal{E}_{\mathcal{T}}(x, y) = \min(f^{-1}(\max(0, f(y) - f(x))), f^{-1}(\max(0, f(x) - f(y)))).$$

Since f is decreasing, it then follows that

$$\begin{aligned} d(x, y) = f(\mathcal{E}_{\mathcal{T}}(x, y)) &= \max(f(y) - f(x), f(x) - f(y), 0) \\ &= |f(x) - f(y)|. \end{aligned}$$

∎

EXAMPLE 2. Consider the Łukasiewicz t-norm W with additive generator $f(x) = 1 - x$; then the metric $d = f \circ \mathcal{E}_W$ on [0, 1] is given by $d(x, y) = |x - y|$.

The foregoing proposition can be generalized as follows.

PROPOSITION 11. *Consider a* t-*norm* $\mathcal{T}^*$ *with additive generator* f *and a left-continuous* t-*norm* $\mathcal{T}$. *If* $\mathcal{T}^* \leq \mathcal{T}$, *then the* $[0,1]^2 \to [0,\infty]$ *mapping* $d = f \circ \mathcal{E}_{\mathcal{T}}$ *is a metric on* [0, 1].

Proof. It follows immediately from Theorems 4 and 6. ∎

The following "converse" proposition is quite remarkable, as it allows one to decide, considering one particular (potential) $\mathcal{T}$-equality, whether one t-norm is weaker than another. Note that for any additive generator f and any t-norm $\mathcal{T}$, since $\mathcal{E}_{\mathcal{T}}$ is always reflexive and symmetric, the mapping $d = f \circ \mathcal{E}_{\mathcal{T}}$ satisfies (P1) and (P2).

PROPOSITION 12. *Consider a* t-*norm with an additive generator* f *and a* t-*norm* $\mathcal{T}$. *If the* $[0,1]^2 \to [0,\infty]$ *mapping* $d = f \circ \mathcal{E}_{\mathcal{T}}$ *is a pseudo-metric on* [0, 1], *then* $\mathcal{T}^* \leq \mathcal{T}$.

Proof. Suppose there exists $(a, b) \in [0, 1]^2$ such that $\mathcal{T}^*(a, b) > \mathcal{T}(a, b)$. By assumption, it holds for any $(x, y, z) \in [0, 1]^3$ that

$$f(\mathcal{E}_{\mathcal{T}}(x, z)) \leq f(\mathcal{E}_{\mathcal{T}}(x, y)) + f(\mathcal{E}_{\mathcal{T}}(y, z)).$$

Since $f^{(-1)}$ is decreasing, it then follows that

$$f^{(-1)}(f(\mathcal{E}_{\mathcal{T}}(x, z))) \geq f^{(-1)}(f(\mathcal{E}_{\mathcal{T}}(x, y))) + f(\mathcal{E}_{\mathcal{T}}(y, z))$$

or, equivalently, that

$$\mathcal{E}_{\mathcal{T}}(x, z) \geq \mathcal{T}^*(\mathcal{E}_{\mathcal{T}}(x, y), \mathcal{E}_{\mathcal{T}}(y, z)). \tag{2}$$

Now choose $x = \mathcal{T}(a, b)$, $y = b$, and $z = 1$; then $x \leq y \leq z$. One easily verifies that for this choice it holds that

$$\mathcal{E}_{\mathcal{T}}(x, z) = \mathcal{I}_{\mathcal{T}}(z, x) = \mathcal{I}_{\mathcal{T}}(1, \mathcal{T}(a, b)) = \mathcal{T}(a, b),$$
$$\mathcal{E}_{\mathcal{T}}(x, y) = \mathcal{I}_{\mathcal{T}}(y, x) = \mathcal{I}_{\mathcal{T}}(b, \mathcal{T}(b, a)) \geq a,$$

and

$$\mathcal{E}_{\mathcal{T}}(y, z) = \mathcal{I}_{\mathcal{T}}(z, y) = \mathcal{I}_{\mathcal{T}}(1, b) = b.$$

Substituting the above results into (2), we obtain that $\mathcal{T}(a, b) \geq \mathcal{T}^*(a, b)$, a contradiction. ∎

Note that in the foregoing proposition it is not necessary to impose left-continuity on the t-norm $\mathcal{T}$, since it is not required that $\mathcal{E}_{\mathcal{T}}$ is a $\mathcal{T}$-equality, but only that $d = f \circ \mathcal{E}_{\mathcal{T}}$ is a pseudo-metric. However, if we do impose left-continuity, then this proposition is a stronger version of the implication (ii) $\Rightarrow$ (i) of Theorem 4, for the special case of $X = [0, 1]$, since it allows one to conclude the comparability of t-norms by considering one particular $\mathcal{T}$-equality. We can then state the following corollary.

COROLLARY 3. *Consider a* t-*norm* $\mathcal{T}^*$ *with an additive generator* f *and a left-continuous* t-*norm* $\mathcal{T}$. *If for the* $\mathcal{T}$*-equality* $\mathcal{E}_{\mathcal{T}}$ *on* $[0, 1]$ *it holds that the* $[0, 1]^2 \to [0, \infty]$ *mapping* $d = f \circ \mathcal{E}_{\mathcal{T}}$ *is a metric on* $[0, 1]$, *then* $\mathcal{T}^* \leq \mathcal{T}$.

6. METRICS AND $\mathcal{T}$-EQUALITIES ON $\mathcal{F}(X)$

6.1. *Two Particular* $\mathcal{T}$*-Equalities on* $\mathcal{F}(X)$

In this subsection, we study two particular $\mathcal{T}$-equalities on $\mathcal{F}(X)$, the class of fuzzy sets on a universe X.

DEFINITION 10 ([4]). Consider a t-norm $\mathcal{T}$. The binary fuzzy relation $E^{\mathcal{T}}$ in $\mathcal{F}(X)$ is defined, for any two fuzzy sets A and B in X, as

$$E^{\mathcal{T}}(A, B) = \inf_{x \in X} \mathcal{E}_{\mathcal{T}}(A(x), B(x)).$$

DEFINITION 11 ([7, 14, 16]). Consider a t-norm $\mathcal{T}$. The binary fuzzy relation $E_{\mathcal{T}}$ in $\mathcal{F}(X)$ is defined, for any two fuzzy sets A and B in X, as

$$E_{\mathcal{T}}(A, B) = \mathcal{T}\left(\inf_{x\in X} \mathcal{I}_{\mathcal{T}}(A(x), B(x)), \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(B(x), A(x))\right),$$

PROPOSITION 13. *Consider a* t-*norm* $\mathcal{T}$; *then it holds that* $E_{\mathcal{T}} \subseteq E^{\mathcal{T}}$.

Proof. Consider two fuzzy sets A and B in X; then it holds that

$$\begin{aligned} E_{\mathcal{T}}(A, B) &= \mathcal{T}\left(\inf_{x\in X} \mathcal{I}_{\mathcal{T}}(A(x), B(x)), \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(B(x), A(x))\right) \\ &\leq \inf_{x\in X} \mathcal{T}(\mathcal{I}_{\mathcal{T}}(A(x), B(x)), \mathcal{I}_{\mathcal{T}}(B(x), A(x))) \\ &= \inf_{x\in X} \mathcal{E}_{\mathcal{T}}(A(x), B(x)) = E^{\mathcal{T}}(A, B). \end{aligned}$$

∎

Note that when $\mathcal{T} = M$, it obviously holds that $E_M = E^M$. Also, if $\#X = 1$, say $X = \{x\}$, then for any t-norm $\mathcal{T}$ it holds that $E_{\mathcal{T}}(A, B) = E^{\mathcal{T}}(A, B) = \mathcal{E}_{\mathcal{T}}(A(x), B(x))$.

In the following theorem, we show that any $\mathcal{T}$-equality on [0, 1] can be extended, by means of the infimum operator, to a $\mathcal{T}$-equality on $\mathcal{F}(X)$.

THEOREM 7. *Consider a* t-*norm* $\mathcal{T}$ *and a binary fuzzy relation* E *in* [0, 1]. *Define the binary fuzzy relation* E' *in* $\mathcal{F}(X)$ *as follows, for any two fuzzy sets* A *and* B *in* X:

$$E'(A, B) = \inf_{x\in X} E(A(x), B(x)).$$

Then the following statements are equivalent:

(i) E *is a* $\mathcal{T}$-*equality on* [0, 1].

(ii) E' *is a* $\mathcal{T}$-*equality on* $\mathcal{F}(X)$.

Proof. We will first prove the implication (i) ⇒ (ii).

(a) Consider two fuzzy sets A and B in X; then the following chain of equivalences holds:

$$\begin{aligned} E'(A, B) = 1 &\Leftrightarrow \inf_{x\in X} E(A(x), B(x)) = 1 \\ &\Leftrightarrow (\forall\, x \in X)(E(A(x), B(x)) = 1) \\ &\Leftrightarrow (\forall\, x \in X)(A(x) = B(x)) \Leftrightarrow A = B. \end{aligned}$$

(b) The symmetry of E' is obvious.

(c) Consider three fuzzy sets A, B, and C in X; then

$$\mathcal{T}(E'(A,B), E'(B,C)) = \mathcal{T}\left(\inf_{x\in X} E(A(x),B(x)), \inf_{x\in X} E(B(x),C(x))\right)$$
$$\leq \inf_{x\in X} \mathcal{T}(E(A(x),B(x)), E(B(x),C(x))).$$

Since E is $\mathcal{T}$-transitive, it follows that

$$\mathcal{T}(E'(A,B), E'(B,C)) \leq \inf_{x\in X} E(A(x),C(x)) = E'(A,C).$$

Next, we prove the implication (ii) $\Rightarrow$ (i). Consider $(a,b,c) \in [0,1]^3$ and the corresponding constant fuzzy sets $A(x) = a$, $B(x) = b$, and $C(x) = c$ in X. It then holds that $E'(A,B) = E(a,b)$, $E'(B,C) = E(b,c)$, and $E'(A,C) = E(a,c)$.

(a) Let $E(a,b) = 1$. Then also $E'(A,B) = 1$, which implies that $A = B$ and also that $a = b$.

(b) The symmetry of E follows immediately from the symmetry of E'.

(c) The $\mathcal{T}$-transitivity of E' implies that

$$\mathcal{T}(E'(A,B), E'(B,C)) \leq E'(A,C)$$

and hence also that

$$\mathcal{T}(E(a,b), E(b,c)) \leq E(a,c).$$

∎

COROLLARY 4. *Consider a* t-*norm* $\mathcal{T}$. *The binary fuzzy relation* $E^{\mathcal{T}}$ *is a* $\mathcal{T}$*-equality on* $\mathcal{F}(X)$ *if and only if* $\mathcal{T}$ *is left-continuous.*

Proof. It follows immediately from Theorems 6 and 7. ∎

THEOREM 8. *Consider a* t-*norm* $\mathcal{T}$. *The binary fuzzy relation* $E_{\mathcal{T}}$ *is a* $\mathcal{T}$*-equality on* $\mathcal{F}(X)$ *if and only if* $\mathcal{T}$ *is left-continuous.*

Proof. The proof from right to left was given by Gottwald [7]. Indeed, he has shown that for a left-continuous t-norm $\mathcal{T}$, $E_{\mathcal{T}}$ is a $\mathcal{T}$-equivalence on $\mathcal{F}(X)$. He further demonstrated that in this case $E_{\mathcal{T}}(A,B) = 1$ if and only if $A = B$. This means that $E_{\mathcal{T}}$ is a $\mathcal{T}$-equality on $\mathcal{F}(X)$.

For the proof from left to right, consider $(a,b,c) \in [0,1]^3$ such that $c < b < a$. Consider x_0 in X and construct the fuzzy sets A, B, and C in X as follows:

$$A(x) = \begin{cases} a, & \text{if } x = x_0, \\ 0, & \text{elsewhere.} \end{cases}$$

$$B(x) = \begin{cases} b, & \text{if } x = x_0, \\ 0, & \text{elsewhere.} \end{cases}$$

$$C(x) = \begin{cases} c, & \text{if } x = x_0, \\ 0, & \text{elsewhere.} \end{cases}$$

One easily verifies that in this case $E_{\mathcal{T}}(A, B) = \mathcal{I}_{\mathcal{T}}(a, b)$, $E_{\mathcal{T}}(B, C) = \mathcal{I}_{\mathcal{T}}(b, c)$, and $E_{\mathcal{T}}(A, C) = \mathcal{I}_{\mathcal{T}}(a, c)$. Since $E_{\mathcal{T}}$ is $\mathcal{T}$-transitive, it then follows that

$$\mathcal{T}(E_{\mathcal{T}}(A, B), E_{\mathcal{T}}(B, C)) \leq E_{\mathcal{T}}(A, C)$$

and hence also that

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}(a, b), \mathcal{I}_{\mathcal{T}}(b, c)) \leq \mathcal{I}_{\mathcal{T}}(a, c).$$

With Lemma 1 and Theorem 5, it then follows that $\mathcal{T}$ is left-continuous. ∎

The fuzzy relation $E_{\mathcal{T}}$ is inspired by the following classical equivalence, for any two sets A and B in X:

$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A.$$

The inclusion of a fuzzy set A in X in a fuzzy set B in X is then measured by $\inf_{x \in X} \mathcal{I}(A(x), B(x))$, with $\mathcal{I}$ a fuzzy implication operator, such as the residual implicator $\mathcal{I}_{\mathcal{T}}$ (see e.g. [3, 14]).

6.2. *Metrics on $\mathcal{F}(X)$ Based on $E_{\mathcal{T}}$*

PROPOSITION 14. *Consider a* t-*norm $\mathcal{T}^*$ with an additive generator f and a left-continuous* t-*norm $\mathcal{T}$ such that $\mathcal{T}^* \leq \mathcal{T}$; then the $\mathcal{F}(X)^2 \to [0, \infty]$ mapping $d = f \circ E_{\mathcal{T}}$ is a metric on $\mathcal{F}(X)$.*

Proof. It follows immediately from Theorems 4 and 8. ∎

Note that for any additive generator f and any t-norm $\mathcal{T}$, since $E_{\mathcal{T}}$ is always reflexive and symmetric, the mapping $d = f \circ E_{\mathcal{T}}$ satisfies (P1) and (P2).

THEOREM 9. *Consider a* t-*norm $\mathcal{T}^*$ with additive generator f and a* t-*norm $\mathcal{T}$. If the $\mathcal{F}(X)^2 \to [0, \infty]$ mapping $d = f \circ E_T$ is a pseudo-metric on $\mathcal{F}(X)$, then $\mathcal{T}^* \leq \mathcal{T}$.*

Proof. Suppose there exists $(a, b) \in [0, 1]^2$ such that $\mathcal{T}^*(a, b) > \mathcal{T}(a, b)$. By assumption, it holds for any $(A, B, C) \in \mathcal{F}(X)^3$ that

$$f(E_{\mathcal{T}}(A, C)) \leq f(E_{\mathcal{T}}(A, B)) + f(E_{\mathcal{T}}(B, C)).$$

As in the proof of Proposition 12, it then follows that

$$E_{\mathcal{T}}(A, C) \geq \mathcal{T}^*(E_{\mathcal{T}}(A, B), E_{\mathcal{T}}(B, C)). \tag{3}$$

Now consider two different elements x_0 and y_0 of X and construct the fuzzy sets A, B, and C in X as

$$A(x) = \begin{cases} 1, & \text{if } x = x_0, \\ b, & \text{if } x = y_0, \\ 0, & \text{elsewhere.} \end{cases}$$

$$B(x) = \begin{cases} a, & \text{if } x = x_0, \\ b, & \text{if } x = y_0, \\ 0, & \text{elsewhere.} \end{cases}$$

$$C(x) = \begin{cases} a, & \text{if } x = x_0, \\ 1, & \text{if } x = y_0, \\ 0, & \text{elsewhere.} \end{cases}$$

One easily verifies that for this choice it holds that $E_{\mathcal{T}}(A, B) = a$, $E_{\mathcal{T}}(B, C) = b$, and $E_{\mathcal{T}}(A, C) = \mathcal{T}(a, b)$. Substituting these results in (3), we obtain that $\mathcal{T}(a, b) \geq \mathcal{T}^*(a, b)$, a contradiction. ∎

COROLLARY 5. *Consider a* t-*norm* $\mathcal{T}^*$ *with an additive generator* f *and a left-continuous* t-*norm* $\mathcal{T}$. *If for the* $\mathcal{T}$-*equality* $E_{\mathcal{T}}$ *on* $\mathcal{F}(X)$ *it holds that the* $\mathcal{F}(X)^2 \to [0, \infty]$ *mapping* $d = f \circ E_{\mathcal{T}}$ *is a metric on* $\mathcal{F}(X)$, *then* $\mathcal{T}^* \leq \mathcal{T}$.

As a corollary of Proposition 14 and Theorem 9, we rediscover the main theorem of Gottwald in [7].

COROLLARY 6 ([7]). *Consider a left-continuous t-norm* $\mathcal{T}$. *The* $\mathcal{F}(X)^2 \to [0, \infty]$ *mapping* $d = 1 - E_{\mathcal{T}}$ *is a metric on* $\mathcal{F}(X)$ *if and only if* $W \leq \mathcal{T}$.

We cite Gottwald here [7]: "The intuition behind that relation comes from the interpretation of $E_{\mathcal{T}}$ as a graded measure of the equality of fuzzy sets or of their indistinguishability. The negation of such an indistinguishability relation $E_{\mathcal{T}}$ hence should be a kind of graded distinguishability and thus (perhaps) even a kind of 'distance'."

An important remark should be made here: In Corollary 6, the operation 1– should not be interpreted as the standard negation, but as an additive generator of the Łukasiewicz t-norm. Only this insight can lead to the more general results presented in this paper.

6.3. *Metrics on* $\mathcal{F}(X)$ *Based on* $E^{\mathcal{T}}$

Propositions similar to those in the previous subsection can be written for the $\mathcal{T}$-equality $E^{\mathcal{T}}$. The first proposition is an extended version of Proposition 10.

PROPOSITION 15. *Consider a continuous Archimedean* t-*norm* $\mathcal{T}$ *with an additive generator* f; *then the* $\mathcal{F}(X)^2 \to [0, \infty]$ *mapping* $d = f \circ E^{\mathcal{T}}$ *is a metric on* $\mathcal{F}(X)$. *Moreover, it holds, for any two fuzzy sets* A *and* B *in* X, *that*

$$d(A, B) = \sup_{x \in X} |f(A(x)) - f(B(x))|.$$

Proof. It follows immediately from Corollary 4 and Theorem 4 that d is a metric on $\mathcal{F}(X)$. Consider $(A, B) \in \mathcal{F}(X)^2$; then

$$d(A, B) = f(E^{\mathcal{T}}(A, B)) = f\left(\inf_{x \in X} \mathcal{E}_{\mathcal{T}}(A(x), B(x))\right).$$

Since f is continuous and decreasing, it then follows that

$$d(A, B) = \sup_{x \in X} f(\mathcal{E}_{\mathcal{T}}(A(x), B(x))).$$

As in the proof of Proposition 10, it then follows that

$$d(A, B) = \sup_{x \in X} |f(A(x)) - f(B(x))|.$$

∎

EXAMPLE 3. Consider the Łukasiewicz t-norm W with an additive generator $f(x) = 1 - x$; then the metric $d = f \circ E_W$ on $\mathcal{F}(X)$ is given by, for any two fuzzy sets A and B in X,

$$d(A, B) = \sup_{x \in X} |A(x) - B(x)|.$$

PROPOSITION 16. *Consider a* t-*norm* $\mathcal{T}^*$ *with an additive generator* f *and a left-continuous* t-*norm* $\mathcal{T}$ *such that* $\mathcal{T}^* \leq \mathcal{T}$; *then the* $\mathcal{F}(X)^2 \to [0, \infty]$ *mapping* $d = f \circ E^{\mathcal{T}}$ *is a metric on* $\mathcal{F}(X)$.

Proof. It follows immediately from Corollary 4 and Theorem 4. ∎

The following theorem (also its proof) is an extended version of Proposition 12.

THEOREM 10. *Consider a* t-*norm* $\mathcal{T}^*$ *with an additive generator* f *and a* t-*norm* $\mathcal{T}$. *If the* $\mathcal{F}(X)^2 \to [0, \infty]$ *mapping* $d = f \circ E^{\mathcal{T}}$ *is a pseudo-metric on* $\mathcal{F}(X)$, *then* $\mathcal{T}^* \leq \mathcal{T}$.

Proof. Suppose there exists $(a, b) \in [0, 1]^2$ such that $\mathcal{T}^*(a, b) > \mathcal{T}(a, b)$. As in the proof of Proposition 12, it follows that for any $(A, B, C) \in \mathcal{F}(X)^3$ it holds that

$$E^{\mathcal{T}}(A, C) \geq \mathcal{T}^*(E^{\mathcal{T}}(A, B), E^{\mathcal{T}}(B, C)). \tag{4}$$

Now consider x_0 in X and construct the fuzzy sets A, B, and C in X as follows:

$$A(x) = \begin{cases} \mathcal{T}(a, b), & \text{if } x = x_0, \\ 0, & \text{elsewhere.} \end{cases}$$

$$B(x) = \begin{cases} b, & \text{if } x = x_0, \\ 0, & \text{elsewhere.} \end{cases}$$

$$C(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{elsewhere.} \end{cases}$$

One easily verifies that for this choice it holds that $E^{\mathcal{T}}(A, C) = \mathcal{T}(a, b)$, $E^{\mathcal{T}}(A, B) \geq a$, and $E^{\mathcal{T}}(B, C) = b$. Substituting these findings in (4) we obtain $\mathcal{T}(a, b) \geq \mathcal{T}^*(a, b)$, a contradiction. ∎

COROLLARY 7. *Consider a* t-*norm* $\mathcal{T}^*$ *with an additive generator* f *and a left-continuous* t-*norm* $\mathcal{T}$. *If for the* $\mathcal{T}$-*equality* $E^{\mathcal{T}}$ *on* $\mathcal{F}(X)$ *the* $\mathcal{F}(X)^2 \to [0, \infty]$ *mapping* $d = f \circ E^{\mathcal{T}}$ *is a metric on* $\mathcal{F}(X)$, *then* $\mathcal{T}^* \leq \mathcal{T}$.

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