The Areal Spread of Matrices*

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Let $\mathscr{C}$ be the class of plane regions (in the complex plane), such as triangles, or circles, or convex sets. For a square matrix A, denote by $\mathscr{C}_A$ the subclass of $\mathscr{C}$ consisting of those regions which contain all characteristic roots of A. How "small" a region can be chosen in $\mathscr{C}_A$? As a natural measure of "smallness," we shall adopt the ratio $\sigma(A)/\|A\|^2$, where $\sigma(A)$ is the minimal area of all regions in $\mathscr{C}_A$, and $\|\cdot\|$ denotes the euclidean matrix norm. (This ratio might be called the "areal spread of A with respect to $\mathscr{C}$.") Our problem, then, is to estimate the supremum of $\sigma(A)/\|A\|^2$ as A ranges over all nonzero $n \times n$ matrices. Stated in these broad terms, the problem seems far from easy. As a possible first step towards a comprehensive discussion, we offer here the solution for the special case when $\mathscr{C}$ is the class of circles.

LEMMA 1. *Let Δ be a closed circular disk of minimal radius which contains the points $P_1, \ldots, P_n$. Then either two of the points $P_1, \ldots, P_n$ are the extremities of a diameter of Δ or three of the points lie on the circumference of Δ and form an acute-angled triangle.*

This result, which is closely related to Jung's "covering problem," [1] is well known.

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* Dedicated to Professor A. M. Ostrowski on his 75th birthday.

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LEMMA 2. *If G is the centroid of an acute-angled triangle ABC of circumradius R, then*

$$AG^2 + BG^2 + CG^2 \geqslant 2R^2.$$

To prove this inequality, we note that

$$AG^2 + BG^2 + CG^2 = \tfrac{1}{3}(a^2 + b^2 + c^2).$$

Moreover, in an acute-angled triangle,

$$a^2 + b^2 + c^2 \geqslant 2\{\max(a, b, c)\}^2 \geqslant 6R^2,$$

and the assertion follows.

THEOREM. *Let $n \geqslant 2$ and let $\sigma(A)$ denote the area of the smallest circular disk which contains all characteristic roots of the complex $n \times n$ matrix A. Then*

$$\sup\{\sigma(A)/||A||^2\} = \tfrac{1}{2}\pi,$$

where the supremum is taken over all nonzero $n \times n$ matrices A.

Let $R(A)$ denote the radius of a minimal circular disk, say Δ, which contains all characteristic roots of A. Then the assertion of the theorem is equivalent to the statement that

$$\sup\{R(A)/||A||\} = 1/\sqrt{2}.$$

Since $R(A)/||A|| = 1/\sqrt{2}$ for the $n \times n$ matrix $A = \operatorname{diag}(1, -1, 0, \ldots, 0)$, it only remains to show that $||A||^2 \geqslant 2R(A)^2$ for every A. We note that, for any complex numbers $\omega_1, \ldots, \omega_n$,

$$\sum_{k=1}^{n} |\omega_k|^2 \geqslant \sum_{k=1}^{n} |\omega_k - \zeta|^2, \tag{1}$$

where $\zeta = (\omega_1 + \cdots + \omega_n)/n$. If $\omega_1, \ldots, \omega_n$ denote the characteristic roots of A, then, by Schur's inequality [2],

$$||A||^2 \geqslant \sum_{k=1}^{n} |\omega_k|^2.$$

It suffices, therefore, to show that

$$\sum_{k=1}^{n} |\omega_k|^2 \geqslant 2R(A)^2. \tag{2}$$

Now, by Lemma 1, either two ω's (say ω_1 and ω_2) lie at the extremities of a diameter of Δ or else three ω's (say ω_1, ω_2, ω_3) lie on the circumference of Δ and form an acute-angled triangle. In the former case, writing $\zeta_2 = (\omega_1 + \omega_2)/2$, we have, by (1),

$$\sum_{k=1}^{n} |\omega_k|^2 \geqslant |\omega_1|^2 + |\omega_2|^2 \geqslant |\omega_1 - \zeta_2|^2 + |\omega_2 - \zeta_2|^2 = 2R(A)^2.$$

In the latter case, writing $\zeta_3 = (\omega_1 + \omega_2 + \omega_3)/3$, we have, by (1) and Lemma 2,

$$\sum_{k=1}^{n} |\omega_k|^2 \geqslant |\omega_1|^2 + |\omega_2|^2 + |\omega_3|^2$$

$$\geqslant |\omega_1 - \zeta_3|^2 + |\omega_2 - \zeta_3|^2 + |\omega_3 - \zeta_3|^2 \geqslant 2R(A)^2.$$

Thus (2) is valid in both cases and the proof is complete.

REFERENCES

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2 I. Schur, Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, *Math. Annalen* **66**(1909), 488–510.

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