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On graphs whose second largest eigenvalue equals 1 – the star complement technique

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Abstract

The star complement technique is a spectral tool recently developed for constructing some bigger graphs from their smaller parts, called star complements. Here we first identify among trees and complete graphs those graphs which can be star complements for 1 as the second largest eigenvalue. Using the graphs just obtained, we next search for their maximal extensions, either by theoretical means, or by computer aided search.

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1. Introduction

We will consider only simple graphs, that is finite, undirected graphs without loops or multiple edges. If G is such a graph with vertex set $V_G = \{1, 2, \ldots, n\}$, the *adjacency matrix* of G is the $n \times n$ matrix $A_G = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and 0 otherwise. The *eigenvalues* of G, denoted by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, are just the eigenvalues of A_G. Note, the eigenvalues of G are real and do not depend on vertex labelling. Additionally, for connected graphs $\lambda_1 > \lambda_2$ holds. The *characteristic polynomial* of G is the characteristic polynomial of its adjacency matrix, so $P_G(\lambda) = \det(\lambda I - A_G)$. For more details on graph spectra, see [3].

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If μ is an eigenvalue of G of multiplicity k, then a *star set* for μ in G is a set X of k vertices taken from G such that μ is not an eigenvalue of $G - X$. The graph $H = G - X$ is then called a *star complement* for μ in G (or a *μ-basic subgraph* of G in [7]). (Star sets and star complements exist for any eigenvalue and any graph; they need not be unique.) The H-neighborhoods of vertices in X can be shown to be non-empty and distinct, provided that $\mu \notin \{-1, 0\}$ (see [5, Chapter 7]). If $t = |V_H|$, then $|X| \leqslant \binom{t}{2}$ (see [1]) and this bound is best possible.

It can be proved that if Y is a proper subset of X then $X - Y$ is a star set for μ in $G - Y$, and therefore H is a star complement for μ in $G - Y$. If G has star complement H for μ, and G is not a proper induced subgraph of some other graph with star complement H for μ, then G is a *maximal graph* with star complement H for μ, or it is an *H-maximal graph* for μ. By the above remarks, there are only finitely many such maximal graphs, provided $\mu \notin \{-1, 0\}$. In general, there will be various different maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph (if so, this graph is characterized by its star complement for μ).

We now mention some results from the literature in order to make the paper more self-contained (they are taken from [4–6]).

The following result is known as the Reconstruction Theorem (see, for example, [5, Theorems 7.4.1 and 7.4.4]).

Theorem 1.1. *Let G be a graph with adjacency matrix*

$$\begin{pmatrix} A_X & B^{\mathrm{T}} \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by the vertex set X. Then X is a star set for μ if and only if μ is not an eigenvalue of C and $\mu I - A_X = B^{\mathrm{T}}(\mu I - C)^{-1}B$.

From the above, we see that if μ, C and B are fixed then A_X is uniquely determined. In other words, given the eigenvalue μ, a star complement H for μ, and the H-neighborhoods of the vertices in the star set X, the graph G is uniquely determined. In the light of these facts, we may next ask to what extent G is determined by only H and μ. Having in mind the observation above, it is sufficient to consider graphs G which are H-maximal for μ.

Following [2], we will now fix some further notation and terminology. Given a graph H, a subset U of $V(H)$ and a vertex u not in $V(H)$, denote by $H(U)$ the graph obtained from H by joining u to all vertices of U. We will say that u $(U, H(U))$ is a *good vertex* (resp. *good set*, *good extension*) for μ and H, if μ is an eigenvalue of $H(U)$ but is not an eigenvalue of H. By Theorem 1.1, a vertex u and a subset U are good if and only if $\mathbf{b}_u^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{b}_u = \mu$, where $\mathbf{b}_u$ is the characteristic vector of u (with respect to $V(H)$) and C is the adjacency matrix of H. Assume now that U_1 and U_2 are not necessarily good sets corresponding to vertices u_1 and u_2, respectively. Let $H(U_1, U_2; 0)$ and $H(U_1, U_2; 1)$ be the graphs obtained by adding to H both vertices, u_1 and u_2, so that they are non-adjacent in the former graph, while adjacent in the latter graph. We say that u_1 and u_2 are *good partners* and that U_1 and U_2 are *compatible sets* if μ is an eigenvalue of multiplicity two either in $H(U_1, U_2; 0)$ or in $H(U_1, U_2; 1)$. (Note, if $\mu \notin \{-1, 0\}$, any good set is non-empty, any two of them if corresponding to compatible sets are distinct; see [5, cf. Proposition 7.6.2.].) By Theorem 1.1, two vertices u_1 and u_2 are good partners (or two sets U_1 and U_2 are compatible) if and only if $\mathbf{b}_{u_1}^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{b}_{u_2} \in \{-1, 0\}$, where $\mathbf{b}_{u_1}$ and $\mathbf{b}_{u_2}$ are defined as above. In addition, it follows (again by Theorem 1.1) that any vertex set X in which all vertices are good, both individually and in pairs, gives rise to a *good extension*, say G, in which X can be viewed as a star set for μ with H as the corresponding star complement.

The above considerations show us how we can introduce a technique called the *star complement technique*, for finding (or constructing) graphs with certain spectral properties. In this context the graphs we are interested in should have some prescribed eigenvalue usually of very large multiplicity. If G is a graph in which μ is an eigenvalue of multiplicity $k > 1$, then G is a good (k-vertex) extension of any one of its star complements, say H (in particular, G is H-maximal for μ). The *star complement* technique consists of the following: In order to find H-maximal graphs for $\mu(\neq -1, 0)$, we form an *extendability graph* whose vertices are good vertices for μ and H, and add an edge between two good vertices whenever they are good partners. Now it is easy to see that the search for maximal extensions is reduced to the search for maximal cliques in the extendability graph (see, for example, [4,6]). Of course, among H-maximal graphs some of them can be mutually isomorphic.

Connected graphs with $\lambda_2 \leqslant 1$ have not bean much studied in the literature. Some known results are related to bipartite graphs and (generalized) line graphs (see [8] for details). Here we focus our attention on trees and complete graphs in the role of star complements for 1 as the second largest eigenvalue. In Section 2, we determine all trees and complete graphs which can be star complements for $\lambda_2 = 1$. The good graphs $H(U)$ where H is a tree or a complete graph are considered in Section 3. Finally, all maximal graphs for some of the possible star complements are given in Section 4.

2. Trees and complete graphs as star complements for 1 as the second largest eigenvalue

We focus our attention on connected star complements. The following lemma considers the diameter of a star complement.

Lemma 2.1. *Let H be a connected star complement for $\lambda_2 = 1$. Then* $\operatorname{diam}(H) \leqslant 3$.

Proof. Suppose the contrary, i.e. $\operatorname{diam}(H) = d \geqslant 4$. Then H contains a path P of length d as an induced subgraph. Since $d \geqslant 4$, we have $\lambda_2(P) \geqslant 1$. Therefore, by the Interlacing Theorem (see [3, p. 19]), $\lambda_2(H) \geqslant 1$. On the other hand, if H is a star complement for $\lambda_2 = 1$ then $\lambda_2(H) < 1$. A contradiction, and the proof follows. □

The next lemma will be useful in sequel.

Lemma 2.2. *Let H be a graph containing at least one edge such that $\lambda_2(H) < 1$ and let u, U and $H(U)$ be as defined in Section 1. If U and $H(U)$ are good with respect to* 1 *then*

$$\mathbf{b}^{\mathrm{T}}(I - C)^{-1}\mathbf{b} = 1, \tag{1}$$

where $\mathbf{b}$ is the characteristic vector of u and C is the adjacency matrix of H. In addition, $\lambda_2(H(U)) = 1$.

Proof. Firstly, (1) arises from Theorem 1.1. Since U and $H(U)$ are good with respect to 1, we have that 1 belongs to the spectrum of $H(U)$. Since H has at least one edge, its largest eigenvalue is greater or equal to 1. Then, $\lambda_2(H) < 1$ and the Interlacing Theorem imply $\lambda_2(H(U)) = 1$. □

The following two theorems give us all trees which can be the star complements for $\lambda_2 = 1$. By S_n we denote a star of order n, while by $S_{m,n}$ we denote a double star, i.e. the graph which we obtain from stars S_m and S_n by joining their centers. If at least one of integers m, n equals 1, then the double star $S_{m,n}$ reduces to an ordinary star. Hence, for a double star we will assume that $m, n \geqslant 2$. By Lemma 2.1, if some tree is a star complement for $\lambda_2 = 1$ then it must be a star or a double star.

Theorem 2.1. *S_6, S_{10} and S_{11} are the only stars which can be star complements for $\lambda_2 = 1$.*

Proof. It is easy to check that S_1 and S_2 cannot be star complements for $\lambda_2 = 1$, so we can assume that $n \geqslant 3$. If C is the adjacency matrix of S_n (where the first row and the first column correspond to the center of S_n) then

$$(I - C)^{-1} = \frac{1}{n-2} \begin{pmatrix} -1 & -1 & \dots & -1 \\ -1 & n-3 & \dots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & n-3 \end{pmatrix}.$$

Firstly, let u be joined to the center and to $k - 1$ of the terminal vertices of S_n, and let $\mathbf{b}$ be the characteristic vector of u. So, exactly k of the coordinates of $\mathbf{b}$ are ones (including the first coordinate). If $k = 1$ then the graph obtained is a star as well, and the second largest eigenvalue of a star does not equal 1. For $k \geqslant 2$, by using Eq. (1), we get

$$k^2 + (2 - n)k + 2n - 4 = 0. \tag{2}$$

Since k is an integer, the discriminant of the equation above must be a perfect square, and $(n - 6)^2 - 16$ is a perfect square only for $n = 10$ and $n = 11$ (and in both cases we get that k is a positive integer). Therefore, we have just obtained stars S_{10} and S_{11} which can be star complements for $\lambda_2 = 1$.

Now, let u be joined to k of the terminal vertices of S_n. Then, exactly k of the coordinates of its characteristic vector are ones (here, the first coordinate is zero). Again, Eq. (1) gives us

$$k^2 + (2 - n)k + n - 2 = 0. \tag{3}$$

The discriminant $(n - 4)^2 - 4$ is a perfect square only for $n = 6$ (which gives a positive integer for k). So, in this case, S_6 is the only solution and the result follows. □

Theorem 2.2. *The double star $S_{m,n}$ $(m, n \geqslant 2)$ can be a star complement for $\lambda_2 = 1$ if and only if at least one of integers m, n is equal to 2.*

Proof. In order to prove that $S_{2,n}$ can be a star complement for $\lambda_2 = 1$, it is sufficient to find at least one set $U \subset V(S_{2,n})$ and a vertex u joined to all vertices of U such that $\lambda_2(S_{2,n}(U)) = 1$. For instance, if U contains exactly one terminal vertex of S_n we obtain the situation as required. (Obviously, a star S_n $(n \geqslant 2)$, considered as an induced subgraph of $S_{2,n}$, contains at least one terminal vertex.) Indeed, by removing the center of S_n from $S_{2,n}(U)$ we get a disconnected graph whose spectrum contains 1 as an eigenvalue of multiplicity 2. Then, by Interlacing Theorem, $\lambda_2(S_{2,n}(U)) = 1$.

On the other hand, we have $\lambda_2(S_{3,3}) = 1$. Thus, $\lambda_2(S_{m,n}) \geqslant 1$ for $m, n \geqslant 3$, which completes the proof. □

The following theorem gives us all complete graphs which can be star complements for $\lambda_2 = 1$.

Theorem 2.3. *K_{10} and K_{11} are the only complete graphs which can be the star complements for $\lambda_2 = 1$.*

Proof. For $n \leqslant 2$, the graph K_n is isomorphic to S_n, so we can assume $n \geqslant 3$. If C is the adjacency matrix of K_n then

$$(I - C)^{-1} = \frac{1}{2n-4}\begin{pmatrix} n-3 & -1 & \dots & -1 \\ -1 & n-3 & \dots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & n-3 \end{pmatrix}.$$

If u is joined to k of the vertices of K_n, then exactly k of the coordinates of its characteristic vector are ones. By using Eq. (1), we arrive at Eq. (2), and therefore, $n = 10$ and $n = 11$ are the only solutions. This completes the proof. □

3. Good sets

We now proceed to identify all good sets U, i.e. to identify the sets U for which graph $H(U)$ has 1 as the second largest eigenvalue, where H is any of graphs S_6, S_{10}, S_{11}, $S_{2,n}$ $(n \geqslant 2)$, K_{10}, K_{11}.

We will use the following notation: the center of S_n will be denoted by v. The center of S_n, as well as the center and the only terminal vertex of S_2, when S_n and S_2 are induced subgraphs of $S_{2,n}$ will be denoted by v, w and w_1, respectively. Also, the set of any k of the terminal vertices of S_n, or any k of the vertices of K_n, will be denoted by T_k.

Lemma 3.1. *The graph $S_6(U)$ is good if and only if $U = T_2$. The graph $S_{10}(U)$ is good if and only if $U = \{v\} \cup T_3$. The graph $S_{11}(U)$ is good if and only if $U = \{v\} \cup T_2$ or $U = \{v\} \cup T_5$. The graph $K_{10}(U)$ is good if and only if $U = T_4$. The graph $K_{11}(U)$ is good if and only if $U = T_3$ or $U = T_6$.*

Proof. We find that Eq. (3) has exactly one pair of integral solutions: $(n, k) = (6, 2)$, and therefore $S_6(U)$ is good if and only if $U = T_2$. Similarly, Eq. (2) has three pairs of integral solutions: $(n, k) = (10, 4)$, $(11, 3)$ or $(11, 6)$ what gives us all good graphs $S_{10}(U)$ and $S_{11}(U)$, as well as $K_{10}(U)$ and $K_{11}(U)$. The proof is complete. □

From the previous lemma we have that every graph with any of star complements S_{10}, S_{11} for $\lambda_2 = 1$ is a cone, since the center of the corresponding star is adjacent to all vertices of star set.

In the following lemma we will use Schwenk's formula (see [3, p. 78]) which can be stated as follows: For a given (simple) graph G, let $\mathscr{C}(v)$ denote the set of all cycles containing a vertex v of G. Then

$$P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{w \sim v} P_{G-v-w}(\lambda) - 2 \sum_{C \in \mathscr{C}(v)} P_{G-V(C)}(\lambda), \tag{4}$$

where $w \sim v$ denotes that w is a vertex adjacent to v, while $G - V(C)$ is the graph obtained from G by removing all vertices belonging to the cycle C.

Lemma 3.2. *The graph $S_{2,n}(U)$ is good if and only if U has one of the following forms*:

1. $U = T_1$;
2. $U = \{v\} \cup T_1$;
3. $U = \{w\} \cup T_{n-3}, \quad n \geqslant 3$;
4. $U = \{w_1\} \cup T_{n-2}, \quad n \geqslant 2$;
5. $U = \{v, w\} \cup T_{n-5}, \quad n \geqslant 5$;
6. $U = \{v, w_1\} \cup T_{n-4}, \quad n \geqslant 4$;
7. $U = \{w, w_1\} \cup T_{\frac{4n-8}{3}}$, *whenever* $\frac{4n-8}{3}$ *is a non-negative integer*;
8. $U = \{v, w, w_1\} \cup T_{\frac{4n-12}{3}}$, *whenever* $\frac{4n-12}{3}$ *is a non-negative integer*.

Sketch proof: The proof is based on the application of the formula (4) to the graph $S_{2,n}(U)$. Here we demonstrate how we get the set $U = \{v, w, w_1\} \cup T_{\frac{4n-12}{3}}$, while all other sets are obtained in the similar way.

Consider the graph $S_{2,n}(U)$ where $U = \{v, w, w_1\} \cup T_k$, $(1 \leqslant k \leqslant n-1)$. When we apply the formula (4) with respect to vertex u adjacent to all vertices of set U, we get

$$P_{S_{2,n}(U)}(1) = P_{S_{2,n}}(1) - P_{S_{2,n}-v}(1) - P_{S_{2,n}-w}(1) - P_{S_{2,n}-w_1}(1) - kP_{S_{2,n-1}}(1) - 2\sum_{C\in\mathscr{C}(u)} P_{S_{2,n}(U)-C}(1).$$

By applying (4) to each graph, we find $P_{S_{2,n}}(1) = -1$, $P_{S_{2,n}-v}(1) = 0$, $P_{S_{2,n}-w}(1) = 2-n$, $P_{S_{2,n}-w_1}(1) = 1-n$, $P_{S_{2,n-1}}(1) = -1$ and $\sum_{C\in\mathscr{C}(u)} P_{S_{2,n}-C}(1) = 4-n+2k$. So $P_{S_{2,n}(U)}(1) = -12+4n-3k$, and therefore $P_{S_{2,n}(U)}(1) = 0$ if and only if $k = \frac{4n-12}{3}$. This completes the proof. $\square$

4. Maximal graphs

We will now determine all H-maximal graphs (for $\lambda_2 = 1$) for some of the star complements H given in Theorems 2.1–2.3. As mentioned in Section 1, finding maximal graphs is equivalent to finding maximal cliques in the extendability graph. As already noted, a necessary and sufficient condition for u_1 and u_2 to be good partners follows from Theorem 1.1 (recall, if $\mathbf{b}_{u_1}$ and $\mathbf{b}_{u_2}$ are the characteristic vectors of U_1 and U_2, respectively then u_1 and u_2 are good partners if and only if $\mathbf{b}_{u_1}^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{b}_{u_2}$ is equal either 0 or -1). (Here, u_1 and u_2 are non-adjacent in the former case, and adjacent in the latter case.) This easy criterion for checking if two good vertices are good partners will be used in the sequel.

We first demonstrate this technique if H is equal to S_6.

Theorem 4.1. *There is a unique maximal graph with star complement S_6 for $\lambda_2 = 1$.*

Proof. Due to Lemma 3.1, here we have exactly 10 good sets. Denote them by $U_1, \ldots, U_{10}$, and the corresponding vertices by $u_1, \ldots, u_{10}$, respectively. We have exactly two cases for the intersection of any pair of good sets U_i, U_j, $1 \leqslant i < j \leqslant 10$: they have no vertices in common or they have exactly one vertex in common. We find that in the former case the corresponding vertices u_i and u_j are good partners if they are adjacent, while in the latter case they are good partners if they are non-adjacent. Therefore, each pair of good sets correspond to two vertices which are good partners. This leads us to the unique maximal graph whose star set contains each of 10 vertices $u_1, \ldots, u_{10}$. The proof is complete. □

Remark 4.1. The maximal graph from the previous theorem is a strongly regular graph on 16 vertices, known as the Clebsch graph. Its spectrum is $[-3^5, 1^{10}, 5]$. (In the exponential notation, exponents stand for the multiplicities of the eigenvalues.) This result is established in [9], as well.

In the following theorem, the vertices of K_{10} will be labelled by numbers $1, \ldots, 10$.

Theorem 4.2. *There are exactly two non-isomorphic maximal graphs with star complement K_{10} for $\lambda_2 = 1$.*

Proof. Due to Lemma 3.1, we have that two good sets can have between zero and three vertices in common, and by inspecting these situations we find that they are compatible in two cases: if they have precisely two vertices in common (then the vertices in the star set corresponding to the good are non-adjacent), and if they have an empty intersection (then the corresponding vertices are adjacent).

Consider a maximal graph with star complement K_{10} for $\lambda_2 = 1$. Let $X = \{u_1, \ldots, u_k\}$ be the star set of this graph, and let $X_U = \{U_1, \ldots, U_k\}$ be the collection of corresponding good sets. The rest of the proof will be separated in three parts.

(I) *Let $U_i, U_j \in X_U$. If $|U_i \cap U_j| = 2$ then $U = (U_i \backslash U_j) \cup (U_j \backslash U_i) \in X_U$.*

With no loss of generality, we can say that $U_i = \{1, 2, 3, 4\}$ and $U_j = \{1, 2, 5, 6\}$, then $U = \{3, 4, 5, 6\}$. Assume to the contrary, $U \notin X_U$. Then there is a good set $U_l \in X_U$ such that $|U \cap U_l|$ is equal to either 1 or 3. (Otherwise, $U \notin X_U$ implies that the graph considered is not maximal.) Suppose, firstly $|U \cap U_l| = 1$. The vertex belonging to $U \cap U_l$ also belongs to exactly one of sets U_i, U_j. Say, $U \cap U_l \cap U_i \neq \emptyset$. Since U_l and U_j are compatible, they must have two vertices in common: the only possibility is $U_l \cap U_j = \{1, 2\}$, but then $|U_l \cap U_i| = 3$ which is not possible since U_l and U_i are also compatible. The case $U \cap U_l \cap U_j \neq \emptyset$ follows analogously. Suppose now $|U \cap U_l| = 3$. Again, we must have $|U_l \cap U_j| = 2$, which implies $|U_l \cap U_i| \neq 2$. A contradiction, and this part is proved.

(II) *There are at least two sets in X_U having an empty intersection.*

Assume the contrary. Then every two sets in X_U have exactly two vertices in common. With no loss of generality, we can say $U_1 = \{1, 2, 3, 4\}$, $U_2 = \{1, 2, 5, 6\} \in X_U$. Then, by part (I), we have $U_3 = \{3, 4, 5, 6\} \in X_U$. The maximality of the considered graph implies that we have more sets in X_U. According to our assumption, every other set in X_U has at least two vertices whose labels are strictly less than 7 (in order for it to have two vertices in common with each of U_1, U_2, U_3). It is easy to conclude that the set $U_4 = \{a, b, c, d\}$, where $1 \leqslant a < b \leqslant 6$, $7 \leqslant c < d \leqslant 10$, cannot have two vertices in common with each of U_1, U_2, U_3. The same holds if we take $1 \leqslant a < b < c < d \leqslant 6$. The remaining situation is $1 \leqslant a < b < c \leqslant 6$, $7 \leqslant d \leqslant 10$. Here, we have more than one possibility for the vertices a, b, c. For instance, let $U_4 = \{2, 3, 5, d\} \in X_U$. (Obviously, $|U_4 \cap U_i| = 2$, $i = 1, 2, 3$.) By part (I), we get $U_5 = \{1, 4, 5, d\}$, $U_6 = \{1, 3, 6, d\}$, $U_7 = \{2, 4, 6, d\} \in X_U$. Finally, we check that set $U_8 = \{a', b', c', d'\}$, where $1 \leqslant a' < b' < c' \leqslant 6$, $7 \leqslant d' \leqslant 10$, cannot have two vertices in common with each of $U_1, \ldots, U_7$, so there are no more sets in X_U having two vertices in common with each other set. But, on the other hand, for $U_8 = \{1, 3, 5, d'\}, d' \neq d$, we get $|U_8 \cap U_i| = 2, i = 1, \ldots, 6$ and $|U_8 \cap U_7| = 0$. Therefore, $U_8 \in X_U$ and X_U contains at least two sets having an empty intersection. A contradiction! If we take some other choice for vertices a, b, c such that U_4 satisfies the condition of our assumption, we get a contradiction in a similar way and the proof of this part follows.

(III) *There are exactly two non-isomorphic maximal graphs.*

Due to part (II), we can assume $U_1 = \{1, 2, 3, 4\}, U_2 = \{7, 8, 9, 10\} \in X_U$. If some set is compatible with each of U_1, U_2 then it contains either both of the vertices 5, 6 or neither of these vertices. Now, we distinguish two cases: there is a set in X_U containing vertices 5 and 6, and every set in X_U does not contain vertices 5 and 6.

In the former case, let U_3 ($U_3 \in X_U$) contain the vertices 5 and 6. Then it must have two vertices in common with one of sets U_1, U_2 and an empty intersection with the other. With no loss of generality we can take $U_3 = \{1, 2, 5, 6\}$. By part (I), we have $U_4 = \{3, 4, 5, 6\} \in X_U$. Now, every other set in X_U has a form $\{a, b, c, d\}$, where $1 \leqslant a < b \leqslant 6, 7 \leqslant c < d \leqslant 10$. Suppose, $\{a, b, c, d\} \in X_U$, for some a, b ($1 \leqslant a < b \leqslant 6$), and $c = 7, d = 8$. Then the following sets are the only ones compatible with each of $U_1, \ldots, U_4$: $U_5 = \{1, 2, 7, 8\}$, $U_6 = \{1, 2, 9, 10\}$, $U_7 = \{3, 4, 7, 8\}$, $U_8 = \{3, 4, 9, 10\}$, $U_9 = \{5, 6, 7, 8\}$ and $U_{10} = \{5, 6, 9, 10\}$. In addition, every pair of $U_5, \ldots, U_{10}$ are compatible, and therefore, we have just obtained a maximal graph (with star set determined by the set $X_U = \{U_1, \ldots, U_{10}\}$). If we take some other choice for vertices c, d, we get an isomorphic graph.

In the latter case, except for U_1 and U_2, every set in X_U has the form $\{a', b', c', d'\}$, where $1 \leqslant a' < b' \leqslant 4, 7 \leqslant c' < d' \leqslant 10$. With no loss of generality, we can say that $U'_3 = \{1, 2, 7, 8\} \in X_U$. Then by part (I), we have $U'_4 = \{1, 2, 9, 10\}$, $U'_5 = \{3, 4, 7, 8\}$, $U'_6 = \{3, 4, 9, 10\} \in X_U$. Now, each of non-compatible sets $\{1, 3, 7, 9\}$, $\{1, 3, 7, 10\}$ is compatible with each of $U_1, U_2, U'_3, \ldots, U'_6$. If we choose $U'_7 = \{1, 3, 7, 9\} \in X_U$, then the following sets are the only ones compatible with each of $U_1, U_2, U'_3, \ldots, U'_7$: $U'_8 = \{1, 3, 8, 10\}$, $U'_9 = \{2, 4, 7, 9\}$, $U'_{10} = \{2, 4, 8, 10\}$, $U'_{11} = \{1, 4, 7, 10\}$, $U'_{12} = \{1, 4, 8, 9\}$, $U'_{13} = \{2, 3, 7, 10\}$ and $U'_{14} = \{2, 3, 8, 9\}$. In addition, every pair of $U'_8, \ldots, U'_{14}$ are compatible, and therefore, we have just obtained a maximal graph (with star set determined by the set $X_U = \{U_1, U_2, U'_3, \ldots, U'_{14}\}$). If we take $U'_7 = \{1, 3, 7, 10\} \in X_U$ (or any other possible choice for U'_7), we get an isomorphic graph. This completes the proof. □

Remark 4.2. We give some data for the two maximal graphs obtained in the previous theorem. The first graph has 20 vertices (10 vertices of degree 7, and 10 vertices of degree 13). Its spectrum is $[-4^4, -1^5, 1^{10}, 11]$. The second graph has 24 vertices (14 vertices of degree 5, 2 vertices of degree 9, and 8 vertices of degree 16). Its spectrum is $[-3.28, -3^7, -1, 1^{14}, 11.28]$.

In order to find all maximal graphs for a given star complement and an eigenvalue μ, we have created an SCL (*star complement library*), i.e. a collection of programs related to the star complement technique. This library includes the programs for identifying good sets, for checking their compatibility, for finding maximal cliques and for identifying isomorphism classes. Some results obtained by making use of SCL facilities are given in the next two theorems. We use v, w and w_1 to denote the same vertices as in Section 3, while the terminal vertices of S_n (in the first theorem S_n is an induced subgraph of $S_{2,n}$) are labelled by numbers $1, \ldots, n-1$.

Theorem 4.3. *There are n non-isomorphic maximal graphs with star complement* $S_{2,n}$ $(2 \leqslant n \leqslant 5)$ *for* $\lambda_2 = 1$. *For each graph in the list below, the following data are given: the number of vertices, the number of edges, the spectrum, followed by good sets*

$S_{2,2}$:
G_1: 5, 5, $[-1.68, -1, -0.54, 1, 2.21]$; $\{v, 1\}$.
G_2: 6, 6, $[-2, -1^2, 1^2, 2]$; $\{1\}, \{w_1\}$.
$S_{2,3}$:
G_3: 7, 8, $[-2, -1^2, -0.73, 1^2, 2.73]$; $\{v, 1\}, \{v, 2\}$.
G_4: 8, 11, $[-2^2, -1^2, 1^3, 3]$; $\{w_1, 1\}, \{w_1, 2\}, \{v, w, w_1\}$.
G_5: 10, 15, $[-2^4, 1^5, 3]$; $\{1\}, \{2\}, \{w\}, \{w_1, 1\}, \{w_1, 2\}$.
$S_{2,4}$:
G_6: 9, 11, $[-2.33, -1^3, -0.82, 1^3, 3.15]$; $\{v, 1\}, \{v, 2\}, \{v, 3\}$.
G_7: 10, 16, $[-2.70, -2, -1^3, 1^4, 3.70]$; $\{1\}, \{v, 2\}, \{v, 3\}, \{w_1, 2, 3\}$.
G_8: 12, 24, $[-3.27, -2^3, -1, 1^6, 4.27]$; $\{1\}, \{2\}, \{v, 3\}, \{w, 3\}, \{w_1, 1, 3\}, \{w_1, 2, 3\}$.
G_9: 16, 40, $[-3^5, 1^{10}, 5]$; $\{1\}, \{2\}, \{3\}, \{v, w_1\}, \{w, 1\}, \{w, 2\}, \{w, 3\}, \{w_1, 1, 2\}, \{w_1, 1, 3\}, \{w_1, 2, 3\}$.
$S_{2,5}$:
G_{10}: 11, 14, $[-2.64, -1^4, -0.86, 1^4, 3.50]$; $\{v, 1\}, \{v, 2\}, \{v, 3\}, \{v, 4\}$.
G_{11}: 12, 21, $[-3.27, -2, -1^4, 1^5, 4.27]$; $\{1\}, \{v, 2\}, \{v, 3\}, \{v, 4\}, \{w_1, 2, 3, 4\}$.
G_{12}: 14, 33, $[-4.22, -2^3, -1^2, 1^7, 5.22]$; $\{1\}, \{2\}, \{v, 3\}, \{v, 4\}, \{w, 3, 4\}, \{w_1, 1, 3, 4\}, \{w_1, 2, 3, 4\}$.
G_{13}: 18, 57, $[-4.66, -3^4, -1, 1^{11}, 6.66]$; $\{1\}, \{2\}, \{3\}, \{v, 4\}, \{v, w_1, 4\}, \{w, 1, 4\}, \{w, 2, 4\}, \{w, 3, 4\}, \{w_1, 1, 2, 4\}, \{w_1, 1, 3, 4\}, \{w_1, 2, 3, 4\}$.
G_{14}: 27, 135, $[-5^6, 1^{20}, 10]$; $\{1\}, \{2\}, \{3\}, \{4\}, \{v, w\}, \{v, w_1, 1\}, \{v, w_1, 2\}, \{v, w_1, 3\}, \{v, w_1, 4\}, \{w, 1, 2\}, \{w, 1, 3\}, \{w, 1, 4\}, \{w, 2, 3\}, \{w, 2, 4\}, \{w, 3, 4\}, \{w_1, 1, 2, 3\}, \{w_1, 1, 2, 4\}, \{w_1, 1, 3, 4\}, \{w_1, 2, 3, 4\}$.

Remark 4.3. The graph G_2 is a cycle C_6; the graph G_9 is isomorphic to the Clebsch graph obtained in Theorem 4.1 which means that this graph contains two different trees as a star complements for $\lambda_2 = 1$; the graphs G_5 and G_{14} are strongly regular, as well.

Theorem 4.4. *There are* 15 *non-isomorphic maximal graphs with star complement* S_{10} *for* $\lambda_2 = 1$. *For each graph listed below, the following data are given: the number of vertices, the number of edges, the spectrum, followed by the reduced good sets (since all good sets contain the center of* S_{10}, *we restrict their representation to the terminal vertices only).*

G_1: 17, 58, $[-3, -2^6, -0.69, 0, 1^7, 8.69]$; $\{1, 4, 5\}, \{1, 6, 7\}, \{1, 8, 9\}, \{4, 6, 8\}, \{4, 7, 9\}, \{5, 6, 9\}, \{5, 7, 8\}$.
G_2: 20, 64, $[-3^6, 0^3, 1^{10}, 8]$; $\{1, 6, 7\}, \{1, 6, 8\}, \{1, 6, 9\}, \{1, 7, 8\}, \{1, 7, 9\}, \{1, 8, 9\}, \{6, 7, 8\}, \{6, 7, 9\}, \{6, 8, 9\}, \{7, 8, 9\}$.
G_3: 20, 72, $[-3^6, -0.90, 0^2, 1^{10}, 8.90]$; $\{1, 5, 7\}, \{1, 6, 8\}, \{1, 6, 9\}, \{1, 7, 8\}, \{1, 7, 9\}, \{1, 8, 9\}, \{5, 8, 9\}, \{6, 7, 8\}, \{6, 7, 9\}, \{7, 8, 9\}$.
G_4: 20, 75, $[-3^6, -1.20, 0^2, 1^{10}, 9.20]$; $\{1, 5, 6\}, \{1, 5, 7\}, \{1, 6, 8\}, \{1, 6, 9\}, \{1, 7, 8\}, \{1, 7, 9\}, \{1, 8, 9\}, \{5, 8, 9\}, \{6, 7, 8\}, \{6, 7, 9\}$.

G_5: 20, 76, $[-3^6, -1.29, 0^2, 1^{10}, 9.29]$; $\{1,5,6\}, \{1,5,8\}, \{1,6,9\}, \{1,7,8\}, \{1,7,9\}, \{1,8,9\}, \{5,7,9\}, \{5,8,9\}, \{6,7,8\}, \{6,8,9\}$.

G_6: 20, 76, $[-3^6, -1.29, 0^2, 1^{10}, 9.29]$; $\{1,5,6\}, \{1,7,8\}, \{1,7,9\}, \{1,8,9\}, \{5,7,8\}, \{5,7,9\}, \{5,8,9\}, \{6,7,8\}, \{6,7,9\}, \{6,8,9\}$.

G_7: 20, 76, $[-3^5, -2^2, -0.42, 0, 1^{10}, 9.42]$; $\{1,4,7\}, \{1,5,8\}, \{1,6,9\}, \{1,7,8\}, \{1,7,9\}, \{1,8,9\}, \{4,8,9\}, \{5,7,9\}, \{6,7,8\}, \{7,8,9\}$.

G_8: 20, 79, $[-3^6, -1.57, 0^2, 1^{10}, 9.57]$; $\{1,5,6\}, \{1,5,7\}, \{1,6,8\}, \{1,7,9\}, \{1,8,9\}, \{5,6,9\}, \{5,7,8\}, \{5,8,9\}, \{6,7,8\}, \{6,7,9\}$.

G_9: 23, 97, $[-5.26, -3^5, -1^3, 1^{13}, 10.26]$; $\{1,i,9\}$, *whenever* $2 \leqslant i \leqslant 6$, $\{1,6,7\}, \{1,6,8\}, \{1,7,8\}, \{1,7,9\}, \{1,8,9\}, \{6,7,9\}, \{6,8,9\}, \{7,8,9\}$.

G_{10}: 23, 101, $[-4.92, -3^5, -1.78, -1^2, 1^{13}, 10.71]$; $\{1,2,9\}, \{1,3,9\}, \{1,4,9\}, \{1,5,7\}, \{1,5,9\}, \{1,6,8\}, \{1,6,9\}, \{1,7,8\}, \{1,7,9\}, \{1,8,9\}, \{5,8,9\}, \{6,7,9\}, \{7,8,9\}$.

G_{11}: 23, 102, $[-4.81, -3^5, -2, -1^2, 1^{13}, 10.81]$; $\{1,2,9\}, \{1,3,9\}, \{1,4,9\}, \{1,5,6\}, \{1,5,7\}, \{1,5,9\}, \{1,6,8\}, \{1,6,9\}, \{1,7,8\}, \{1,7,9\}, \{1,8,9\}$.

G_{12}: 26, 134, $[-6, -4.25, -3^4, -2^3, 1^{16}, 12.25]$; $\{1,i,j\}$, *whenever* $2 \leqslant i \leqslant 7, 8 \leqslant j \leqslant 9$, $\{1,6,7\}, \{1,8,9\}, \{6,7,9\}, \{6,8,9\}$.

G_{13}: 29, 175, $[-6^2, -3^7, 1^{19}, 14]$; $\{1,i,j\}$, *whenever* $2 \leqslant i \leqslant 6, 7 \leqslant j \leqslant 9$, $\{1,7,8\}, \{1,7,9\}, \{1,8,9\}, \{7,8,9\}$.

G_{14}: 29, 175, $[-6^2, -3^7, 1^{19}, 14]$; $\{1,i,j\}$, *whenever* $2 \leqslant i \leqslant 7, 8 \leqslant j \leqslant 9$, $\{i,8,9\}$, *whenever* $1 \leqslant i \leqslant 7$.

G_{15}: 38, 331, $[-6^7, -3, -2.19, 1^{28}, 19.19]$; $\{1,i,j\}$, *whenever* $2 \leqslant i < j \leqslant 9$.

Remark 4.4. G_5 and G_6 are cospectral cones over cospectral graphs. G_{13} and G_{14} are cospectral cones, but not cones over cospectral graphs. The graph G_2 is a cone over $G \cup 4K_1$, where G is a strongly regular graph.

The maximal graphs with other star complements obtained in Section 2, are not felt to be interesting for presentation (there are too many such graphs). However, if we are looking for integral or cospectral graphs, or graphs with small number of distinct eigenvalues, then these graphs (or their subgraphs) should be considered in future research. For instance, we found 15 non-isomorphic cospectral graphs having 28 vertices and 174 edges, where each of them is a maximal graph with star complement $S_{2,6}$ for $\lambda_2 = 1$. Their spectrum is $[-5^6, -3, 1^{20}, 13]$.

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