Note

Firing squad synchronization problem in reversible cellular automata

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Abstract

We studied the Firing Squad Synchronization Problem (FSSP) on reversible (i.e., backward deterministic) cellular automata (RCA). First we proved that, in the case of RCA, there is no solution under the usual condition for FSSP where the firing state is only one. So we defined a little weaker condition suitable for RCA in which finite number of firing states are allowed. We showed that Minsky’s solution in time $3n$ can be embedded in an RCA with 99 states that satisfies the new condition. We used the framework of partitioned CA (PCA), which is regarded as a subclass of CA, for making ease of constructing RCA.

1. Introduction

The Firing Squad Synchronization Problem (FSSP) was first devised by Myhill and was introduced by Moore [4]. This is the problem to construct a one-dimensional three-neighbor cellular automata of arbitrary finite length such that one of the end cell (general) makes all the other cells (soldiers) be in a particular state (firing state) at a certain time.

The problem was first solved by Minsky and McCarthy. They constructed a solution synchronizing $n$ cells in $3n$ steps using a divide-and-conquer method. Until now, a 7-state solution of this method has been shown by Yunès [8]. A minimal time, i.e., $(2n - 2)$-step solution was given by Goto. Then Waksman [7] and Balzar [1] reduced the number of states. At present 6-state minimal time solution has been given by Mazoyer [3]. Some other variations such as FSSP on two-dimensional arrays [2] have also been studied.

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In this paper, we studied FSSP on reversible (i.e., backward deterministic) cellular automata (RCA). In general, it is difficult to synchronize cells on RCA, because generation and extinction of signal are restricted. First we proved that, in the case of RCA, there is no solution under the usual condition for FSSP where the firing state is only one. So we defined a little weaker condition suitable for RCA in which finite number of firing states are allowed. We showed that Minsky's solution in time $3n^2$ can be embedded in an RCA with 99 states that satisfies the new condition. We used the framework of partitioned CA (PCA) [5], which can be regarded as a subclass of CA, for making ease of constructing RCA.

2. Cellular automata and synchronizing conditions

2.1. One-dimensional cellular automata

A deterministic one-dimensional cellular automaton (CA) is a system defined by

$A = (Z, Q, N, \varphi_A, \#)$,

where $Z$ is the set of all integers, $Q$ is a nonempty finite set of internal states of each cell, $N = (m_1, m_2, \ldots, m_k)$ ($m_i \in Z$) is a neighborhood index, $\varphi_A : Q^k \rightarrow Q$ is a mapping called a local function, and $\# \in Q$ is a quiescent state which satisfies $\varphi_A(\#, \ldots, \#) = \#$. A CA is called a three-neighbor CA if $N = (-1, 0, 1)$.

A configuration over $Q$ is a mapping $c : Z + Q$. Let $\text{Conf}(Q)$ denote the set of all configurations over $Q$, i.e., $\text{Conf}(Q) = \{c \mid c : Z \rightarrow Q\}$. The function $\Phi_A : \text{Conf}(Q) \rightarrow \text{Conf}(Q)$ defined as follows is called the global function of $A$.

$$\Phi_A(c)(i) = \varphi_A(c(i + m_1), c(i + m_2), \ldots, c(i + m_k)).$$

We say $A$ is reversible (or injective) iff $\Phi_A$ is one-to-one.

2.2. Standard synchronizing condition (SC1)

Let $A = (Z, Q, (-1, 0, 1), \varphi_A, \#)$ be a three-neighbor CA. A standard synchronizing condition SC1 for $A$ is stated as follows.

(SC1) There exist three distinct states $g, s, f \in Q - \{\#\}$ that satisfy the following ($g, s$ and $f$ correspond to general, soldier, and firing states, respectively).

1. $\varphi_A(\#, s, s) = s$, $\varphi_A(s, s, s) = s$, and $\varphi_A(s, s, \#) = s$.
2. Let $c_{s}^{(n)}$ be a configuration defined by

$$c_{s}^{(n)}(x) = \begin{cases} 
    g & \text{if } x = 1, \\
    s & \text{if } x = 2, \ldots, n, \\
    \# & \text{if } x \leq 0 \text{ or } x \geq n + 1.
\end{cases}$$

Then, there is a function $t_{f} : Z_{+} \rightarrow Z_{+}$, where $Z_{+}$ is the set of all positive integers, that satisfies
$\forall n \in \mathbb{Z}_+ \forall x \in \mathbb{Z}$

$((1 \leq x \leq n \Rightarrow \Phi_A^{f(n)}(c_s^{(n)})(x) = f) \land ((x < 1 \lor x > n) \Rightarrow \Phi_A^{f(n)}(c_s^{(n)})(x) = \#),$

and

$\forall n \in \mathbb{Z}_+ \forall i \in \mathbb{Z} \forall x \in \mathbb{Z}(0 \leq i < t_f(n) \Rightarrow (\Phi_A^{f(n)}(c_s^{(n)})(x) \neq f)).$

If $A$ satisfies the above condition, we say $A$ is a solution of FSSP under the synchronizing condition (SC1). The configuration $c_s^{(n)}$ is called an initial configuration for FSSP, where cells $c_s^{(n)}(1), c_s^{(n)}(2), \ldots, c_s^{(n)}(n)$ should be synchronized. The function $t_f(n)$ is called a firing time function which gives the steps to synchronize these $n$ cells. Further, $c_f^{(n)} = \Phi_A^{f(n)}(c_s^{(n)})$ is called a firing configuration.

2.3. Reversible CA and Synchronizing Condition (SC1)

Synchronizing condition (SC1) requires all $n$ cells to become single firing state $f$ on firing configuration. We can prove, in reversible CA, this type of solutions do not exist. It is proved using the next result by Richardson [6].

**Proposition 2.1** (Richardson [6]). Let $A = (\mathbb{Z}, Q, (-1, 0, 1), \varphi_A, \#)$ be any CA, and $\Phi_A$ be the global function of $A$. If $\Phi_A$ is an injection, then there is a CA $B = (\mathbb{Z}, Q, N, \varphi_B, \#)$ whose global function is $\Phi_B = \Phi_A^{-1}$.

**Theorem 2.1.** There is no one-dimensional three-neighbor reversible CA which satisfies the synchronizing condition (SC1).

**Proof.** Suppose there is a one-dimensional reversible CA $A = (\mathbb{Z}, Q, (-1, 0, 1), \varphi_A, \#)$ which satisfies (SC1), and let $t_f(n)$ be the firing time function of $A$. By Proposition 2.1, there is a CA $B = (\mathbb{Z}, Q, N, \varphi_B, \#)$ whose global function is $\Phi_B = \Phi_A^{-1}$. Without loss of generality, we can assume $N = (m_1, m_2, \ldots, m_p) \subset \mathbb{Z}$ and $|m_1| \leq |m_2| \leq \cdots \leq |m_p|$. Assume, at time $t = 0$, $B$ is in the configuration $c_f^{(n)} (= \Phi_A^{f(n)}(c_s^{(n)}))$. We consider the transition process from $c_f^{(n)}$ to $c_s^{(n)}$ on $B$, which is the reverse process from $c_s^{(n)}$ to $c_f^{(n)}$ on $A$. At time $t = 1$, all the $n - 2|m_p|$ cells of $B$ at the positions from $|m_p| + 1$ to $n - |m_p|$ are in the same state $\varphi_B(f, \ldots, f)$, because $c_f^{(n)}(x) = f$ holds for $1 \leq x \leq n$ and the next state of each cell depends only on the present states of the cells within the distance $|m_p|$. That is,

$|m_p| + 1 \leq x \leq n - |m_p| \Rightarrow \Phi_B(c_f^{(n)})(x) = f^{(1)},$

where $f^{(1)} = \varphi_B(f, \ldots, f)$. Generally,

$i|m_p| + 1 \leq x \leq n - i|m_p| \Rightarrow \Phi_B(c_f^{(n)})(x) = f^{(i)} \quad (1)$

holds for $i = 0, 1, \ldots, \lfloor (n-1)/2|m_p| \rfloor$, where $f^{(0)} = f$ and $f^{(j)} = \varphi_B(f^{(j-1)}, \ldots, f^{(j-1)}) (j = 1, 2, \ldots)$. 

Let \( s \) be the number of states of \( B \) (i.e., \( s = |Q| \)). Then, apparently there are \( i_1, i_2 \in \mathbb{Z} \) that satisfy \( 0 < i_1 < i_2 < s \) and \( f^{(i_1)}(0) = f^{(i_2)}(0) \) for large enough \( n \). We claim that there exits \( k (1 < k < i_2) \) such that \( f^{(0)} = f^{(k)} \). If \( i_1 = 0 \) it is done. So consider the case \( i_1 > 0 \). Suppose \( n \geq 2i_2|_m, + 3 \). Then, from (1), the following hold for all \( i \in \{0, 1, 2, \ldots, i_2\} \).

\[
\begin{align*}
\Phi_{B}^{i}(c_{f}^{(n)}([n/2] - 1)) &= f^{(i)}, \\
\Phi_{B}^{i}(c_{f}^{(n)}([n/2])) &= f^{(i)}, \\
\Phi_{B}^{i}(c_{f}^{(n)}([n/2] + 1)) &= f^{(i)}.
\end{align*}
\]

Further, since \( A \) is a three-neighbor CA such that \( \Phi_{B} = \Phi_{A}^{-1} \),

\[
\Phi_{B}^{i-1}(c_{f}^{(n)}([n/2])) = \varphi_{A}(\Phi_{B}^{i}(c_{f}^{(n)}([n/2] - 1)), \Phi_{B}^{i}(c_{f}^{(n)}([n/2])), \Phi_{B}^{i}(c_{f}^{(n)}([n/2] + 1)))
\]

for all \( i \in \{1, 2, \ldots, i_2\} \). Therefore \( f^{(i-1)} = \varphi_{A}(f^{(i)}(f^{(i)}(f^{(i)})), f^{(i)}(f^{(i)}(f^{(i)}))), f^{(i)}(f^{(i)}(f^{(i)}))) \) holds. Thus \( f^{(i-1)} = f^{(i-1)} \) is obtained. By repeating this, we can conclude \( f^{(0)} = f^{(k)} \) for \( k = i_2 - i_1 \). Consequently, the firing state \( f \) appears at time \( t = k \) on \( B \), or equivalently, appears at time \( t = t_{f}(n) - k \) on \( A \). Since \( t_{f}(n) \geq 2n - 2 \) and \( 0 < k \leq i_2 < s \), the relation \( 0 < t_{f}(n) - k < t_{f}(n) \) holds for large enough \( n \). This contradicts the fact that \( A \) satisfies the synchronizing condition (SC1).

\[\square\]

2.4. Synchronizing condition for reversible CA (SC2)

As shown above, in reversible CA, there is no solution with single firing state. So we define a set \( F \subset Q \) of finite number of firing states, and regard that the cells synchronize if the state of each cell is in \( F \) at time \( t_{f}(n) \). Moreover, we assume that not only the \( n \) cells but the other cells are allowed to change its internal states.

Let \( A = (\mathbb{Z}, Q, (-1, 0, 1), \varphi_{A}, #) \) be a reversible CA. A synchronizing condition (SC2) for reversible CA is as follows.

(SC2) There exist two distinct states \( g, s \in Q - \{#\} \) and a state set \( F \subset Q - \{#, g, s\} \) that satisfy the following.

1. \( \varphi_{A}(#, s, s) = s, \varphi_{A}(s, s, s) = s, \) and \( \varphi_{A}(s, s, #) = s. \)
2. Let \( c_{s}^{(n)}(x) \) be a configuration defined by

\[
c_{s}^{(n)}(x) = \begin{cases} 
g & x = 1, \\
s & x = 2, \ldots, n, \\
# & x \leq 0, x \geq n + 1. \end{cases}
\]

Then, there is a function \( t_{f} : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \), that satisfies
\( \forall n \in \mathbb{Z}_+, \forall x \in \mathbb{Z} \)

\[ ((1 \leq x \leq n \Rightarrow \Phi^{(n)}_A(c^{(n)}_s)(x) \in F) \land ((x < 1 \lor x > n) \Rightarrow \Phi^{(n)}_A(c^{(n)}_s)(x) \notin F)), \]

and

\[ \forall n \in \mathbb{Z}_+, \forall i \in \mathbb{Z}, \forall x \in \mathbb{Z}(0 \leq i < t_f(n) \Rightarrow (\Phi^{(n)}_A(c^{(n)}_s)(x) \notin F)). \]

### 3. Construction of a solution on reversible CA

#### 3.1. Definition of partitioned CA

In order to construct a solution for FSSP on reversible space, we use a partitioned CA (PCA).

One-dimensional three-neighbor PCA \( P \) is regarded as the subclass of normal one-dimensional CA, where each cell is partitioned into three parts \( L, C, \) and \( R \). It is defined by

\[ P = (\mathbb{Z}, L, C, R, \varphi_P, (\#, \#, \#)), \]

where \( \mathbb{Z} \) is the set of all integers, \( L, C, R \) is a non empty finite sets of left, center and right internal states of each cell, \( \varphi_P : R \times C \times L \rightarrow L \times C \times R \) is a mapping called a local function, and \( (\#, \#, \#) \in L \times C \times R \) is a quiescent state which satisfies \( \varphi_P(\#, \#, \#) = (\#, \#, \#) \).

A configuration over \( L \times C \times R \) is a mapping \( c : \mathbb{Z} \rightarrow L \times C \times R \), and let \( \text{Conf}(L \times C \times R) \) denote the set of all configurations over \( L \times C \times R \).

\[ \text{Conf}(L \times C \times R) = \{ c : \mathbb{Z} \rightarrow L \times C \times R \}. \]

Global function

\[ \Phi_P : \text{Conf}(L \times C \times R) \rightarrow \text{Conf}(L \times C \times R) \]

is also defined by

\[ \Phi_P(c)(x) = \varphi_P(\text{RIGHT}(c(x - 1)), \text{CENTER}(c(x)), \text{LEFT}(c(x + 1))) \]

where \( \text{LEFT} \) (CENTER, RIGHT, respectively) is the projection function which picks out the left (center, right) element of a triple in \( L \times C \times R \). It has been proved that \( P \) is reversible iff \( \varphi_P \) is one-to-one [5]. Using PCA, we can construct reversible CA with ease.

#### 3.2. 3n time solution on reversible PCA (99-state)

We present a 99-state solution where the numbers of elements of each state sets \( L, C, R \) are 3, 11, 3.

**Solution on reversible PCA (99-state)**

- \( P = (\mathbb{Z}, L, C, R, \varphi_P, (\#, \#, \#)), C = \{ \#, \bar{s}, \bar{s}, \bar{e}, \bar{e}, \bar{u}, \bar{w}, \bar{w}, v, f \}, I = R = \{ \#, \bar{+}, \bar{*} \}. \)
Table 1
Transition table of 99-state reversible PCA \( P \) (\( s \) and \( w \) are symmetric with \( \bar{s} \) and \( \bar{w} \)).

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\[ v | \# | * | + \]

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\[ w | \# | * | + \]

- General, soldier, and quiescent states are \((+, s, +), (#, s, #), \) and \((#, #, #)\) respectively. Initial configuration is

\[ e^o(x) = \begin{cases} 
(+, s, +) & x = 1, \\
(#, s, #) & x = 2, \ldots, n, \\
(#, #, #) & x < 0, x > n + 1.
\end{cases} \]

- Firing states set \( F \) is

\[ F = \{(#, f, +), (+, f, #), (*, f, +), (+, f, *), (#, f, #), (+, f, +), (*, f, #), (#, f, *), \} \]

- Transition table of local function \( \varphi_P \) is shown in Table 1. Fig. 1 shows transition of configurations \((n = 9)\).

If we regard PCA \( P \) as a normal CA (denoted by \( A \)), then \((SC2-1)\) becomes

\[ \varphi_A((#, s, #), (#, s, #), (#, #, #)) = (#, s, #), \]

\[ \varphi_A((#, #, #), (#, s, #), (#, #, #)) = (#, s, #), \]

\[ \varphi_A((#, #, #), (#, s, #), (#, #, #)) = (#, s, #). \]

Since \( \varphi_{#}(#, s, #) = (#, s, #) \) holds, the above condition \((SC2-1)\) is satisfied. The reversibility of \( P \) is concluded by the fact that \( \varphi_P \) is one-to-one.

This solution is based on Minsky's \( 3n \) time one. First, the leftmost cell (general) emits two kinds of signals of velocity \( 1 \) and \( \frac{1}{2} \) to the right. At time \( 3n/2 \), they collide at the center of the \( n \) cells, and thus the problem is divided into two subproblems of synchronizing \([n/2]\) cells. Then the center cell(s) becomes a new general, and
Fig. 1. Synchronization of 9 cells by the 99-state reversible PCA P.

emits these signals in both directions. Repeating this process, it is finally reduced to the problem of synchronizing single cell. Propagation and bouncing of signals can be easily performed reversively on a PCA. But, in order to make it completely reversible, the following information must be kept in the cells where the collisions of signals have occurred: (1) the direction of the velocity $\frac{1}{3}$ signal, and (2) the phase of the collision of two signals (it depends on the parity of the length of the array). They are memorized by an “arrow” of the center part state and the state “+” of the left and right parts. They can be thought as “garbage” information.

From the construction of $P$, we can see the following facts. If the length $n$ of the array is even, the $(n+2)/2$-th cell from the general becomes a new general for the two arrays of length $n/2$ at time $3n/2$ (hereafter the old general acts as a “wall” that
bounces the signal of velocity 1). If the length $n$ is odd, the $(n + 1)/2$-th and the $(n + 3)/2$-th cells from the general become new generals for the two arrays of length $(n - 1)/2$ at time $3(n + 1)/2$. By above, $t_f(n) = 3n$ is concluded (by a simple induction on $n$).

In general, simulating an irreversible CA by a reversible one, "garbages" spread over. But in this solution, all signals (including signals introduced to keep reversibility) are confined to the $n + 2$ cells and do not spread over the entire cell space. We call it garbage preserving reversible CA.

4. Conclusion

We defined the firing squad synchronization problem for reversible CA and constructed a 99-state solution based on Minsky’s $3n$ time one. Of course the number of states (99-state) may not be minimum. Finding simpler solutions is left for future study.

$3n$ time solution uses velocity 1 and $\frac{1}{3}$ signals. By using velocity $\frac{1}{7}$ signal in addition, we can construct $(7/3)n$ time solution. Moreover, using large enough but finite number of signals of velocity $(1, 1/3, 1/7, 1/15, \ldots)$, $(2 + \varepsilon)n$ time solution can be constructed for any $\varepsilon > 0$. But it is an open problem whether there is a $2n - 2$ time (or even $2n + c$ time) solution on RCA.

References