Variational iteration method for coupled nonlinear Schrödinger equations

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Received 28 September 2006; accepted 20 December 2006

Abstract

In this paper, we apply the variational iteration method proposed by Ji-Huan He to simulate numerically a system of two coupled nonlinear one-dimensional Schrödinger equations subjected initially to a prescribed periodic wave solution. Test examples are given to demonstrate the accuracy and capability of the method with different wave–wave interaction coefficients. The accuracy of the method is verified by ensuring that the energy conservation remains almost constant. The numerical results obtained with a minimum amount of computation show that the variational iteration method is much easier, more convenient and efficient for solving nonlinear partial differential equations.

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Keywords: Coupled nonlinear Schrödinger equations; Variational iteration method

1. Introduction

In recent years much attention has been devoted to simulating real life problems which can be described by weakly (or strongly) nonlinear partial differential equations using reliable and more efficient methods; for more details on these methods see [1]. The variational iteration method (VIM) proposed by Ji-Huan He (see [1,2] and the references cited therein) is one of the methods which have received much concern. It is based on the Lagrange multiplier and it has the merits of simplicity and easy execution. Unlike the traditional numerical methods, VIM needs no discretization, linearization, transformation or perturbation. Many authors (see [1–10], and the references cited therein) pointed out that VIM can overcome the difficulties arising in the calculation of Adomian polynomials in the Adomain decomposition method (see [11–13] and the references therein).

In this work we apply VIM to simulate the following system of two coupled one dimensional nonlinear Schrödinger equations:

\begin{align}
\text{i}A_t + A_{xx} + (|A|^2 + \beta|B|^2)A &= 0, \quad x \in \mathbb{R}, \quad t \geq 0 \\
\text{i}B_t + B_{xx} + (\beta|A|^2 + |B|^2)B &= 0,
\end{align}

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with initial conditions \( A(x, 0) = f(x) \), \( B(x, 0) = g(x) \) and homogenous boundary conditions, where \( i = \sqrt{-1} \). The unknowns \( A(x, t) \), \( B(x, t) \) are the envelopes of the wave packets, and \( \beta \) is the wave–wave interaction coefficient which describes the cross modulation of the wave packets; for more details see [14]. The assumption of homogenous boundary conditions is for simplicity only and it is not essential: the method can be easily designed for arbitrary domain and non-homogenous boundary conditions. The system of two coupled nonlinear Schrödinger equations has wide applications in physics, e.g., nonlinear optics and geophysical fluid dynamics [14,15]. The system (1.1) and (1.2) was first analyzed in [16]. It is known that this system is completely integrable, i.e., it can be solved by the inverse scattering method [14,17–21]. Numerical study for the long time behavior of the solution of the system (1.1) and (1.2) is an interesting and important problem in applications [14]. There are many previous works on the solitary wave solutions for the system (1.1) and (1.2), for more details see [15,22–28].

2. Variational iteration method

In this section, we will illustrate the VIM for solving the following non-homogeneous, nonlinear coupled system of partial differential equations:

\[
L_1 u(x, t) + N_1(u(x, t), v(x, t)) = p(x, t),
\]
\[
L_2 v(x, t) + N_2(u(x, t), v(x, t)) = q(x, t),
\]

where \( L_1, L_2 \) are linear differential operators with respect to time and \( N_1, N_2 \) are nonlinear operators and \( p(x, t), q(x, t) \) are arbitrary (smooth) nonlinear given functions. According to the variational iteration methodology [1–29], we can construct correct functionals as follows:

\[
u_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(\tau) [L_2 v_n(x, \tau) + N_2(\tilde{u}_n(\tau), \tilde{v}_n(\tau)) - q(x, \tau)]d\tau,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are general Lagrange multipliers, which can be identified optimally via variational theory. The second term on the right-hand side in (2.3) and (2.4) are called the corrections, the subscript \( n \) denotes the \( n \)th order approximation. Under suitable restricted variational assumptions (i.e. \( \tilde{u}_n \) and \( \tilde{v}_n \) are considered as a restricted variation) we can assume that the above correctional functionals are stationary (i.e., \( \delta u_{n+1} = 0 \) and \( \delta v_{n+1} = 0 \)), then the Lagrange multipliers can be identified. Now we can start with the given initial approximation and by the above iteration formulas we can obtain the approximate solutions.

2.1. Implementation of VIM to the model problem

Consider the system (1.1) and (1.2) and by using VIM we can construct the following correction variational functionals in the region \( R = [x_L < x < x_R] \times [t > 0] \) with its boundary \( \partial R \) which consists of the ordinates \( x = x_L, x = x_R \) and the axis \( t \geq 0 \):

\[
A_{n+1}(x, t) = A_n(x, t) + \int_0^t \lambda_1(\tau) [A_n - i\hat{A}_{n}\hat{A}_n + i(\hat{A}_n)^2 + \beta|\tilde{B}_n|^2]\hat{A}_n d\tau,
\]

\[
B_{n+1}(x, t) = B_n(x, t) + \int_0^t \lambda_2(\tau) [B_n - i\hat{B}_{n}\hat{B}_n + i(\hat{B}_n)^2 + |\tilde{A}_n|^2]\hat{B}_n d\tau,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are general Lagrange multipliers, \( \hat{A}_{n}, \hat{B}_{n} \) denote restricted variations i.e. \( \delta \hat{A}_{n} = 0 \). Making the above correction functionals stationary, we obtain the following stationary conditions:

\[
\begin{align*}
1 + \lambda_1(\tau)|_{t=t} &= 0, & \dot{\lambda}_1(\tau) &= 0, \\
1 + \lambda_2(\tau)|_{t=t} &= 0, & \dot{\lambda}_2(\tau) &= 0.
\end{align*}
\]

The Lagrange multiplier, therefore, can be defined in the following form:

\[
\lambda_1(\tau) = -1, \quad \lambda_2(\tau) = -1.
\]
Substituting (3.3) into the correction functional equations (3.1) and (3.2) results in the following iteration formulas:

\[ A_{n+1}(x, t) = A_n(x, t) - \int_0^t [A_{nt} - iA_{nx} - i(|A_n|^2 + \beta |B_n|^2)A_n]d\tau \]  
\[ B_{n+1}(x, t) = B_n(x, t) - \int_0^t [B_{nt} - iB_{nx} - i(\beta |A_n|^2 + |B_n|^2)B_n]d\tau. \]

(3.4)

(3.5)

In an algorithmic form, the solution procedure is as follows.

**Algorithm.** Let \( n \) be the iteration index, set a suitable value for the tolerance (Tol)

Step 1: compute \( A_0 = A(x, 0) \) and \( B_0 = B(x, 0) \) given by (3.6) and (3.7), set \( n = 0 \);

Step 2: use the calculated values of \( A_n \) and \( B_n \) to compute \( A_{n+1}(x, t) \) from (3.4);

Step 3: define \( A_n := A_{n+1} \); 

Step 4: use the calculated values of \( A_n \) and \( B_n \) to compute \( B_{n+1}(x, t) \) from (3.5);

Step 5: if \( \max |A_{n+1} - A_n| < \text{Tol} \) and \( \max |B_{n+1} - B_n| < \text{Tol} \) stop, otherwise continue;

Step 6: set \( A_{n+1} := A_n \);

Step 7: set \( n = n + 1 \), Return to step 2;

2.2. Numerical experiment

Using the above algorithm with the following initial approximations [14]:

\[ A(x, 0) = f(x) = a_0(1 - \varepsilon \cos(\alpha x)), \]  
\[ B(x, 0) = g(x) = b_0(1 - \varepsilon \cos(\alpha x)), \]

(3.6)

(3.7)

where \( a_0, b_0 \) are the initial amplitudes of the two perturbed periodic waves, respectively, \( \varepsilon \ll 1 \) is a small parameter, it represents the strength of the perturbation and \( \alpha \) is the wave number of the perturbation. The other components can be obtained directly:

\[ A_1(x, t) = a_0(1 - \varepsilon \cos(\alpha x)) + ia_0(\alpha^2 \varepsilon \cos(\alpha x) - (a_0^2 + \beta b_0^2)(-1 + \varepsilon \cos(\alpha x))^3), \]

\[ B_1(x, t) = b_0(1 - \varepsilon \cos(\alpha x)) + ib_0(\alpha^2 \varepsilon \cos(\alpha x) - (\beta a_0^2 + b_0^2)(-1 + \varepsilon \cos(\alpha x))^3). \]

The rest of components of the iteration formulas (3.4) and (3.5) were obtained in the same manner using the Mathematica Package. The numerical simulation are represented in Figs. 1–3 at different time values from \( t = 0 \) to \( t = 40 \) in the region \( 0 \leq x \leq 115 \), where \( \beta = 1, 2/3, 2 \) and \( a_0 = 0.08, b_0 = 0.1, \alpha = 0.05, \varepsilon = 0.05 \). The results show that few iterations (about two to seven iterations) in the above algorithm lead to fast converge, the accuracy researches Tol = 10\(^{-5}\) for five iterations. It is evident that the overall errors can be made smaller for more iterations.

3. Conserved quantities

To demonstrate the higher accuracy of VIM, we use the same procedure as that in [28] which emphasizes that a good numerical scheme should have excellent long-time numerical behavior, as well as energy conservation properties. To verify the accuracy we consider the following two conserved quantities:

\[ E(A) := \int_0^s |A(x, t)|^2dx \quad \text{and} \quad E(B) := \int_0^s |B(x, t)|^2dx, \]

where \( s := \frac{2\pi}{\alpha} = 40\pi \) is the spatial period of the solution [19] for \( \alpha = 0.05 \). The calculated values of \( E(A) \) and \( E(B) \) for different wave–wave interaction coefficient \( \beta \), where \( \beta = 1, 2, 2/3 \), are given in Tables 1, 2 and 3 respectively. The nearly constant values of both \( E(A) \) and \( E(B) \) show that the method is very effective. Also, following the stability analysis suggested by Tan and Boyd [14], the wave solution is linearly stable only if the perturbation wave number \( \alpha \) is above the critical value \( \alpha_c \), otherwise the wave solution is unstable. We choose the constants as \( a_0 = 0.08, b_0 = 0.1, \alpha = 0.05 \), and find that \( \alpha_c = 0.181108 \), therefore, the wave solution in this case is unstable. The amplitude of \( A \) and \( B \) undergoes oscillations between the near-uniform state and the one-hume state, see Fig. 1.
Fig. 1a. The solution $|A(x, t)|$ at $(\beta = 1)$.

Fig. 1b. The solution $|B(x, t)|$ at $(\beta = 1)$.

Fig. 2a. The solution $|A(x, t)|$ at $(\beta = 2/3)$. 
Fig. 2b. The solution $|B(x,t)|$ at ($\beta = 2/3$).

Fig. 3a. The solution $|A(x,t)|$ at ($\beta = 2$).

Fig. 3b. The solution $|B(x,t)|$ at ($\beta = 2$).
Table 1
(\(\beta = 1\))

<table>
<thead>
<tr>
<th>Time</th>
<th>(E(A))</th>
<th>(E(B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.824855</td>
<td>1.28884</td>
</tr>
<tr>
<td>6</td>
<td>0.831142</td>
<td>1.29866</td>
</tr>
<tr>
<td>9</td>
<td>0.841619</td>
<td>1.31503</td>
</tr>
<tr>
<td>12</td>
<td>0.856288</td>
<td>1.33795</td>
</tr>
<tr>
<td>15</td>
<td>0.875147</td>
<td>1.36742</td>
</tr>
<tr>
<td>18</td>
<td>0.898198</td>
<td>1.40343</td>
</tr>
</tbody>
</table>

The critical value in this case: \(\alpha_c = \sqrt{2(a_0^2 + b_0^2)} = 0.181108\).

Table 2
(\(\beta = 2/3\))

<table>
<thead>
<tr>
<th>Time</th>
<th>(E(A))</th>
<th>(E(B))</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>0.824087</td>
<td>1.28804</td>
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<tr>
<td>6</td>
<td>0.828069</td>
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<td>15</td>
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</tr>
<tr>
<td>18</td>
<td>0.870548</td>
<td>1.37465</td>
</tr>
</tbody>
</table>

The critical value in this case: \(\alpha_c = \sqrt{2(a_0^2 + b_0^2 + \sqrt{(a_0^2 + b_0^2)^2 - \left(\frac{209}{37}a_0^2b_0^2\right)}} = 0.166306\).

Table 3
(\(\beta = 2\))

<table>
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<th>Time</th>
<th>(E(A))</th>
<th>(E(B))</th>
</tr>
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<tbody>
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<tr>
<td>15</td>
<td>0.958936</td>
<td>1.44414</td>
</tr>
<tr>
<td>18</td>
<td>1.018850</td>
<td>1.51391</td>
</tr>
</tbody>
</table>

The critical value in this case: \(\alpha_c = \sqrt{2(a_0^2 + b_0^2 + \sqrt{(a_0^2 + b_0^2)^2 + 12a_0^2b_0^2)}} = 0.220458\).

4. Conclusions

In this paper, the variational iteration method with its merits of simplicity and easy execution has been successfully applied to simulate a system of two coupled nonlinear Schrödinger equations on different ranges of the wave–wave interaction coefficient. An algorithm is also given to simulate the solution of the coupled nonlinear Schrödinger equations. The accuracy of the method is verified by ensuring that the conserved quantities remain almost constant. A clear conclusion can be drawn from the numerical results that the variational iteration method provides with highly accurate numerical solutions without spatial discretizations for nonlinear partial differential equations.

References


