On the Metrization of the Natural Topology on the Space $A^B$

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1. Let $A$ be a set and $B$ a well-ordered set. In [10] Steiner and Steiner considered the so-called natural topology $\mathcal{N}$ on $A^B$, which is an extension of the natural topology on the space of sequences on two symbols [1]. Two elements $x$ and $y$ of $A^B$ are equal if and only if for each $\alpha \in B$, $x_\alpha = y_\alpha$. For each $x \in A^B$ and $\alpha \in B$ define

$$x(\alpha) = \{y \in A^B | y_\beta = x_\beta \text{ for all } \beta \in B, \beta \leq \alpha\}.$$  

Then the natural topology $\mathcal{N}$ on $A^B$ is defined to be the topology generated by the base $\{x(\alpha) | x \in A^B, \alpha \in B\}$. $(A^B, \mathcal{N})$ is a normal, totally disconnected $T_1$-space. $\mathcal{N}$ is the product topology on $A^B$ if and only if $A$ has at most one element or $B$ has order-type $\leq \omega$. It is easy to show: for any two sets $x(\alpha)$, $y(\beta)$ in the base their intersection is either empty or is equal to one of them. $(A^B, \mathcal{N})$ is compact [separable] if and only if $A$ is finite [countable] and $B$ has order-type $\leq \omega$. Furthermore the authors showed in [10]: $(A^B, \mathcal{N})$ is metrizable if $B$ is countable. The purpose of this note is to prove a converse assertion:

**Theorem.** $(A^B, \mathcal{N})$ is metrizable if and only if there exists a set $C$ and a countable well-ordered set $D$, such that $(A^B, \mathcal{N})$ is homeomorphic to a subspace $P \subseteq (C^D, \mathcal{N})$.

As a by-product we get a corollary, which completely characterizes the non-archimedeanly metrizable spaces (sometimes called ultrametric spaces [2]). A metric space $(X, d)$ is said to be a non-archimedean (n.-a.) metric space, if its metric $d$ satisfies the strong triangle inequality

$$d(x, y) \leq \max(d(x, z), d(z, y))$$

for all $x, y, z$ in $X$. These spaces were investigated extensively, especially by Monna [6], de Groot and de Vries [3, 4]. (In [7, 12] the authors suggested and discussed several ways of an applicability of n.-a. metrics in natural science.)
It will be shown that the class of n.-a. metrizable spaces is exactly the class of spaces, which are homeomorphic to subspaces $P \subset (A^B, \mathcal{N})$, where $A$ is any set and $B$ is a countable well-ordered set.

$\mathbb{N}$ always denotes the set of non-negative integers.

2. The proof of the theorem will be given by three steps. Remember, first, the definition of the so-called Baire's 0-dimensional spaces $(S(\Omega), d)$: let $\Omega$ be a set and denote by $S(\Omega)$ the set of all sequences of elements in $\Omega$. Defining the distance $d$ of two points $x = (x_1, x_2, ...), y = (y_1, y_2, ...)$ in $S(\Omega)$ by

$$d(x, y) = \frac{1}{\min \{k : x_k \neq y_k \}}$$

we get the 0-dimensional metric space $(S(\Omega), d)$.

Lemma 1 is almost obvious:

**Lemma 1.** For each set $\Omega$, $(\Omega^\mathbb{N}, \mathcal{N})$ is homeomorphic to $(S(\Omega), d)$.

**Proof.** By definition, $\Omega^\mathbb{N} = S(\Omega)$. For each $x \in S(\Omega)$ and $l \in \mathbb{N}$ let

$$B \left( x, \frac{1}{l} \right) = \left\{ y \in S(\Omega) : d(x, y) < \frac{1}{l} \right\}.$$

Hence

$$B \left( x, \frac{1}{l} \right) = \left\{ y \in S(\Omega) : \frac{1}{\min \{k : x_k \neq y_k \}} < \frac{1}{l} \right\}$$

$$= \left\{ y \in S(\Omega) : y_i = x_i, i \leq l \right\} = x(l).$$

Thus the family $\{x(l) : x \in \Omega^\mathbb{N}, l \in \mathbb{N}\}$, which is a base for the natural topology $\mathcal{N}$, coincides with the family of all open $d$-balls in $(S(\Omega), d)$. Therefore $(S(\Omega), d) \cong (\Omega^\mathbb{N}, \mathcal{N})$.

**Lemma 2.** If the space $(A^B, \mathcal{N})$ is metrizable, it is non-archimedeanly metrizable.

**Proof.** De Groot proved the following theorem: a metric space $(X, d)$ is n.-a. metrizable if and only if $\text{Ind} X = 0$. ($\text{Ind} X$ denotes the strong inductive dimension of $X$. As to the definition of $\text{Ind} X$ see e.g. [9]). So we have to show $\text{Ind}(A^B, \mathcal{N}) = 0$. Let $E \subset A^B$, $G \subset A^B$, be two closed non-void sets, $E \cap G = \emptyset$, and let $H$ be the union of all sets $x(a)$ with $x(a) \cap G \neq \emptyset$, $x(a) \cap E = \emptyset$. Define $F$ to be the union of all sets $x(a)$, which satisfy
\(x(\alpha) \cap G = \emptyset\) and \(x(\alpha) \subseteq H\). \(F\) and \(H\) are open sets and \(F \cup H = A^B\). As mentioned above for each two sets \(x(\alpha), y(\beta)\) we have

\[x(\alpha) \cap y(\beta) \neq \emptyset \Rightarrow x(\alpha) \subseteq y(\beta) \quad \text{or} \quad x(\alpha) \supseteq y(\beta).\]

From this follows \(F \cap H = \emptyset\). Thus \(F\) and \(H\) are open and closed sets and since \(E \subseteq F\), \(G \subseteq H\) we have \(\text{Ind}(A^B, \mathcal{N}) = 0\).

**Corollary.** A topological space \((X, \tau)\) is non-archimedeanly metrizable if and only if there exists a set \(A\) and a countable well-ordered set \(B\) such that \((X, \tau)\) is homeomorphic to a subspace \(P \subseteq (A^B, \mathcal{N})\).

**Proof.** If \(B\) is countable, then \(\bigcup \{x(\alpha) / x \in A^B, \alpha \in B\}\) is a \(\sigma\)-locally discrete base for \(\mathcal{N}\) [10] and, since \((A^B, \mathcal{N})\) is a normal \(T_1\)-space, the metrizability of \((A^B, \mathcal{N})\) follows directly from the Nagata-Smirnov theorem [5]. Consequently, by Lemma 2, \((A^B, \mathcal{N})\) is n.-a. metrizable.

Conversely, let \((X, \tau)\) be a n.-a. metrizable space, thus \(\text{Ind} X = 0\). Morita [8] proved the following theorem: a metric space \(Y\) has \(\dim Y \leq n\) if and only if there exists a subspace \(P\) of \((S(\Omega), d)\) for a suitable set \(\Omega\) and a closed continuous mapping \(f\) of \(P\) onto \(X\) such that for each point \(y \in Y\), \(f^{-1}(y)\) consists of at most \(n + 1\) points.

Now let us apply this theorem to the space \((X, \tau)\). As for all metric spaces \(Y\): \(\dim Y = \text{Ind} Y\), we have \(\dim X = 0\). In this case the mapping \(f\) is a homeomorphism and therefore there exists a set \(\Omega\) such that \((X, \tau)\) is homeomorphic to a suitable subspace \(P\) of \((S(\Omega), d)\). Finally, by Lemma 1, \((S(\Omega), d) \cong (\Omega^N, \mathcal{N})\), and the proof is completed.

3. The Proof of the Theorem is a consequence of the preceding statements. As pointed out in the introduction, \((C^D, \mathcal{N})\) is metrizable, if \(D\) is countable. Thus any subspace \(P \subseteq (C^D, \mathcal{N})\) is metrizable, too. Now let \((A^B, \mathcal{N})\) be metrizable. Accordingly, by Lemma 2, \((A^B, \mathcal{N})\) is n.-a. metrizable and, as the proof of the corollary showed, there exists a set \(\Omega\) such that \((A^B, \mathcal{N})\) is homeomorphic to a subspace of \((\Omega^N, \mathcal{N})\).

**References**