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On lower bounds for the L_2-discrepancy

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ABSTRACT

The L_2-discrepancy measures the irregularity of the distribution of a finite point set. In this note, we prove lower bounds for the L_2-discrepancy of arbitrary N-point sets. Our main focus is on the two-dimensional case. Asymptotic upper and lower estimates of the L_2-discrepancy in dimension 2 are well known, and are of the sharp order $\sqrt{\log N}$. Nevertheless, the gap in the constants between the best-known lower and upper bounds is unsatisfactorily large for a two-dimensional problem. Our lower bound improves upon this situation considerably. The main method is an adaption of Roth's method, using the Fourier coefficients of the discrepancy function with respect to the Haar basis. We obtain the same improvement in the quotient of lower and upper bounds in the general d-dimensional case. Our lower bounds are also valid for the weighted discrepancy.

1. Introduction

The L_2-discrepancy is a measure for the irregularity of the distribution of a finite point set with respect to the uniform distribution. If $\mathcal{P}$ is an N-point subset of the d-dimensional unit cube $\mathbb{Q}^d = [0,1)^d$, the *discrepancy function* $D_{\mathcal{P}}$ is defined as

$$D_{\mathcal{P}}(x) := \sum_{z \in \mathcal{P}} \mathbb{1}_{C_z}(x) - N|B_x|. \tag{1}$$

$|B_x| = x_1, \dots, x_d$ denotes the volume of the rectangular box $B_x = [0, x_1) \times \cdots \times [0, x_d)$ for $x = (x_1, \dots, x_d) \in \mathbb{Q}^d$, and $\mathbb{1}_{C_z}$ is the characteristic function of the rectangular box $C_z = (z_1, 1) \times \cdots \times (z_d, 1)$ for $z = (z_1, \dots, z_d) \in \mathcal{P}$. Observe that the sum in this definition is just the number of points of $\mathcal{P}$ in the box B_x. So the discrepancy function measures the deviation of this number from the fair

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number of points $N|B_x|$ which would be achieved by a perfect (but impossible) uniform distribution of the points of $\mathcal{P}$. The L_2-discrepancy of $\mathcal{P}$ is the L_2-norm of the discrepancy function given by

$$\|D_{\mathcal{P}}|L_2\|^2 = \int_{\mathbb{Q}^d} D_{\mathcal{P}}(x)^2 \mathrm{d}x. \tag{2}$$

In this note, we are mainly interested in the two-dimensional case, i.e., $d = 2$. Only at the very end do we add some remarks about the general d-dimensional case. So, from now on until further notice, fix $d = 2$. In this case, the asymptotic behavior of the minimal possible L_2-discrepancy of an N-point set for $N \to \infty$ is well known. There is a constant c such that, for all $N \in \mathbb{N}$ and all N-point subsets $\mathcal{P} \subset \mathbb{Q}^2$,

$$\|D_{\mathcal{P}}|L_2\| \geq c\sqrt{\log N}, \tag{3}$$

and there is a constant C such that, for all $N \in \mathbb{N}$, there exists an N-point subset $\mathcal{P} \subset \mathbb{Q}^2$ with

$$\|D_{\mathcal{P}}|L_2\| \leq C\sqrt{\log N}. \tag{4}$$

The lower bound in (3) is the celebrated result of Roth [9]. Constructions of point sets satisfying (4) are plenty, the first one was given by Davenport [1]. For further constructions and the general theory of discrepancy, we refer the reader to the books [2,6–8].

We are interested in the constants in (3) and (4) for large N, so let us define

$$\underline{c} = \liminf_{N\to\infty} \inf_{\#\mathcal{P}=N} \frac{\|D_{\mathcal{P}}|L_2\|}{\sqrt{\log N}} \quad \text{and} \quad \overline{c} = \limsup_{N\to\infty} \inf_{\#\mathcal{P}=N} \frac{\|D_{\mathcal{P}}|L_2\|}{\sqrt{\log N}}. \tag{5}$$

The best estimates for the constants $\underline{c}$ and $\overline{c}$ known so far are

$$0.0046918\ldots = \sqrt{\frac{1}{2^{16}\log 2}} \leq \underline{c} \leq \overline{c} \leq \sqrt{\frac{278\,629}{2\,811\,072\log 22}} = 0.17907\ldots. \tag{6}$$

The bound for $\underline{c}$ is from a modification of the proof by Roth, and can be found in [5, Chapter 2, proof of lemma 2.5]. With this constant, the estimate (3) even holds for all $N \in \mathbb{N}$. The bound for $\overline{c}$ is from a recent construction in [3] using generalized scrambled Hammersley point sets. For a two-dimensional problem, the gap between the constants is huge.

The main purpose of this note is to improve the lower bound. Our main result is as follows.

Theorem 1. *For all $N \in \mathbb{N}$, and all N-point sets $\mathcal{P} \subset \mathbb{Q}^2$, the inequality*

$$\|D_{\mathcal{P}}|L_2\| \geq c\sqrt{\log N}$$

holds with

$$c = \frac{7}{216\sqrt{\log 2}} = 0.038925\ldots.$$

The proof is still a variant of Roth's method, which uses the information that certain dyadic rectangles do not contain any points of a given point set and adds those local discrepancies up with the help of orthogonal functions. Our improvement is due to the fact that we consider different levels of dyadic rectangles. A convenient method to do this is to compute the Fourier coefficients with respect to the Haar system and then use Parseval's formula. This method has already been used in a recent paper [4] by the first author to prove optimal upper estimates for the discrepancy of Hammersley-type point sets measured in spaces of dominating mixed smoothness.

The next section contains the necessary tools concerning the Haar basis in L_2. In Section 3, we prove the lower bound. In the final section, we show that our lower bound remains valid for the weighted discrepancy, and we comment on what can be done with the Haar function method in higher dimensions.

2. Haar coefficients of the discrepancy function

A dyadic interval of length 2^{-j}, $j \in \mathbb{N}_0$, in $[0, 1)$ is an interval of the form $I = I_{j,m} := \big[2^{-j}m, 2^{-j}(m+1)\big)$ for $m = 0, 1, \ldots, 2^j - 1$. The left half and the right half of $I = I_{j,m}$ are the dyadic intervals $I^+ = I^+_{j,m} = I_{j+1,2m}$ and $I^- = I^-_{j,m} = I_{j+1,2m+1}$, respectively. The Haar function $h_I = h_{j,m}$ with support I is the function on $[0, 1)$ which is $+1$ on the left half of I, -1 on the right half of I, and 0 outside of I. The L_∞-normalized Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_0$ and $m = 0, 1, \ldots, 2^j - 1$ together with the indicator function $h_{-1,0}$ of $[0, 1)$. After normalization in $L_2(\mathbb{Q})$, we obtain the orthonormal Haar basis of $L_2(\mathbb{Q})$.

Let $\mathbb{N}_{-1} = \{-1, 0, 1, 2, \ldots\}$, and define $\mathbb{D}_j = \{0, 1, \ldots, 2^j - 1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$ for $j = -1$. For $j = (j_1, j_2) \in \mathbb{N}^2_{-1}$ and $m = (m_1, m_2) \in \mathbb{D}_j := \mathbb{D}_{j_1} \times \mathbb{D}_{j_2}$, the Haar function $h_{j,m}$ is given as the tensor product $h_{j,m}(x) = h_{j_1,m_1}(x_1)\, h_{j_2,m_2}(x_2)$ for $x = (x_1, x_2) \in [0, 1)^2$. We will call the rectangles $I_{j,m} = I_{j_1,m_1} \times I_{j_2,m_2}$ dyadic rectangles. The L_∞-normalized tensor Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}^2_{-1}$ and $m \in \mathbb{D}_j$. After normalization in $L_2(\mathbb{Q}^2)$, we obtain the orthonormal Haar basis of $L_2(\mathbb{Q}^2)$.

Now, Parseval's equation shows that the L_2-norm of a function $f \in L_2(\mathbb{Q}^2)$ can be computed as

$$\|f|L_2\|^2 = \sum_{j\in\mathbb{N}^2_{-1}} 2^{\max(0,j_1)+\max(0,j_2)} \sum_{m\in\mathbb{D}_j} |\mu_{j,m}|^2, \tag{7}$$

where

$$\mu_{j,m} = \mu_{j,m}(f) = \int_{\mathbb{Q}^2} f(x)h_{j,m}(x)\,\mathrm{d}x \tag{8}$$

are the Haar coefficients of f.

The following two crucial lemmas are easy to verify, and have already been used in [4].

Lemma 2. *Let $f(x) = x_1 x_2$ for $x = (x_1, x_2) \in \mathbb{Q}^2$. Let $j \in \mathbb{N}^2_0$, $m \in \mathbb{D}_j$, and let $\mu_{j,m}$ be the Haar coefficient of f given by (8). Then*

$$\mu_{j,m} = 2^{-2j_1-2j_2-4}.$$

Lemma 3. *Fix $z = (z_1, z_2) \in \mathbb{Q}^2$, and let $f(x) = \mathbb{1}_{C_z}(x)$ for $x = (x_1, x_2) \in \mathbb{Q}^2$. Let $j \in \mathbb{N}^2_0$, $m \in \mathbb{D}_j$, and let $\mu_{j,m}$ be the Haar coefficient of f given by (8). Then $\mu_{j,m} = 0$ whenever z is not contained in the interior of the dyadic rectangle $I_{j,m}$ supporting $h_{j,m}$.*

3. The lower bound

We are now ready to prove Theorem 1. Let $N \in \mathbb{N}$ with $N \geq 2$, and let $\mathcal{P} \subset \mathbb{Q}^2$ be an N-point set. Let $j = (j_1, j_2) \in \mathbb{N}^2_0$, $m \in \mathbb{D}_j$ be such that no point of $\mathcal{P}$ lies in the interior of the dyadic rectangle $I_{j,m}$ supporting $h_{j,m}$. Let $\mu_{j,m}$ be the Haar coefficient of the discrepancy function (1). Now, Lemmas 2 and 3 imply that

$$\mu_{j,m} = -N2^{-2j_1-2j_2-4}.$$

Observe that for fixed $j = (j_1, j_2) \in \mathbb{N}^2_0$ the cardinality of $\mathbb{D}_j$ is $2^{j_1+j_2}$, and the interiors of the dyadic boxes $I_{j,m}$ supporting $h_{j,m}$ are mutually disjoint. This implies that there are at least $2^{j_1+j_2} - N$ such $m \in \mathbb{D}_j$ for which no point of $\mathcal{P}$ lies in the interior of the dyadic rectangle $I_{j,m}$ supporting $h_{j,m}$.

We abbreviate $M = \lceil \log_2 N \rceil$. Then, we obtain from (7) that

$$\begin{aligned}\|D_{\mathcal{P}}|L_2\|^2 &\geq N^2 \sum_{j_1+j_2\geq M} 2^{j_1+j_2}(2^{j_1+j_2} - N)2^{-4j_1-4j_2-8}\\ &= 2^{-8}N^2 \sum_{j_1+j_2\geq M} 4^{-(j_1+j_2)} - 2^{-8}N^3 \sum_{j_1+j_2\geq M} 8^{-(j_1+j_2)}.\end{aligned}$$

Now, for any $q > 1$, we have

$$\sum_{j_1+j_2\geq M} q^{-(j_1+j_2)} = \sum_{k=M}^{\infty}(k+1)q^{-k} = q^{-M+1}\left(\frac{M}{q-1}+\frac{q}{(q-1)^2}\right),$$

which leads to

$$\|D_{\mathcal{P}}|L_2\|^2 \geq 2^{-6}(N2^{-M})^2\left(\frac{M}{3}+\frac{4}{9}\right) - 2^{-5}(N2^{-M})^3\left(\frac{M}{7}+\frac{8}{49}\right)$$
$$\geq 2^{-6}(N2^{-M})^2\frac{M}{3} - 2^{-5}(N2^{-M})^3\frac{M}{7},$$

where the last estimate easily follows from $0 < N2^{-M} \leq 1$.

Now, let $t = M - \log_2 N$ so that $0 \leq t < 1$ and $N2^{-M} = 2^{-t}$. Then, we have proved that

$$\|D_{\mathcal{P}}|L_2\|^2 \geq \gamma \log_2 N$$

if we can verify that

$$2^{-6}2^{-2t}\frac{M}{3} - 2^{-5}2^{-3t}\frac{M}{7} \geq \gamma(M-t)$$

for all $M \in \mathbb{N}$ and $0 \leq t < 1$. The last inequality is equivalent to

$$\left(\gamma - 2^{-6}3^{-1}2^{-2t} + 2^{-5}7^{-1}2^{-3t}\right)M \leq \gamma t,$$

which is certainly satisfied whenever $\gamma \geq 0$ and

$$\gamma \leq 2^{-6}3^{-1}2^{-2t} - 2^{-5}7^{-1}2^{-3t}$$

for all $0 \leq t < 1$ or, alternatively,

$$\gamma \leq 2^{-6}3^{-1}y^2 - 2^{-5}7^{-1}y^3$$

for all $1/2 < y \leq 1$. The maximal value of the right-hand side is easily seen to be $\frac{49}{46\,656}$ for $y = \frac{7}{9}$. So we can choose

$$\gamma = \frac{49}{46\,656} = \left(\frac{7}{216}\right)^2,$$

which leads to the value for c in the theorem. This finishes the proof. □

4. Final remarks

Our lower bound is also valid for the weighted discrepancy, which can be defined as follows. Let $a = (a_z)_{z\in\mathcal{P}}$ be a system of real numbers associating a weight a_z with a point $z \in \mathcal{P}$. Then the weighted discrepancy function is defined as

$$D_{\mathcal{P},a}(x) := \sum_{z\in\mathcal{P}} a_z \mathbb{1}_{C_z}(x) - N|B_x|.$$

The discrepancy function defined by (1) is obtained in the case that all points of $\mathcal{P}$ have weight 1. Thanks to Lemma 3, the Haar coefficient with respect to a Haar function whose support does not intersect $\mathcal{P}$ does not depend on the weights. So one gets the same lower bound with the same constant for the weighted L_2-discrepancy as in the case without weights. Hence, we have the following generalization of Theorem 1 to the weighted discrepancy.

Theorem 4. *For all $N \in \mathbb{N}$, all N-point sets $\mathcal{P} \subset \mathbb{Q}^2$, and all weights $a = (a_z)_{z\in\mathcal{P}}$, the inequality*

$$\|D_{\mathcal{P},a}|L_2\| \geq c\sqrt{\log N}$$

holds with

$$c = \frac{7}{216\sqrt{\log 2}} = 0.038925\ldots.$$

We now consider point sets in higher dimensions. Then, for all $N \in \mathbb{N}$, and all N-point subsets $\mathcal{P} \subset \mathbb{Q}^d$, there is a known lower bound of the form

$$\|D_{\mathcal{P}}|L_2\| \geq c_d (\log N)^{\frac{d-1}{2}},$$

where the constant is known from [2] as

$$c_d = \frac{1}{2^{2d+4}\sqrt{(d-1)!}(\log 2)^{\frac{d-1}{2}}}.$$

Our intention is to use the method above to improve this constant. The idea of tensor product Haar bases can easily be transferred to higher dimensions, i.e., $d > 2$. For $j = (j_1, \ldots, j_d) \in \mathbb{N}_{-1}^d$ and $m = (m_1, \ldots, m_d) \in \mathbb{D}_j := \mathbb{D}_{j_1} \times \cdots \times \mathbb{D}_{j_d}$, the Haar function $h_{j,m}$ is given as the tensor product $h_{j,m}(x) = h_{j_1,m_1}(x_1) \ldots h_{j_d,m_d}(x_d)$ for $x = (x_1, \ldots, x_d) \in [0, 1)^d$. We will call the rectangles $I_{j,m} = I_{j_1,m_1} \times \cdots \times I_{j_d,m_d}$ dyadic boxes. The L_∞-normalized tensor Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_{-1}^d$ and $m \in \mathbb{D}_j$. After normalization in $L_2(\mathbb{Q}^d)$, we obtain the orthonormal Haar basis of $L_2(\mathbb{Q}^d)$.

Now, Parseval's equation shows that the L_2-norm of a function $f \in L_2(\mathbb{Q}^d)$ can be computed as

$$\|f|L_2\|^2 = \sum_{j \in \mathbb{N}_{-1}^d} 2^{\max(0,j_1)+\cdots+\max(0,j_d)} \sum_{m \in \mathbb{D}_j} |\mu_{j,m}|^2, \tag{9}$$

where

$$\mu_{j,m} = \mu_{j,m}(f) = \int_{\mathbb{Q}^d} f(x) h_{j,m}(x)\, \mathrm{d}x \tag{10}$$

are the Haar coefficients of f.

Analogously to the case when $d = 2$, we can state the following two lemmas. They are easy to verify.

Lemma 5. *Let $f(x) = x_1 \ldots x_d$ for $x = (x_1, \ldots, x_d) \in \mathbb{Q}^d$. Let $j \in \mathbb{N}_0^d$, $m \in \mathbb{D}_j$, and let $\mu_{j,m}$ be the Haar coefficient of f given by (10). Then*

$$\mu_{j,m} = 2^{-2j_1-\cdots-2j_d-2d}.$$

Lemma 6. *Fix $z = (z_1, \ldots, z_d) \in \mathbb{Q}^d$, and let $f(x) = \mathbb{1}_{C_z}(x)$ for $x = (x_1, \ldots, x_d) \in \mathbb{Q}^d$. Let $j \in \mathbb{N}_0^2$, $m \in \mathbb{D}_j$, and let $\mu_{j,m}$ be the Haar coefficient of f given by (10). Then $\mu_{j,m} = 0$ whenever z is not contained in the interior of the dyadic box $I_{j,m}$ supporting $h_{j,m}$.*

Now, let $N \in \mathbb{N}$ with $N \geq 2$, and let $\mathcal{P} \subset \mathbb{Q}^d$ be an N-point set. Let $j \in \mathbb{N}_0^d$, $m \in \mathbb{D}_j$ be such that no point of $\mathcal{P}$ lies in the interior of the dyadic box $I_{j,m}$ supporting $h_{j,m}$. Let $\mu_{j,m}$ be the Haar coefficient of the discrepancy function (1). Then Lemmas 2 and 3 imply that

$$|\mu_{j,m}| = N 2^{-2j_1-\cdots-2j_d-2d}$$

for $j = (j_1, \ldots, j_d) \in \mathbb{N}_0^2$. We obtain the following result.

Theorem 7. *For all $N \in \mathbb{N}$, and all N-point sets $\mathcal{P} \subset \mathbb{Q}^d$, the inequality*

$$\|D_{\mathcal{P}}|L_2\| \geq c_d (\log N)^{\frac{d-1}{2}}$$

holds with

$$c_d = \frac{7}{27 \cdot 2^{2d-1}\sqrt{(d-1)!}(\log 2)^{\frac{d-1}{2}}}.$$

In comparison with the known result, the constant is improved by a factor of $\frac{224}{27} = 8.296296\ldots$. The calculation of the constant is analogous to the case when $d = 2$. First, one obtains

$$\|D_{\mathcal{P}}|L_2\|^2 \geq 2^{-4d}N^2 \sum_{j_1+\cdots+j_d\geq M} 4^{-(j_1+\cdots+j_d)} - 2^{-4d}N^3 \sum_{j_1+\cdots+j_d\geq M} 8^{-(j_1+\cdots+j_d)}.$$

Then, one checks that the coefficient of M^{d-1} in

$$\sum_{j_1+\cdots+j_d\geq M} q^{-(j_1+\cdots+j_d)}$$

for any $q > 1$ is

$$\frac{q^{-M+1}}{(q-1)(d-1)!}.$$

Finally, one obtains

$$\|D_{\mathcal{P}}|L_2\|^2 \geq 2^{-4d}(N2^{-M})^2 \frac{4}{3}\frac{M^{d-1}}{(d-1)!} - 2^{-4d}(N2^{-M})^3 \frac{8}{7}\frac{M^{d-1}}{(d-1)!}.$$

Then, analogously to the two-dimensional case, we get the estimate

$$\|D_{\mathcal{P}}|L_2\|^2 \geq \gamma(\log_2 N)^{d-1}$$

if

$$\gamma \leq \frac{1}{2^{4d}(d-1)!}\left(\frac{4}{3}y^2 - \frac{8}{7}y^3\right)$$

for all $1/2 < y \leq 1$. The maximal value of the right-hand side is reached for $y = \frac{7}{9}$, and is

$$\gamma = \frac{1}{2^{4d}(d-1)!}\left(\frac{14}{27}\right)^2.$$

This leads to the value of c_d in the theorem.

Analogously to the two-dimensional case, one obtains in the d-dimensional case the same bounds for the weighted discrepancy as for the unweighted.

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