Fermat–Reyes method in the ring of Fermat reals

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Abstract

To discover derivatives, Pierre de Fermat used to assume a non-zero increment \( h \) in the incremental ratio and, after some calculations, to set \( h = 0 \) in the final result. This method, which sounds as inconsistent, can be perfectly formalized with the Fermat–Reyes theorem about existence and uniqueness of a smooth incremental ratio. In the present work, we will introduce the cartesian closed category where to study and prove this theorem and describe in general the Fermat method. The framework is the theory of Fermat reals, an extension of the real field containing nilpotent infinitesimals which does not need any knowledge of mathematical logic. This key theorem will be essential in the development of differential and integral calculus for smooth functions defined on the ring of Fermat reals and also for infinite-dimensional operators like derivatives and integrals.

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1. Introduction

Infinitesimals are commonly perceived by “working mathematicians” as banned from present rigorous mathematics. We are all forced to deal with long weird sequences of \( \varepsilon \) and \( \delta \), whose choice is not easily teachable. Someone in search has also find that there are rigorous modern theories of infinitesimals, but, to be fully understood and used, they need a non-trivial amount of mathematical logic or category theory or non-trivial set theory. For example, on the one hand it is true that nonstandard analysis can be used, at a first level, without the full trans-
fer theorem (see, e.g., [35]); on the other hand, if one wants to prove that infinitesimals \((s_n)_n \in \ast \mathbb{R}\) correspond exactly to hypernumbers generated by some kind of null sequences, one need the continuum hypotheses and the use of transfinite induction to chose a particular type of ultrafilter (called P-point, see [16]). Moreover, this intuitively desirable property is true if and only if one takes a P-point in the construction of \(\ast \mathbb{R}\), and the existence of this particular type of ultrafilters cannot be proved without assuming the mentioned set-theoretical hypotheses. Finally, this result says only that \((s_n)_n \in \ast \mathbb{R}\) is infinitesimal (in the sense of nonstandard analysis) if and only if\(^2\) \(s_{\sigma_n} \to 0\), for \(n \to +\infty\), on a subsequence \((\sigma_n)_n\), and there is no way to improve this result so as to obtain that infinitesimals correspond to real null sequences, i.e. that \(s_n \to 0\) for \(n \to +\infty\). For others examples in this direction, also for Synthetic Differential Geometry, see [23]; for approaches to nonstandard analysis without requiring a background of formal logic, see, e.g. [35,29,27,7]. On the other hand, it is opinion of the author that nonstandard analysis is the most powerful and flexible theory of infinitesimals, and Synthetic Differential Geometry is surely the most beautiful and powerful theory of differential geometry that uses infinitesimals, though these theories sometimes lack in the intuitive meaning, which is an unexpected property for a theory of infinitesimals.

The theory of Fermat reals has been developed aiming to be a non-trivial theory of infinitesimals easily acceptable by working mathematicians, physicists, differential geometer and engineers. It doesn’t need any background of logic to work with, and its truth is always supported by a strong dialectic between intuitive geometrical interpretation and the corresponding formal translation. For example, Fermat reals can be drawn, see [26,22]. The whole construction takes strong inspiration from synthetic differential geometry (SDG, see, e.g., [30,32,37,6]) and hence it has strong analogies, but also deep differences, with respect to that theory. The theory of Fermat reals deals with nilpotent infinitesimals and is best suitable for smooth functions. For the simple definition of the ring \(\ast \mathbb{R}\) of Fermat reals, see [23]; for some applications in physics, see [24]; for the total order relation on \(\ast \mathbb{R}\), see [26,22]; for the relation between Fermat reals and intuitionistic logic, first applications to infinite-dimensional spaces of mappings and a comparison with other theories of infinitesimals, see [22].

The main aim of the present work is to prove the Fermat–Reyes theorem in the context of Fermat reals, which essentially is a formalization of the informal methods originally used by P. de Fermat to discover derivatives. This theorem is a key in the development of both differential and integral calculus of smooth functions defined on \(\ast \mathbb{R}^d\). These are more general than extensions \(\ast g\) of ordinary smooth functions \(g\) defined on \(\mathbb{R}^d\) and, essentially, are functions of the form \(\ast \alpha(p, -)\), where \(p \in \ast \mathbb{R}^d\) is a fixed parameter. The non-trivial case, of course, being if \(p\) is nonstandard, i.e. \(p \notin \mathbb{R}^d\). Even if this type of functions is essentially all those appearing in applications, their calculus is not trivial due to the possibility that \(p\) is a nilpotent infinitesimal. For example, if \(f := \ast \alpha(p, -)\), then the natural definition \(f'(x) := \ast (\partial_2 \alpha)(p, x)\) is correct only if \(f\) is defined in an open neighbourhood of \(x\) of the form \(\ast U\), where \(U\) is open in \(\mathbb{R}^d\). This is not the case if the function \(f\) is defined only on an infinitesimal set, e.g. on \(D^d = D \times \cdots \times D\), where \(D\) is the ideal of first order infinitesimals, see [23]

The calculus for these “quasi-standard” smooth functions \(\ast \alpha(p, -)\) will be performed proving two basic instruments: the Fermat–Reyes theorem (i.e. existence and uniqueness of smooth

\(^2\) Of course, the “if” part is always true, for every ultrafilter.
incremental ratio, Theorem 22; see also [32]) and existence and uniqueness of primitives, which will be presented in a future work.

Prerequisites for the present work are [23] for basic properties of $\mathbb{R}$ and [26] for its order relation (which is described also in Chapter 4 of [22]).

Let us recall that our notion of quasi-standard function (see the next Definition 9) has a counterpart in an analogous notion used in the early days of nonstandard analysis, and from which we borrowed the same name [35, Chapter 5, pp. 115–121]. Later, this notion was abandoned and replaced by the more general (and more flexible) notion of “internal function” (see, e.g., [17]).

1.1. Cartesian closedness

Another augmented value of the present work is to introduce, in a simple way, a cartesian closed framework to study infinite-dimensional operators. In this framework, it is easy to prove that operators like differentiation, integration, smooth incremental ratio, composition and evaluation are all smooth, in a precise sense we will define later. In this way we will be able to include all the operators used in the calculus of variations. In this introductory section, we only want to motivate and define the notion of cartesian closure. In the study of infinite-dimensional spaces, this is the key concept shared by several authors like [4,6,11–15,19,20,30–34,37,38,40,42,43].

We firstly fix the notations for the notions of adjoint of a map.

**Definition 1.** If $X, Y, Z$ are sets and $f : X \to Z^Y$, $g : X \times Y \to Z$ are maps, then

$$\forall (x, y) \in X \times Y : f^\land(x, y) := \left[f(x)\right](y) \in Z,$$

$$\forall x \in X : g^\lor(x) := g(x, -) \in Z^Y,$$

hence

$$f^\land : X \times Y \to Z,$$

$$g^\lor : X \to Z^Y.$$

The map $f^\land$ is called the adjoint of $f$ and the map $g^\lor$ is called the adjoint$^3$ of $g$.

Let us note that $(f^\land)^\lor = f$ and $(g^\lor)^\land = g$, that is the two applications

$$(-)^\land : (Z^Y)^X \to Z^{X \times Y},$$

$$(-)^\lor : Z^{X \times Y} \to (Z^Y)^X$$

are one the inverse of the other and hence represent in explicit form the bijection of sets $(Z^Y)^X \simeq Z^{X \times Y}$ i.e. $\text{Set}(X, \text{Set}(Y, Z)) \simeq \text{Set}(X \times Y, Z)$.

As we hinted above, we shall start with two problems: the first one is to give a precise definition of quasi-standard smooth function, the second one is to give this definition so as to include

$^3$ Here we are using the notations of [2], but some authors, e.g. [31], used opposite notations for the adjoint maps.
also infinite-dimensional operators. Both problems can be stated saying that we want to generalize the notions of smooth manifold and of smooth map between two manifolds so as to obtain a new category “with good properties”, e.g. from the point of view of infinite-dimensional spaces. If we denote by $\mathcal{C}^\infty$ this new category (that, of course, we have still to define), and we call smooth maps its morphisms and smooth spaces its objects, then this category must be cartesian closed, i.e. it has to verify the following properties for every pair of smooth spaces $X, Y \in \mathcal{C}^\infty$:

1. $\mathcal{C}^\infty(X, Y)$ is a smooth space, i.e. $\mathcal{C}^\infty(X, Y) \in \mathcal{C}^\infty$.
2. The maps $(-)^\vee$ and $(-)^\wedge$ are smooth, i.e. they realize in the category $\mathcal{C}^\infty$ the bijection $\mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z)) \cong \mathcal{C}^\infty(X \times Y, Z)$.

Property 1 is another way to state that the category we want to construct must contain as objects the space of all the smooth maps between two generic objects $X, Y \in \mathcal{C}^\infty$:

$$\mathcal{C}^\infty(X, Y) = \{ f \mid X \overset{f}{\to} Y \text{ is smooth} \} = \{ f \mid X \overset{f}{\to} Y \text{ is a morphism of } \mathcal{C}^\infty \}.$$ 

Moreover, let us note the following consequence of (2)

$$X \overset{f}{\to} \mathcal{C}^\infty(Y, Z) \text{ is smooth} \iff X \times Y \overset{f^\vee}{\to} Z \text{ is smooth}, \quad (1)$$

$$X \times Y \overset{g}{\to} Z \text{ is smooth} \iff X \overset{g^\wedge}{\to} \mathcal{C}^\infty(Y, Z) \text{ is smooth}. \quad (2)$$

The importance of (1) and (2) can be explained saying that if we want to study a smooth map having values in the space $\mathcal{C}^\infty(Y, Z)$, then it suffices to study its adjoint map $f^\vee$. If, e.g., the spaces $X, Y$ and $Z$ are finite-dimensional manifolds, then $\mathcal{C}^\infty(Y, Z)$ is infinite-dimensional, but $f^\vee : X \times Y \to Z$ is a standard smooth map between finite-dimensional manifolds, and hence we have a strong simplification. Conversely, if $g : X \times Y \to Z$ is a smooth map, then it generates a smooth map with values in $\mathcal{C}^\infty(Y, Z)$, and all the smooth maps with values in this type of space can be generated in this way. Of course, this idea is frequently used, even if informally, in the calculus of variations. Let us note explicitly that the cartesian closure of the category $\mathcal{C}^\infty$, i.e. properties 1 and 2, does not say anything about smooth maps with a domain of the form $\mathcal{C}^\infty(Y, Z)$, but it reformulates in a convenient way the problem of smoothness of maps with codomain of this type. For a more abstract notion of cartesian closed category, see, e.g., [36,9,3,2].

In this context, we shall also prove that the isomorphism

$$\langle -, - \rangle : ([x]_\sim, [y]_\sim) \in \star \mathbb{R} \times \star \mathbb{R} \mapsto [(x, y)]_\sim \in \star(\mathbb{R} \times \mathbb{R}) \quad (3)$$

is a smooth mapping. In (3) the equivalence relation $\sim$ is the equality in the ring $\star \mathbb{R}$, see Definition 5 of [23], and $[x]_\sim$ is the corresponding equivalence class generated by the little-oh polynomial $x \in \mathbb{R}_o[t]$, whereas $(x, y) : t \in \mathbb{R}_{\geq 0} \mapsto (x_t, y_t) \in \mathbb{R}^2$. Note the importance of this map to perform passages like the following

$$\mathbb{R} \times \mathbb{R} \overset{f}{\to} Y \text{ in } \mathcal{C}^\infty,$$
\[ \bullet (\mathbb{R} \times \mathbb{R}) \xrightarrow{f} \bullet Y \text{ in } \bullet C^\infty, \]
\[ \bullet \mathbb{R} \times \bullet \mathbb{R} \xrightarrow{f} \bullet Y \text{ in } \bullet C^\infty \text{ (identification via } \langle -, - \rangle), \]
\[ \bullet \mathbb{R} \xrightarrow{f^*} \bullet Y^{\bullet \mathbb{R}} \text{ using cartesian closedness.} \]

This isomorphism is also implicitly used in notations like \( \bullet \alpha(p, -) \), where \( g : \mathbb{R}^2 \to \mathbb{R} \) is smooth and \( p \in \bullet \mathbb{R} \). Using a more precise notation, we have to write \( \bullet \alpha(p, -) \).

1.2. What are smooth functions on the ring of Fermat reals?

We can motivate the need for a more general notion of smooth function analyzing, e.g., the usual informal derivation of the wave equation. We have to consider a string making small transversal oscillations around its equilibrium position located on the interval \([a, b]\) of the \( x \) axis, for \( a, b \in \mathbb{R}, a < b \). Usually, the position of the string is represented by the graph of a curve \( \gamma : [a, b] \times [0, +\infty) \to \mathbb{R}^2 \) of the form \( \gamma(x, t) = (x, u(x, t)) \). Using the notation \( \varphi(x, t) \) for the non-oriented angle between the tangent unit vector \( \vec{t}(x, t) \) at the point \( \gamma(x, t) \) and the \( x \) axis, the small oscillations hypothesis can be formalized assuming

\[ \varphi(x, t) \in D, \quad \forall x, t. \]

This enables us to reproduce the classical derivation in the most faithful way. Indeed, we have \( \frac{\partial u}{\partial x} \cdot \cos \varphi = \sin \varphi \), i.e. \( \frac{\partial u}{\partial x} = \varphi \in D \). Hence, \( (\frac{\partial u}{\partial x})^2 = 0 \), and the total length of the string becomes

\[ L = \int_a^b \sqrt{1 + \left( \frac{\partial u}{\partial x}(x, t) \right)^2} \, dx = b - a, \quad \forall t \in [0, +\infty). \quad (4) \]

By Hooke’s law, this proves that the tension can be assumed to have constant modulus depending on neither the position \( x \) nor the time \( t \). Anyway, let us note explicitly that the only standard fourth continuous function verifying the equality \( L = b - a \) is constant and we obtain a contradiction. In the informal deduction, one uses an approximate equality \( \simeq \) in (4), but this must be magically changed in the equality sign for the final wave equation. A physically meaningful idea is, better, to consider a “quasi-standard smooth function” like \( u(x, t) = u_0 \cdot \sin(x + \omega \cdot t) \), where the maximum amplitude \( u_0 \) is a first order infinitesimal, i.e. \( u_0 \in D \). Let us note that the function \( u \) is obtained from the standard smooth function \( \alpha(x_1, x_2, x_3, x_4) = x_1 \cdot \sin(x_2 + x_3 \cdot x_4), x_i \in \mathbb{R} \), by extending it to \( \bullet \mathbb{R} \) and by fixing one of its variables to be a nonstandard parameter \( x_1 = u_0 \in D \). Actually, these maps are simply \( C^\infty \) functions with some fixed parameter \( p \), which could be an infinitesimal distance (e.g. in the potential of the electric dipole, see [24]), an infinitesimal coefficient associated to a metric (like, e.g., in Einstein’s formula \( \sqrt{1 - h_{44}(x)} = 1 - \frac{1}{2}h_{44}(x) \), where \( h_{44}(x) \in D \)), or a side \( \vdash = s(a, -) \) of an infinitesimal surface \( s : [a, b] \times [c, d] \to \bullet \mathbb{R} \), where \( a \in \bullet \mathbb{R} \setminus \mathbb{R} \). For a more general and complete approach to the wave equation in the context of Fermat reals, see [24].

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4 In the present work, the term “standard” is used as synonymous of “defined on the usual real field” and the term “nonstandard” is used as precised in the text. Hence, there is no relation with the corresponding concepts of Nonstandard Analysis.
2. Infinitesimals in $\mathcal{R}(\mathbb{R}^d)$

The necessity of isomorphisms of the type $\mathcal{R} \times \mathcal{R} \simeq \mathcal{R}(\mathbb{R}^2)$ can be understood only if we define the extension $\mathcal{R}(\mathbb{R}^d)$ directly, i.e. without using the product $\mathcal{R} \times \mathcal{R} \cdots \times \mathcal{R}$. Therefore, we start generalizing to arbitrary dimension $d \in \mathbb{N}_{>0}$ the definition of $\mathcal{U}$, for $U$ open subset of $\mathbb{R}^d$.

2.1. Nilpotent paths

The generalization of the notion of nilpotent path is straightforward:

**Definition 2.** Let $U$ be an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}_{>0}$, and let $x = (x_t)_t : \mathbb{R}_{\geq 0} \to U$ be a path continuous at the origin, then we say that $x$ is nilpotent iff we can find $k \in \mathbb{N}$ such that

$$\|x_t - x_0\|^k = o(t) \quad \text{as} \ t \to 0^+.$$ 

Moreover, we define

$$C_0(U) := \{ x : \mathbb{R}_{\geq 0} \to U \mid x \text{ is continuous at } t = 0^+ \},$$

$$\mathcal{N}_U := \mathcal{N}(U) := \{ x \in C_0(U) \mid x \text{ is nilpotent} \}.$$ 

Of course, if $f \in C^\infty(U, V)$ is a smooth map, then it preserves nilpotent paths, i.e. $f \circ x \in \mathcal{N}_V$ if $x \in \mathcal{N}_U$. Moreover, it is not difficult to prove the following theorem, concerning the relations between the product of two open sets $U, V$ and nilpotent paths.

**Theorem 3.** Let $U, V$ be open sets in $\mathbb{R}^u$ and $\mathbb{R}^v$ respectively, and $x : \mathbb{R}_{\geq 0} \to U$, $y : \mathbb{R}_{\geq 0} \to V$ be two maps, then

$$x \in \mathcal{N}_U \quad \text{and} \quad y \in \mathcal{N}_V \quad \iff \quad (x, y) \in \mathcal{N}_{U \times V},$$

where we set $(x, y)_t := (x_t, y_t)$.

2.2. Little-oh polynomials

We can proceed in a similar way with respect to the generalization of the notion of little-oh polynomial.

**Definition 4.** Let $U$ be an open set of $\mathbb{R}^d$, we say that $x$ is a little-oh polynomial in $U$, and we write $x \in U_0[t]$, iff:

1. $x : \mathbb{R}_{\geq 0} \to U$.
2. We can write

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad \text{as} \ t \to 0^+$$

for suitable
Now, we want to prove that little-oh polynomials are preserved using cartesian product and by smooth functions.

**Theorem 5.** Let $U$, $V$ be open set of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let $x : \mathbb{R}_{\geq 0} \to U$ and $y : \mathbb{R}_{\geq 0} \to V$ be two maps, then

$$x \in U_o[t] \quad \text{and} \quad y \in V_o[t] \iff (x, y) \in (U \times V)_o[t].$$

**Proof.** $\Rightarrow$: Let us fix some notations for the little-oh polynomials $x$ and $y$:

$$x_t = r + \sum_{i=1}^{K} \alpha_i \cdot t^{a_i} + o_1(t),$$

$$y_t = s + \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + o_2(t),$$

where $r, \alpha_1, \ldots, \alpha_K \in \mathbb{R}^m$ and $s, \beta_1, \ldots, \beta_N \in \mathbb{R}^n$. Define $u := (r, s) \in \mathbb{R}^{m+n}$, $\gamma_i := (\alpha_i, 0) \in \mathbb{R}^{m+n}$, $\gamma_{j+K} := (0, \beta_j) \in \mathbb{R}^{m+n}$, $c_i := a_i$ and $c_{j+K} := b_j$, then

$$(x_t, y_t) = \left( r + \sum_{i=1}^{K} \alpha_i \cdot t^{a_i} + o_1(t), s + \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + o_2(t) \right)$$

$$= (r, s) + \sum_{i=1}^{K} (\alpha_i, 0) \cdot t^{a_i} + (0, o_1(t)) + \sum_{j=1}^{N} (0, \beta_j) \cdot t^{b_j} + (0, o_2(t))$$

$$= u + \sum_{i=1}^{K} \gamma_i \cdot t^{c_i} + \sum_{i=K+1}^{K+N} \gamma_i \cdot t^{c_i} + (o_1(t), o_2(t)),$$

and this proves the conclusion because $(o_1(t), o_2(t)) = o(t)$.

$\Leftarrow$: By hypotheses we can write

$$(x_t, y_t) = u + \sum_{k=1}^{H} \gamma_k \cdot t^{c_k} + o(t).$$

We only have to reverse the previous ideas defining

$$r := (u_1, \ldots, u_m), \quad s := (u_{m+1}, \ldots, u_{m+n}),$$

$$\alpha_k := (\gamma_{k}^1, \ldots, \gamma_{k}^m), \quad \beta_k := (\gamma_{k}^{m+1}, \ldots, \gamma_{k}^{m+n}),$$

$$a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}.$$
\[ o_1(t) := (o^1(t), \ldots, o^m(t)), \quad o_2(t) := (o^{m+1}(t), \ldots, o^{m+n}(t)), \]
\[ a_i := c_i, \quad b_k := c_k, \]
where we have used the notations
\[ \gamma_k = (\gamma^1_k, \ldots, \gamma^{m+n}_k), \]
\[ o(t) = (o^1(t), \ldots, o^{m+n}(t)) \]
for the components. Then
\[ (x_t, y_t) = \left( r, 0 \right) + \sum_{k=1}^{H} (\alpha_k, 0) \cdot t^c_k + (o_1(t), 0) + (0, s) + \sum_{k=1}^{H} (0, \beta_k) \cdot t^c_k + (0, o_2(t)) \]
\[ = \left( r + \sum_{i=1}^{H} \alpha_i \cdot t^c_i + o_1(t), s + \sum_{j=1}^{H} \beta_j \cdot t^c_j + o_2(t) \right), \]
and hence the conclusion follows. \( \square \)

From this result, if \( x \in \mathbb{R}^d_o[t] \), then each component is a 1-dimensional little-oh polynomial: \( x_i \in \mathbb{R}^o_o[t] \), for \( i = 1, \ldots, d \). But we know (see Section 4.1 in [23]) that these polynomials are nilpotent, i.e. \( x_i \in \mathcal{N} \). Therefore, from Theorem 3 it follows that \( x \in \mathcal{N}_{\mathbb{R}^d}, \) i.e.
\[ \mathbb{R}^d_o[t] \subseteq \mathcal{N}_{\mathbb{R}^d}. \]

In the following result, we prove that little-oh polynomial in several dimensions are preserved by smooth functions.

**Theorem 6.** Let \( x \in U_o[t] \) and \( f \in C^\infty(U, \mathbb{R}^p) \), with \( U \) open in \( \mathbb{R}^d \), then \( f \circ x \in \mathbb{R}^p_o[t] \).

**Proof.** Let us fix some notations:
\[ x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^a_i + w(t) \quad \text{with} \quad w(t) = o(t), \]
\[ h(t) := x(t) - x(0), \quad \forall t \in \mathbb{R}_{\geq 0} \]
hence \( x_t = x(0) + h_t = r + h_t \). The function \( t \mapsto h(t) = \sum_{i=1}^{k} \alpha_i \cdot t^a_i + w(t) \) belongs to \( \mathbb{R}^d_o[t] \subseteq \mathcal{N}_{\mathbb{R}^d} \), so we can write \( \|h_t\|^N = o(t) \) for some \( N \in \mathbb{N} \). From Taylor’s formula we have
\[ f(x_t) = f(r + h_t) = f(r) + \sum_{i \in \mathbb{N}^d} \frac{\partial^i f}{\partial x_i^i}(r) \cdot \frac{h_t^i}{i!} + o(\|h_t\|^N). \quad (5) \]
But
hence $o(||h_l||^N)/|t| = o(t) \in \mathbb{R}_o^P[t]$. Now, we have to note that, for a multi-index $i \in \mathbb{N}^d$, it results $h_l^i = h_l^1(t) \cdot \ldots \cdot h_l^d(t) \in \mathbb{R}_o[t]$ because, from the previous Theorem 5, each function $h_j(t) \in \mathbb{R}_o[t]$ and because $\mathbb{R}_o[t]$ is an algebra. Moreover, if $\beta \in \mathbb{R}^P$ and $h \in \mathbb{R}_o^P[t]$, then $\beta \cdot h \in \mathbb{R}_o^P[t]$, so each addend $\frac{\partial f}{\partial x^i}(r) \cdot h_l^i$ is a little-oh polynomial of $\mathbb{R}_o^P[t]$. From (5) and the closure of little-oh polynomials $\mathbb{R}_o^P[t]$ with respect to linear operations, the conclusion $f \circ x \in \mathbb{R}_o^P[t]$ follows. 

2.3. The Fermat extension of open sets and functions

It is now natural the generalization of the definition of equality in $^\ast \mathbb{R}$ (see Definition 5 in [23]) to a generic open set $U$ of $\mathbb{R}^d$.

**Definition 7.** Let $U$ be an open subset of $\mathbb{R}^d$, and let $x$, $y \in U_o[t]$ be two little-oh polynomials, then we say that $x \sim y$ in $U$, or simply $x = y$ in $^\ast U$, iff $x_t = y_t + o(t)$ as $t \to 0^+$. Obviously we will write $^\ast U := U_o[t]/\sim$ and $^\ast f(x) := f \circ x$ if $f \in \mathcal{C}(U,V)$ and $x \in ^\ast U$ (or, using the notation with equivalence classes, $^\ast f([x]_\sim) := [f \circ x]_\sim$) and we will call them the Fermat extension of $U$ and of $f$ respectively. As usual, we will also define the standard part of $x \in ^\ast U$ as $\circ x := x_0 \in U$.

Because any such function $f$ is locally Lipschitz, we have that the definition of $^\ast f$ is correct, and is also an extension of $f$. If $i : U \hookrightarrow \mathbb{R}^k$ is the inclusion map, it is easy to prove that its Fermat extension $^\ast i : ^\ast U \to ^\ast \mathbb{R}^k$ is injective. We will always identify $^\ast U$ with $^\ast i(\mathbb{R}^k)$, so we simply write $^\ast U \subseteq ^\ast \mathbb{R}^k$. According to this identification, if $U$ is open in $\mathbb{R}^k$, we can also prove that

$$^\ast U = \{ x \in ^\ast \mathbb{R}^k \mid \circ x \in U \}. \quad (6)$$

This property says that the preliminary definition of $^\ast U$ given in Definition 19 of [23] is equivalent to the previous Definition 7 of extension. Finally, it is not hard to prove that $(x,y) = (x',y')$ in $^\ast (U \times V)$ iff $x = x'$ in $^\ast U$ and $y = y'$ in $^\ast V$. From this conclusion and from Theorem 5 we can prove that the following applications

$$a_{UV} := (-,-) : ([x]_\sim, [y]_\sim) \in ^\ast U \times ^\ast V \mapsto [(x,y)]_\sim \in ^\ast (U \times V), \quad (7)$$

$$b_{UV} : [z]_\sim \in ^\ast (U \times V) \mapsto ([z \cdot p_U]_\sim, [z \cdot p_V]_\sim) \in ^\ast U \times ^\ast V \quad (8)$$

(for clarity we have used the notation with the equivalence classes) are well-defined bijections with $a_{UV}^{-1} = b_{UV}$ (obviously $p_U, p_V$ are the projections). We shall use explicit notations like $^\ast (\mathbb{R}^d)$ and $(^\ast \mathbb{R})^d$ until we will have proved that these bijections are smooth.

3. The category of Fermat spaces

The bijection $a_{UV} = (-,-)$ is used in the following definition of quasi-standard smooth function.

$$\frac{o(||h_f||^N)}{|t|} = \frac{o(||h_l||^N)}{||h_f^i||} \cdot \frac{||h_f^i||}{|t|} \to 0$$
Remark 8. To simplify the presentation, in case the context will be sufficiently clear, we shall consider the coupling of variables $(S, s), (T, t), (p, p), (q, q)$ etc. in properties of the form $S \subseteq \mathbb{R}^S, T \subseteq \mathbb{R}^T, p \in \mathbb{R}^P$ or $q \in \mathbb{R}^Q$ respectively. In fact, in these cases we have that the second variable in the pairing, e.g. the number $s \in \mathbb{N}$ in the pairing $(S, s)$, is uniquely determined by the first variable $S$. E.g. the number $p \in \mathbb{N}$ is uniquely determined by the point $p \in \mathbb{R}^P$.

Therefore, if we denote by $\sigma(V) \in \mathbb{N}$ the unique $v \in \mathbb{N}$ in a pairing $(V, v)$, then any formula of the form $\mathcal{P}(V, v)$ can be interpreted as

$$v = \sigma(V) \Rightarrow \mathcal{P}(V, v).$$

Definition 9. If $S \subseteq \mathbb{R}^S$ and $T \subseteq \mathbb{R}^T$ then we say that $S \xrightarrow{f} T$ is (quasi-standard) smooth iff $f$ maps $S$ in $T$ and for every point $s \in S$ in the domain, we can write

$$f(x) = \alpha(p, x), \quad \forall x \in V \cap S$$

for some $V$ open in $\mathbb{R}^S$ such that $s \in V$, $p \in U$, where $U$ is open in $\mathbb{R}^P$, $\alpha \in C^\infty(U \times V, \mathbb{R}^T)$.

Sometimes, where it will be clear from the context, we will omit the specification “quasi-standard” and we will simply say that $f : S \to T$ is smooth.

In other words, locally a smooth function $f : S \to T$, where $S \subseteq \mathbb{R}^S$ and $T \subseteq \mathbb{R}^T$, is constructed in the following way:

1. start with an ordinary standard function $\alpha \in C^\infty(U \times V, \mathbb{R}^T)$, with $U$ open in $\mathbb{R}^P$ and $V$ open in $\mathbb{R}^S$. The space $\mathbb{R}^P$ has to be thought as a space of parameters for the function $\alpha$;
2. consider its Fermat extension obtaining $\alpha : (U \times V) \to \mathbb{R}^T$;
3. consider the composition $\alpha \circ (\cdot, -) : U \times V \to \mathbb{R}^T$;
4. fix a parameter $p \in U$ as a first variable of the previous composition, i.e. consider $\alpha(p, -) : V \to \mathbb{R}^T$. Locally, the map $f$ is of this form: $f = \alpha(p, -)$.

A direct verification suffices to see that if $f \in C^\infty(S, T)$ is a standard smooth function, where $S \subseteq \mathbb{R}^S, T \subseteq \mathbb{R}^T$ are sets of real numbers, then $f$ is also quasi-standard smooth. Another particular, but useful, way to construct quasi-standard smooth functions is given by the following.

Theorem 10. Let $f \in C^\infty(\mathbb{R}^k, \mathbb{R}^h)$ be a standard $C^\infty$ function and $H \subseteq \mathbb{R}^h$ and $K \subseteq \mathbb{R}^k$ be subsets of Fermat reals. If the function $f$ verifies $\alpha(f|_K(K)) \subseteq H$, then

$$\alpha(f|_K : K \to H$$

is nonstandard smooth.
Proof. It suffices to define $\alpha(x,y) := f(y)$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^k$ to obtain that

$$\alpha(0,k)_t = \alpha(0,k_t) = f(k_t) = f(k)t,$$

that is $\alpha|_K = \alpha(0,k) \in H$ for every $k \in K$. □

Now, our problem is to define a notion of smooth function which is able to include the previous one, and also meaningful examples of infinite-dimensional operators. The idea is to substitute the notion of chart of a manifold (which can be thought as a particular figure traced on the space we are considering) with the set of all the figures, for any possible finite dimension, that can be traced on the given space. This is, essentially, a generalization of the idea of diffeological space, see [14,40,41,28,22,25].

We start defining what is a category of figures.

Definition 11. A category of figures $\mathcal{F}$ is a subcategory of the category of topological spaces $\textbf{Top}$ satisfying the following conditions:

1. $\mathcal{F}$ contains all the constant maps $c : H \to X$, where $X \in \mathcal{F}$, and all the open subspaces $U \subseteq H$ (with the induced topology) of every object $H \in \mathcal{F}$. The corresponding inclusion $i : U \hookrightarrow H$ is also an arrow of $\mathcal{F}$, i.e. $i \in \mathcal{F}(U,H)$.

In the following, we will denote by $|\cdot| : \mathcal{F} \to \textbf{Set}$ the forgetful functor, which associates to any $H \in \mathcal{F}$ its support set $|H| \in \textbf{Set}$. Moreover, with $\tau_H$ we will denote the topology of $H$ and with $(U \prec H)$ the topological subspace of $H$ induced on the open set $U \in \tau_H$. The remaining conditions on $\mathcal{F}$ are the following:

2. The category $\mathcal{F}$ is closed with respect to restrictions to open sets, that is if $f \in \mathcal{F}_{HK}$ and $U$, $V$ are open sets in $H$, $K$ resp. and finally $f(U) \subseteq V$, then $f|_U \in \mathcal{F}(U \prec H, V \prec K)$.

3. Every topological space $H \in \mathcal{F}$ has the following “sheaf property”: let $H$, $K \in \mathcal{F}$ be two objects of $\mathcal{F}$, $(H_i)_{i \in I}$ an open cover of $H$ and $f : |H| \to |K|$ a map such that

$$\forall i \in I : \quad f|_{H_i} \in \mathcal{F}(H_i \prec H, K),$$

then $f \in \mathcal{F}_{HK}$.

The arrows $f \in \mathcal{F}_{HK}$ are called figures and their domain $H \in \mathcal{F}$ is called type of that figure.

For example, $\mathcal{F} = \mathcal{O}\mathbb{R}^n$, the category having as objects open sets $U \subseteq \mathbb{R}^u$ (with the induced topology), for some $u \in \mathbb{N}$ depending on $U$, and with hom-set the usual space $\mathcal{C}^n(U,V)$ of $\mathcal{C}^n$ functions between the open sets $U \subseteq \mathbb{R}^u$ and $V \subseteq \mathbb{R}^v$, is the category of figures used in the definition of diffeological space (see, e.g., [25,22]). We can think at $\mathcal{O}\mathbb{R}^n$ as all the possible finite-dimensional figures (defined on standard open sets) of $\mathcal{C}^n$ regularity. In general, what type of category $\mathcal{F}$ we have to choose depends on the setting we need: e.g., in case we want to consider spaces with boundary, we have to take the analogous of the above mentioned category $\mathcal{O}\mathbb{R}^n$, but having as objects sets of type $U \subseteq \mathbb{R}^u_+ = \{x \in \mathbb{R}^u \mid x_u \geq 0\}$.

For our goal, we surely need figures of the type $\mathcal{O}\mathbb{R}^\infty$, but also infinitesimal figures defined, e.g., in the ideal of first order infinitesimal $D \subset \mathbb{R}$. We will hence consider the following.
Definition 12. We call $S^\bullet \mathbb{R}^\infty$ the category whose objects are topological spaces $(S, \tau_S)$, with $S \subseteq \bullet (\mathbb{R}^s)$ for some $s \in \mathbb{N}$ which depends on $S$, and with the topology $\tau_S$ generated by sets of the form $U = \bullet U \cap S$, for $U$ open in $\mathbb{R}^s$ (in this case we will say that the open set $U$ is defined by $U$ in $S$). In other words $A \in \tau_S$ if and only if
$$A = \bigcup \{ \bullet U \cap S \subseteq A \mid U \text{ is open in } \mathbb{R}^s \}.$$  
(10)
As arrows of $S^\bullet \mathbb{R}^\infty$ we take smooth functions $f: S \to T$ (Definition 9).

In the following, we will frequently use the simplified notation $S$ instead of the complete $(S, \tau_S)$. Moreover, we will consider on $S^\bullet \mathbb{R}^\infty$ the forgetful functor given by the inclusion $|−|: S^\bullet \mathbb{R}^\infty \hookrightarrow \text{Set}$, i.e. $(S, \tau_S) := S$. The category $S^\bullet \mathbb{R}^\infty$ will be called the category of subsets of $^\bullet \mathbb{R}^\infty$ (but note that here $\infty$ indicates the class of regularity of the functions we are considering).

Remark.

1. Because in Definition 9 we ask $s \in \bullet V$ we have that $V := \bullet V \cap S$ is a neighborhood of $s$ defined by $V$ in $S$ (see (10)). Analogously $\bullet U$ is a neighborhood of the parameter $p$.
2. We have chosen to define only locally the equality (9) because of the property 3 of Definition 11, which states that being a figure is a local property.

For the proof of the following theorem, we refer to [25,22].

Theorem 13. $S^\bullet \mathbb{R}^\infty$ is a category of figures.

Finally, starting from a category of figures $\mathcal{F}$, we can define its cartesian closure $\bar{\mathcal{F}}$, which can be thought as the category of spaces $X$ whose geometry is given specifying all the figures $f: U \to X$ of type $U \in \mathcal{F}$ traced on the space $X$. Considering the case $\mathcal{F} = \mathcal{O} \mathbb{R}^n$, we can also think $\mathcal{F}$ as a category which represents a well-known notion of regular space and regular function: with the cartesian closure $\bar{\mathcal{F}}$, we want to extend this notion to a more general type of space (e.g. spaces of mappings). In our case, we will set $\mathcal{F} := S^\bullet \mathbb{R}^\infty$, i.e. we want to extend the notion of quasi-standard smooth map to a more general type of space.

Definition 14. In the sequel, we will frequently use the notation $f \cdot g := g \circ f$ for the composition of maps so as to facilitate the lecture of diagrams, but we will continue to evaluate functions “on the right” hence $(f \cdot g)(x) = g(f(x))$.

The ideas related to the cartesian closure $\bar{\mathcal{F}}$ frequently simply generalize analogous notions of the diffeological setting (see, e.g., [28]).

Definition 15. If $X$ is a set, then we say that $(D, X)$ is an object of $\bar{\mathcal{F}}$ (or simply an $\bar{\mathcal{F}}$-object) if $D = \{ D_H \}_{H \in \mathcal{F}}$ is a family with
$$D_H \subseteq \text{Set}(|H|, X), \quad \forall H \in \mathcal{F}.$$  
We indicate by the notation $\mathcal{F}_{JH} \cdot D_H$ the set of all the compositions $f \cdot d$ of functions $f \in \mathcal{F}_{JH}$ and $d \in D_H$. The family $D$ has finally to satisfy the following conditions:
1. $\mathcal{F}_J H : D_H \subseteq D_J$.

2. $D_H$ contains all the constant maps $d : |H| \to X$.

3. Let $H \in \mathcal{F}$, $(H_i)_{i \in I}$ be an open cover of $H$ and $d : |H| \to X$ a map such that $d|_{H_i} \in D_{(H_i \prec H)}$, then $d \in D_H$.

Finally, we set $|(D, X)| := X$ to denote the underlying set of the space $(D, X)$.

Because of condition 1, we can think of $D_H$ as the set of all the regular functions defined on the “well-known” object $H \in \mathcal{F}$ and with values in the new space $X$; in fact this condition says that the set of figures $D_H$ is closed with respect to re-parametrizations with a generic $f \in \mathcal{F}_J H$.

Condition 3 is a sheaf property, and asserts that the property of being a figure $d \in D_H$ has a local character depending on $\mathcal{F}$.

We will frequently write $d \in H X$ to indicate that $d \in D_H$, and we can read it\textsuperscript{6} saying that $d$ is a figure of $X$ of type $H$ or $d$ belongs to $X$ at the level $H$ or $d$ is a generalized element of $X$ of type $H$.

The definition of arrow $f : X \to Y$ between two spaces $X, Y \in \mathcal{F}$ is the usual one for diffeological spaces, that is $f$ takes, through composition, generalized elements $d \in H X$ of type $H$ in the domain $X$ to generalized elements of the same type in the codomain $Y$:

**Definition 16.** Let $X, Y$ be $\mathcal{F}$-objects, then we will write

$$f : X \to Y$$

or, more precisely if needed\textsuperscript{7}

$$\mathcal{F} \models f : X \to Y$$

iff $f$ maps the support set of $X$ into the support set of $Y$:

$$f : |X| \to |Y|$$

and

$$d \cdot f \in H Y$$

for every type of figure $H \in \mathcal{F}$ and for every figure $d$ of $X$ of that type, i.e. $d \in H X$. In this case, we will also use the notation $f(d) := d \cdot f$.

Note that we have $f : X \to Y$ in $\mathcal{F}$ iff

$$\forall H \in \mathcal{F}, \forall x \in H X : f(x) \in H Y,$$

\textsuperscript{6} The following are common terminologies used in topos theory, see [33,30,37].

\textsuperscript{7} We shall frequently use notations of type $\mathcal{C} \models f : A \to B$ if we need to specify better the category $\mathcal{C}$ we are considering.
moreover $X = Y$ iff

$$\forall H \in \mathcal{F}, \forall d : d \in_H X \iff d \in_H Y.$$ 

These and many other properties justify the notation $\in_H$ and the name “generalized elements”.

**Definition 17.** The category of Fermat spaces is the cartesian closure of the category $S^\bullet \mathbb{R}^\infty$, i.e.

$$\mathcal{C}^\infty := S^\bullet \mathbb{R}^\infty.$$ 

The arrows of this category will be simply called smooth.

The justification of the name “smooth” is given by the following theorem, for the proof of which we refer to [25,22].

**Theorem 18.** The category of Fermat spaces $\mathcal{C}^\infty$ has the following properties:

1. $\mathcal{C}^\infty$ is cartesian closed, complete and co-complete, so that in it we can freely form, e.g., infinite products, sums, quotient sets and equalizers. Limits and co-limits can be constructed as lifting and co-lifting from Set to $\mathcal{C}^\infty$.
2. Set-theoretical operations like compositions and evaluations are arrows of the category $\mathcal{C}^\infty$, i.e. they are smooth.
3. Every Fermat space $X \in \mathcal{C}^\infty$ is a topological space: we say that a subset $U \subseteq |X|$ is open in $X$, and we will write $U \in \tau_X$, iff $d^{-1}(U) \in \tau_H$ for any $H \in S^\bullet \mathbb{R}^\infty$ and any $d \in_H X$. This topology has the following properties:
   (a) $\tau_X/\sim = \tau_X/\sim$, where $\sim$ is an equivalence relation and $\tau_X/\sim$ is the quotient topology;
   (b) $\tau_X \times \tau_Y \subseteq \tau_{X \times Y}$, where $\tau_X \times \tau_Y$ is the product topology;
   (c) $\tau(\text{colim}_{i \in I} X_i) = \text{colim}_{i \in I} \tau_{X_i}$, where the right-hand side is the colimit topology.
4. Every type of figure $K \in S^\bullet \mathbb{R}^\infty$ can be viewed as a Fermat space using $\bar{K} = (S^\bullet \mathbb{R}^\infty(\cdot, K), |K|)$. For the Fermat space $\bar{K} \in \mathcal{C}^\infty$ we have that

$$\mathcal{C}^\infty \models \bar{K} d \rightarrow X \iff d \in_K X.$$ 

Moreover, $S^\bullet \mathbb{R}^\infty(H, K) = \mathcal{C}^\infty(\bar{H}, \bar{K})$. Therefore, $\mathcal{O}^\infty$ and $S^\bullet \mathbb{R}^\infty$ are fully embedded in $\mathcal{C}^\infty$, so that both ordinary and quasi-standard smooth mappings are arrows of $\mathcal{C}^\infty$.
5. Every subset $S \subseteq X$ of a Fermat space $X \in \mathcal{C}^\infty$ defines a subspace denoted $(S \prec_X) := (D, S)$, where, for every type of figure $H \in S^\bullet \mathbb{R}^\infty$, we have set

$$d \in D_H :\iff d : |H| \rightarrow S \quad \text{and} \quad d \cdot i \in_H X.$$ 

Here $i : S \rightarrow |X|$ is the inclusion map. This subspace has the following properties:
   (a) Its topology $\tau_{S \prec_X}$ contains the topology induced by $\tau_X$ on the subset $S$;
   (b) $\tau_{S \prec_X} \subseteq \tau_X$ if $S$ is open in $X$, hence in this case we have on $(S \prec_X)$ exactly the induced topology;
   (c) Let $f : X \rightarrow Y$ be an arrow of $\mathcal{C}^\infty$ and $U$, $V$ be subsets of $|X|$ and $|Y|$ respectively, such that $f(U) \subseteq V$, then $(U \prec_X) \xrightarrow{f|U} (V \prec_Y)$ in $\mathcal{C}^\infty$. 

(d) \( (U \prec \tilde{H}) = (U < H) \) for \( U \) open in \( H \in S^\bullet \mathbb{R}^\infty \) (recall the definition of \( \tilde{H} \in \bullet C^\infty \), for \( H \in S^\bullet \mathbb{R}^\infty \), given in 4);

(e) \( i : (S < X) \hookrightarrow X \) is the lifting of the inclusion \( i : S \hookrightarrow |X| \) from \textbf{Set} to \( \bullet C^\infty \);

(f) \( (S < X) \times (T < Y) = (S \times T < X \times Y) \);

(i) If \( S \subseteq \bullet (\mathbb{R}^d) \), then \( \overline{S} = (S \prec \bullet \mathbb{R}^d) \).

6. Let \( X, Y \in \bullet C^\infty \) be Fermat spaces, \( (U_i)_{i \in I} \) an open cover of \( X \) and \( f : |X| \to |Y| \) a map from the support set of \( X \) to that of \( Y \) such that \( (U_i < X) \xrightarrow{f|U_i} Y \) is smooth (i.e. it is in \( \bullet C^\infty \)) for every \( i \in I \), then \( X \xrightarrow{f} Y \) is smooth.

Really, the category of smooth manifolds can be embedded in \( \bullet C^\infty \) and we can also extend every manifold (and, more generally, every diffeological space) generalizing the extension \( \mathbb{R}^d \mapsto \bullet (\mathbb{R}^d) \) and obtaining a suitable functor having very good properties. These generalizations will be subject of future works and articles.

3.1. The Fermat functor preserves product of open sets

We want to prove that the bijective applications \( a_{UV} \), defined in (7), and \( b_{UV} \), defined in (8), are arrows of \( \bullet C^\infty \).

**Theorem 19.** Let \( U, V \) be open sets as above, then in \( \bullet C^\infty \) the maps \( a_{UV} \) and \( b_{UV} \) realize in \( \bullet C^\infty \) the isomorphism

\[
\bullet(U \times V) \simeq \bullet U \times \bullet V.
\]

**Proof.** In this statement each subset, e.g. \( \bullet U \), is identified with the corresponding \( \bullet C^\infty \) space \( \overline{U} \) described in 4 of Theorem 18. We have to prove that both the maps of (7) and (8):

\[
a_{UV} : \bullet U \times \bullet V \to \bullet(U \times V),
\]

\[
b_{UV} : \bullet(U \times V) \to \bullet U \times \bullet V,
\]

are smooth. For simplicity, in this proof, we will use the simplified notations \( a := a_{UV} \) and \( b := b_{UV} \). We start from the first one, and, because of Definition 16, the application \( a \) is smooth in \( \bullet C^\infty \) if it takes, through composition, a generic figure of the domain \( \delta \in \bullet C^\infty(S, \bullet(U \times V)) \) into a figure \( \delta \cdot a \in \bullet C^\infty(S, (U \times V)) = S^\bullet \mathbb{R}^\infty(S, (U \times V)) \) of the codomain. Here, \( S \subseteq (\mathbb{R}^d) \) is a generic type of figure in \( S^\bullet \mathbb{R}^\infty \). Because projections \( p_1 : \bullet U \times \bullet V \to \bullet U \) and \( p_2 : \bullet U \times \bullet V \to \bullet V \) are smooth, we have that \( \delta \cdot p_1 \in S^\bullet \mathbb{R}^\infty(S, U) \) and \( \delta \cdot p_2 \in S^\bullet \mathbb{R}^\infty(S, V) \). Therefore, because of Definition 9 of quasi-standard smooth function, for every \( s \in S \) we can write

\[
p_1(\delta(x)) = \alpha(p, x), \quad \forall x \in \bullet A \cap S,
\]

\[
p_2(\delta(x)) = \beta(q, x), \quad \forall x \in \bullet B \cap S
\]

for suitable
A open in \( \mathbb{R}^a \) such that \( s \in {}^aA \),

\( B \) open in \( \mathbb{R}^b \) such that \( s \in {}^bB \),

\( p \in {}^aP \), where \( P \) is open in \( \mathbb{R}^p \),

\( q \in {}^bQ \), where \( Q \) is open in \( \mathbb{R}^q \),

\( \alpha \in \mathcal{C}^\infty(P \times A, \mathbb{R}^u) \),

\( \beta \in \mathcal{C}^\infty(Q \times B, \mathbb{R}^v) \).

We can hence write

\[
a(\delta(x)) = \langle {}^a\alpha(p, x), {}^b\beta(q, x) \rangle.
\]

Define

\[
\xi(x_1, x_2, x_3) = (\alpha(x_1, x_3), \beta(x_2, x_3)), \quad \forall x_1 \in P, \ x_2 \in Q, \ x_3 \in A \cap B.
\]

We have \( \xi \in \mathcal{C}^\infty(P \times Q \times (A \cap B), \mathbb{R}^{u+v}) \), \( r := (p, q) \in {}^a(\mathbb{R}^{b+d}) \) and \( s \in {}^a(A \cap B) \) because \( s \in \mathcal{C}^\infty(A \cap B) \) from \( s \in {}^aA \cap {}^bB \). A direct verification proves

\[
a(\delta(x)) = {}^a\xi(r, x), \quad \forall x \in {}^a(A \cap B) \cap S.
\]

This proves that \( \delta \cdot a : S \to {}^a(U \times V) \) is quasi-standard smooth, which is our first conclusion.

To prove that the map \( b \) is an arrow of \( {}^a\mathcal{C}^\infty \) is simpler. Indeed, take \( d \in \mathcal{H} \) \( {}^a(U \times V) \) to prove that \( d \cdot b \in \mathcal{H} \) \( {}^aU \times {}^aV \). Due to the universal property of the product \( {}^aU \times {}^aV \), it suffices to consider the composition of this map \( d \cdot b \) with the projections of this product. But, if \( p_U : U \times V \to U \) is the projection on \( U \), then

\[
{}^aU \times {}^aV \xrightarrow{a} {}^a(U \times V) \xrightarrow{\cdot p_U} {}^aU
\]

and \( {}^a p_U(a(x, y)) = {}^a p_U((x, y)) = x \), so \( a \cdot {}^a p_U \) is the projection of the product \( {}^aU \times {}^aV \) on \( {}^aU \).

Therefore the conclusion \( d \cdot b \in \mathcal{H} \) \( {}^aU \times {}^aV \) is equivalent to

\[
d \cdot b \cdot a \cdot {}^a p_U = d \cdot {}^a p_U \in \mathcal{H} \] \( {}^aU \),

\[
d \cdot b \cdot a \cdot {}^a p_V = d \cdot {}^a p_V \in \mathcal{H} \] \( {}^aV \)

which are true since \( {}^a p_U \) and \( {}^a p_V \) are arrows of \( {}^a\mathcal{C}^\infty \). \( \square \)

In the following, we shall always use the isomorphism \( a_{UV} \) to identify these spaces, hence we write \( {}^aU \times {}^aV = {}^a(U \times V) \), e.g., \( {}^a(\mathbb{R}^d) = ({}^a\mathbb{R})^d =: {}^a\mathbb{R}^d \).
4. The Fermat–Reyes theorem

In this section, we want to introduce the basic theorems and ideas that permits the development of the calculus for $\mathcal{C}^\infty$ functions of the form $f: \cdot U \to \cdot \mathbb{R}^n$, where $U$ is open in $\mathbb{R}^d$. The calculus for functions defined on infinitesimal sets, like $g: D_k^d \to \cdot \mathbb{R}^n$ will be presented in a subsequent article.

Using the infinitesimal Taylor’s formula, as stated in Theorems 25 and 26 of [23], we have a powerful instrument to manage derivatives of functions $\cdot f$ obtained as extensions of ordinary smooth functions $f: \mathbb{R}^d \to \mathbb{R}^n$. But this is not the case if $f: \cdot \mathbb{R}^d \to \cdot \mathbb{R}^n$ is a generic $\mathcal{C}^\infty$ arrow, that is if we can write locally $f(x) = \cdot \alpha(p, x)$, where $p \in \cdot \mathbb{R}^d$ and $\alpha$ is smooth, because, generally speaking, $f$ does not have standard derivatives. Therefore, it arises the problem to define the derivatives of this type of functions in our setting. On the one hand, we would like to set e.g. $f'(x) := \cdot \frac{\partial \alpha}{\partial x}(p, x)$ (if $d = n = 1$, for simplicity), and so the problem would become the independence in this definition from both the function $\alpha$ and the (nonstandard) parameter $p$.

For example, for functions defined on an infinitesimal domain we can see that this problem of independence is not trivial. Let us consider two first order infinitesimals $p, p' \in D$, with $p \neq p'$. Because the product of first order infinitesimals is always zero for Fermat reals, we have that the null function $f(x) = 0$, for $x \in D$, can be written both as $f(x) = p \cdot x =: \cdot \alpha(p, x)$ and as $f(x) = p' \cdot x = \cdot \alpha(p', x)$. But $\cdot \left(\frac{\partial \alpha}{\partial x}(p, x) = p \neq p' = \cdot \left(\frac{\partial \alpha}{\partial x}(p', x)\right)$. For functions defined on an open set, this independence can be established, using the method originally used by Fermat and studied by G.E. Reyes (see [37]; see also [8,39] for analogous ideas in a context different from that of synthetic differential geometry).

In all this section we will use the notations for intervals as subsets of $\cdot \mathbb{R}$, e.g. $[a, b) := \{x \in \cdot \mathbb{R} | a \leq x < b\}$. Notations of the type

$$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$$

will be used to specify that the interval has to be understood as a subset of $\mathbb{R}$. For the properties of the total order relation on the ring of Fermat reals, see [26,22]; here we only recall the following.

**Definition 20.** In the following, we will use the useful notation

$$\forall^0 t \geq 0: \quad \mathcal{P}(t)$$

and we will read the quantifier $\forall^0 t \geq 0$ saying “for every $t \geq 0$ (sufficiently) small”, to indicate that the property $\mathcal{P}(t)$ is true for all $t$ in some right neighborhood of $t = 0$, i.e.

$$\exists \delta > 0: \forall t \in [0, \delta): \quad \mathcal{P}(t).$$

Moreover, the order relation on $\cdot \mathbb{R}$ is defined as follows: Let $x, y \in \cdot \mathbb{R}$, then we say

$$x \leq y$$

iff we can find $z \in \cdot \mathbb{R}$ such that $z = 0$ in $\cdot \mathbb{R}$ and

$$\forall^0 t \geq 0: \quad x_t \leq y_t + z_t.$$
The method used by Fermat to calculate derivatives consists firstly to assume $h \neq 0$, secondly to construct the incremental ratio

$$\frac{f(x + h) - f(x)}{h},$$

and then to set $h = 0$ in the final result. This idea can be perfectly understood if we think that the incremental ratio can be extended with continuity at $h = 0$ if the function $f$ is differentiable at $x$. In our smooth context, we need a theorem confirming the existence of a “smooth version” of the incremental ratio. We firstly introduce the notion of segment in a $d$-dimensional space $\ast \mathbb{R}^d$. As we will prove later, for $d = 1$ it coincides with the notion of interval in $\ast \mathbb{R}$.

**Definition 21.** If $a, b \in \ast \mathbb{R}^d$, then

$$[a, b] := \{a + s \cdot (b - a) \mid s \in [0, 1]\}$$

is the segment of $\ast \mathbb{R}^d$ going from $a \in \ast \mathbb{R}^d$ to $b \in \ast \mathbb{R}^d$.

The Fermat–Reyes theorem, generalized to a generic open set $U$, is stated in the following.

**Theorem 22.** Let $U$ be an open set of $\mathbb{R}$, and $f : \ast U \to \ast \mathbb{R}$ be a $\ast C^\infty$ function. Let us define the thickening\(^8\) of $\ast U$ along the $x$-axis by

$$\widetilde{\ast U} := \{(x, h) \in \ast \mathbb{R}^2 \mid [x, x + h] \subseteq \ast U\}.$$  

Then $\widetilde{\ast U}$ is open in $\ast \mathbb{R}^2$ and there exists one and only one $\ast C^\infty$ map $r : \widetilde{\ast U} \to \ast \mathbb{R}$ such that

$$f(x + h) = f(x) + h \cdot r(x, h) \text{ in } \ast \mathbb{R}, \forall (x, h) \in \widetilde{\ast U}.$$  

Hence, we define $f'(x) := r(x, 0) \in \ast \mathbb{R}$ for every $x \in \ast U$. Moreover, if $f(x) = \ast \alpha(p, x)$, $\forall x \in V \subseteq \ast U$ with $\alpha \in C^\infty(A \times B, \mathbb{R})$, then

$$f'(x) = \left(\frac{\partial \alpha}{\partial x}\right)(p, x) \text{ in } \ast \mathbb{R}.$$  

We anticipate the proof of this theorem, which is not trivial because all the equalities stated are in $\ast \mathbb{R}$, by the following lemmas.

**Lemma 23.** Let $U$ be an open set of $\mathbb{R}^d$ and $v \in \ast \mathbb{R}^d$, then the thickening of $\ast U$ along $v$ defined as

$$\widetilde{\ast U}_v := \{(x, h) \in \ast \mathbb{R}^d \times \ast \mathbb{R} \mid [x, x + hv] \subseteq \ast U\}$$

is open in $\ast \mathbb{R}^d \times \ast \mathbb{R}$.

\(^8\) This name is used by [1].
Proof. Let us take a generic point \((x, h) \in \mathring{U}_v\); we want to prove that \((x, h) \in (A \times B) \subseteq \mathring{U}_v\) for some subsets \(A\) of \(\mathbf{R}^d\) and \(B\) of \(\mathbf{R}\). Because the point \((x, h)\) is in the thickening, we have that

\[
\forall s \in [0, 1]: \quad x + s \cdot hv \in \mathring{U}.
\]

Taking the standard parts we obtain

\[
\forall s \in [0, 1]: \quad ^o x + s \cdot ^o h \cdot ^o v =: \varphi(s) \in U.
\]

The function \(\varphi: [0, 1]_{\mathbf{R}} \to U\) is continuous and thus

\[
\varphi([0, 1]_{\mathbf{R}}) = [^o x, ^o x + ^o h ^o v] =: K
\]

is compact in \(\mathbf{R}^d\). But \(K \subseteq U\) and \(U\) is open, so the distance of \(K\) from the complement \(\mathbf{R}^d \setminus U\) is strictly positive; let us call \(2a := d(K, \mathbf{R}^d \setminus U) > 0\) this distance, so that for every \(c \in K\) we have that

\[
B_a(c) := \{x \in \mathbf{R}^d \mid d(x, c) < a\} \subseteq U.
\]

Now, set \(A := B_{a/2}(^o x)\) and \(B := B_{a}(^o h)\), where we have fixed \(b \in \mathbf{R}_{> 0}\) such that \(b \cdot \|v\| \leq \frac{a}{2}\). We have \(x \in \mathbf{r}A\) because \(^o x \in A\) and \(A\) is open; analogously \(h \in \mathbf{r}B\) and thus \((x, h) \in \mathbf{r}(A \times B)\). We have finally to prove that taking a generic point \((y, k) \in \mathbf{r}(A \times B)\), the whole segment \([y, y + kv]\) is contained in \(\mathbf{r}U\); so, let us take also a Fermat number \(0 \leq s \leq 1\). Since \(U\) is open, to prove that \(y + skv \in \mathbf{r}U\) is equivalent to prove that the standard part \(y + skv \in U\), i.e. \(^o y + s ^o k ^o v \in U\). For, let us observe that

\[
\| ^o y + s ^o k ^o v - ^o x - s ^o h ^o v \| \leq \| ^o y - ^o x \| + s \cdot \| ^o v \| \cdot \| k - ^o h \| \leq \frac{a}{2} + 1 \cdot \| ^o v \| \cdot b \leq a.
\]

Therefore, \(^o y + s ^o k ^o v \in B_a(c) \subseteq U\), where \(c = ^o x + s ^o h ^o v \in K\) from our definition of the compact set \(K\). \(\square\)

**Lemma 24.** If \(a, b \in \mathbf{r} \mathbf{R}\), then

\[
a < b \quad \Rightarrow \quad [a, b] = [a, b],
\]

\[
b \leq a \quad \Rightarrow \quad [a, b] = [b, a].
\]

**Proof.** We will prove the first implication, the second being a simple consequence of the first one. To prove the inclusion \([a, b] \subseteq [a, b]\) take \(x = a + s \cdot (b - a)\) with \(0 \leq s \leq 1\), then \(0 \leq s \cdot (b - a) \leq b - a\) because \(b - a > 0\). Adding \(a\) to these inequalities we get \(a \leq x \leq b\). For the proof of the opposite inclusion, let us consider \(a \leq x \leq b\). If we prove the inclusion for \(a = 0\) only, we can prove it in general: in fact, \(0 \leq x - a \leq b - a\), so that if \([0, b - a] \subseteq [0, b - a]\) we can derive the existence of \(s \in [0, 1]\) such that \(x - a = 0 + s \cdot (b - a)\), which is our conclusion. So, let us assume that \(a = 0\). If \(^o b \neq 0\), then \(b\) is invertible and it suffices to set \(s := \frac{a}{b}\) to have
the conclusion. Otherwise, \( b = 0 \) and hence also \( x = 0 \). Let us consider the decompositions of \( x \) and \( b \)

\[
\begin{align*}
x &= \sum_{i=1}^{k} x_i \cdot dt_{\omega_i(x)}, \\
b &= \sum_{j=1}^{h} b_j \cdot dt_{\omega_j(b)}.
\end{align*}
\]

We have to find a number \( s = \sum_{n=1}^{N} s_n \cdot d\omega_n(x) \) such that \( s \cdot b = x \). It is interesting to note that the attempt to find the solution \( s \in [0, 1] \) directly from these decompositions and from the property \( s \cdot b = x \) is not as easy as to find the solution using directly little-oh polynomials. In fact for \( t > 0 \): \( bt > 0 \) because \( b > 0 \) and hence for \( t > 0 \) sufficiently small, we can form the ratio

\[
\frac{x_t}{b_t} = \frac{\sum_{i=1}^{k} x_i \cdot t_{\omega_i(x)}}{\sum_{j=1}^{h} b_j \cdot t_{\omega_j(b)}}.
\]

Let us note that from \( x > 0 \) we can deduce \( x_1 > 0 \) (see Theorem 25 of [26] or Theorem 4.2.6 of [22]) and hence also \( \omega(b) = \omega_1(b) > \omega(x) \geq \omega_i(x) \) because \( x < b \). From (12) we have

\[
\frac{x_t}{b_t} = \frac{\sum_{i=1}^{k} x_i \cdot t_{\omega_i(x)}}{\sum_{j=1}^{h} b_j \cdot t_{\omega_j(b)}} \cdot \frac{1}{\sum_{j=2}^{h} b_j \cdot t_{\omega_j(b)}}.
\]

Writing, for simplicity, \( a \odot b := \frac{a \cdot b}{a + b} \) we can write the previous little-oh polynomial using the common notation with \( dt_a \):

\[
\frac{x_t}{b_t} = \frac{1}{\sum_{j=1}^{h} b_j \cdot t_{\omega_j(b)}} \cdot \left( \sum_{i=1}^{k} x_i \cdot t_{\omega_i(x)} \right)
\]

As usual, the series in this formula is really a finite sum if it is interpreted in \( *\mathbb{R} \), because \( D_\infty \) is an ideal of nilpotent infinitesimals. Going back in these passages, it is quite easy to prove that the previously defined \( s \in *\mathbb{R} \) verifies the desired equality \( s \cdot b = x \). Moreover, from Definition 20 of order, and from \( 0 \leq x \leq b \), the relations \( 0 \leq s \leq 1 \) follow.

It is interesting to make some considerations based on the proof of this lemma. Indeed, we have just proved that in the Fermat reals every equation of the form \( a + x \cdot b = c \) with \( a <
$c < a + b$ has a solution. If $b$ is invertible, this is obvious and we have a unique solution. If $b$ is a nilpotent infinitesimal, a possible solution is given by a formula like (13), but we do not have uniqueness. E.g. if $a = 0$, $c = dt_2 + dt$ and $b = dt_3$, then $x = dt_6 + dt_3/2$ is a solution of $a + x \cdot b = c$, but $x + dt$ is another solution because $dt \cdot dt_a = 0$ for every $a \geq 1$. Among all the solutions in the case $b \in D_\infty$, we can choose the simplest one, i.e. that “having no useless addends in its decomposition”, that is such that

$$\frac{1}{\omega_i(x)} + \frac{1}{\omega(b)} \leq 1$$

for every addend $x_i \cdot d t_\omega_i(x)$ in the decomposition of $x$. Otherwise, if for some $i$ we have the opposite inequality, we can apply Theorem 13 of [23] to have that we can delete some “useless addend” in the decomposition of $x$. We can thus understand that this algebraic problem is strictly tied with the definition of derivative $f'(x)$, which is the solution of the linear equation $f(x + h) = f(x) + h \cdot f'(x)$: if $f$ is defined only on an infinitesimal set like $D_n$, this equation has not a unique solution in the ring of Fermat reals, and we can define the derivative $f'(x)$ only by considering “the simplest solution” of this equation. We will present the definition of $f'(x)$, where the function $f$ is defined on an infinitesimal set, in a next article.

The uniqueness of the smooth incremental ratio stated in Theorem 22 is tied with the following lemma, for the proof of which we decided to introduce nilpotent paths (see Definition 2) instead of continuous paths at $t = 0$, like in [21]. We will call this lemma the cancellation law of non-infinitesimal functions.

**Lemma 25 (Cancellation law of non-infinitesimal functions).** Let $U$ be an open neighborhood of $0$ in $\mathbb{R}$, and let

$$f, g : \mathcal{U} \rightarrow \mathbb{R}$$

be two smooth functions such that

$$\forall x \in \mathcal{U} : \text{ } x \text{ is invertible } \Rightarrow \text{ } g(x) \text{ is invertible and } g(x) \cdot f(x) = 0. \quad (14)$$

Then $f$ is the null function, i.e. $f = 0$.

**Proof.** We can apply Definition 9 of quasi-standard smooth function at the point $0 \in \mathcal{U}$ obtaining that the function $f$ can be written as

$$f(x) = \alpha(p, x), \quad \forall x \in \mathcal{B} \cap \mathcal{U} = \mathcal{V},$$

where $\alpha \in C^\infty(A \times B, \mathbb{R})$, $p \in A$, $A$ is an open set of $\mathbb{R}^p$ and $B$ is an open neighborhood of $0$ in $\mathbb{R}$. We can always assume that $\circ p = 0$ because, otherwise, we can consider the standard smooth function $(y, x) \mapsto \alpha(y - \circ p, x)$.

---

9 Let us note explicitly, that this is not in contradiction with the non-Archimedean property of $\mathbb{R}$ (let $a = 0$ and $b \in D_\infty$) because of the inequalities that $c$ must verifies to have a solution.
We firstly want to prove that \( \alpha(pt, xt) = o(t) \) for every \( x \in \mathcal{V} \). From our main hypotheses (14) we get \( f(x) = \ast \alpha(p, x) = 0 \) if \( x \in \mathcal{V} \) is invertible. Therefore, considering a generic non-zero \( r \in U \cap B \setminus \{0\} \), then \( x := h + r \in \ast U \), because \( o x = r \in U \), and \( x \) is invertible because its standard part is \( r \neq 0 \). We hence obtain \( \ast \alpha(p, x) = 0 \) in \( \ast \mathbb{R} \), i.e.

\[
\lim_{t \to 0^+} \frac{\alpha(pt, ht + r)}{t} = 0.
\] (15)

Now we have to prove (15) for \( r = 0 \) too. Let us take a generic infinitesimal \( h \in D_\infty \) and choose a \( k \in \mathbb{N}_{>0} \) such that\(^{10}\)

\[
h^k = 0 \quad \text{in} \quad \ast \mathbb{R},
\]

\[
p^k = 0 \quad \text{in} \quad \ast \mathbb{R}^p.
\]

Let us consider the Taylor’s formula of order \( k \) with the function \( \alpha \) at the point \( (0, r) \) (which obviously is true for \( r = 0 \) too):

\[
\frac{\alpha(0 + pt, r + h_t)}{t} = \frac{1}{t} \left[ \sum_{q \in \mathbb{N}^p+1 \atop |q| \leq k} \frac{\partial^q \alpha}{\partial (p, x)^q}(0, r) \cdot (pt, h_t)^q + \sum_{q \in \mathbb{N}^p+1 \atop |q| = k + 1} \frac{\partial^q \alpha}{\partial (p, x)^q}(\xi_t, \eta_t) \cdot (pt, h_t)^q \right]
\] (16)

with \( \xi_t \in (0, pt) \) and \( \eta_t \in (r, r + h_t) \). But \( h^k = 0 \) and \( p^k = (p_1, \ldots, p_p)^k = (p_1^k, \ldots, p_p^k) = 0 \), hence \( h, p_i \in D_k \). Moreover, if \( |q| = k + 1 \), then

\[
\sum_{i=1}^{p+1} q_i \frac{k + 1}{k + 1} = 1,
\]

so that from Corollary 17 of [23] we get

\[
(pt, h_t)^q = p_1(t)^q_1 \cdot \ldots \cdot p_p(t)^q_p \cdot h(t)^{q_{p+1}} = o(t).
\]

Therefore, from (15) and (16) we obtain

\[
\lim_{t \to 0^+} \sum_{q \in \mathbb{N}^p+1 \atop |q| \leq k} \frac{\partial^q \alpha}{\partial (p, x)^q}(0, r) \cdot \frac{1}{q!} \cdot \frac{(pt, h_t)^q}{t} = 0, \quad \forall r \in (U \cap B) \neq 0.
\] (17)

Now, let \( \{q_1, \ldots, q_N\} \) be an enumeration of all the \( q \in \mathbb{N}^p+1 \) such that \( |q| \leq k \), and for simplicity set

\(^{10}\) This passage is possible exactly because we are considering nilpotent paths as elements of \( \ast \mathbb{R} \).
\[
b_i(r) := \frac{\partial^{q_i} \alpha}{\partial (p, x)^{q_i}}(0, r) \cdot \frac{1}{q_i!}, \quad \forall r \in U \cap B,
\]
\[
s_i(t) := \frac{(p_t, h_t)^{q_i}}{t}, \quad \forall t \in \mathbb{R}_{\geq 0},
\]
so that we can write (17) as
\[
\forall r \in (U \cap B) \neq 0: \lim_{t \to 0^+} \sum_{i=1}^{N} b_i(r) \cdot s_i(t) = 0. \tag{18}
\]
If all the functions \(b_i\) are identically zero, then \(b_i(\bar{r}) = b_i(0)\) where \(\bar{r} \in U \cap B \setminus \{0\}\), which always exists because \(U \cap B\) is open in \(\mathbb{R}\). Therefore, (18) (and hence also (17)) is true for \(r = 0\) too. Otherwise, taking a base of the subspace of \(\mathcal{C}^\infty(U \cap B, \mathbb{R})\) generated by the smooth functions \(b_1, \ldots, b_N\) and expressing all the \(b_i\) in this base, we can suppose to have in (18) only linearly independent functions.

Now, we can use the following lemma:

**Lemma 26.** Let \(U\) be an open neighborhood of 0 in \(\mathbb{R}\) and \(b_1, \ldots, b_N : U \to \mathbb{R}\) be linearly independent functions continuous at 0. Then, we can find
\[
r_1, \ldots, r_N \in U \setminus \{0\}
\]
such that
\[
\det \begin{bmatrix}
b_1(r_1) & \ldots & b_N(r_1) \\
\vdots & \ddots & \vdots \\
b_1(r_N) & \ldots & b_N(r_N)
\end{bmatrix} \neq 0.
\]
From (18) we can write
\[
\lim_{t \to 0^+} \begin{bmatrix}
b_1(r_1) & \ldots & b_N(r_1) \\
\vdots & \ddots & \vdots \\
b_1(r_N) & \ldots & b_N(r_N)
\end{bmatrix} \cdot \begin{bmatrix}
s_1(t) \\
\vdots \\
s_N(t)
\end{bmatrix} = 0
\]
and hence, from this lemma, we can deduce that \(s_i(t) \to 0\) for \(t \to 0^+\). Because these limits exist, we can take the limit for \(r \to 0\) of (18) and proceed in the following way
\[
\lim_{r \to 0} \lim_{t \to 0^+} \sum_{i=1}^{N} b_i(r) \cdot s_i(t) = \sum_{i=1}^{N} \lim_{r \to 0} b_i(r) \cdot \lim_{t \to 0^+} s_i(t) \\
= \lim_{t \to 0^+} \sum_{i=1}^{N} b_i(0) \cdot s_i(t) = 0
\]
(let us note that we do not exchange the limit signs). This proves that (18) is true for \(r = 0\) too.
From (16), for \( r = 0 \), we obtain
\[
\lim_{t \to 0^+} \frac{\alpha(p_t, h_t)}{t} = 0.
\]
This proves that \( f(x) = 0 \) in \( \mathbb{T}_{\mathbb{R}} \) for every \( x \in \mathcal{V} \). Finally, if \( x \in \mathbb{T}U \setminus \mathcal{V} \) then \( \circ x \neq 0 \), because otherwise we would have \( x \in \mathcal{V} = \mathbb{T}(B \cap \mathcal{U}) \). So, \( x \) is invertible and hence also \( g(x) \) is invertible, so that from \( g(x) \cdot f(x) = 0 \) we can easily deduce \( f(x) = 0 \) also in this case. \( \square \)

**Proof of Lemma 26.** We prove the converse by induction on \( N \geq 2 \), i.e. if all the determinants cited in the statement are zero, then the functions \( (b_1, \ldots, b_N) \) are linearly dependent. First, let us suppose \( N = 2 \) and that all these determinants are zero, that is
\[
\alpha_1(r, s) \cdot \alpha_2(r, s) = \alpha_2(r, s) \cdot \alpha_1(r, s), \quad \forall r, s \in U_{\neq 0}.
\]
If the functions \( b_i, i = 1, 2 \), are both zero then they are trivially linearly dependent, hence let us suppose, e.g., that \( b_1(\bar{s}) \neq 0 \) for some \( \bar{s} \in U \). Due to the continuity of \( b_1 \) at 0 we can suppose \( \bar{s} \neq 0 \), hence from (19)
\[
b_2(r) = b_1(r) \cdot \frac{b_2(\bar{s})}{b_1(\bar{s})} = : b_1(r) \cdot a, \quad \forall r \in U_{\neq 0}.
\]
From the continuity of \( b_1 \) at 0 we have that \( b_2 = b_1 \cdot a \), that is \( (b_1, b_2) \) are linearly dependent.

Now, suppose that the implication is true for any matrix of \( N \) functions and we prove the conclusion for matrices of order \( N + 1 \) too. By Laplace’s formula with respect to the first row, for every \( r_1, \ldots, r_{N+1} \in U_{\neq 0} \) we have
\[
\begin{vmatrix}
  b_2(r_2) & \cdots & b_{N+1}(r_2) \\
  \vdots & \ddots & \vdots \\
  b_2(r_{N+1}) & \cdots & b_{N+1}(r_{N+1}) \\
\end{vmatrix}
- \cdots
\begin{vmatrix}
  b_1(r_2) & \cdots & b_N(r_2) \\
  \vdots & \ddots & \vdots \\
  b_1(r_{N+1}) & \cdots & b_N(r_{N+1}) \\
\end{vmatrix}
+ (-1)^{N+2} \cdot b_{N+1}(r_1)
\begin{vmatrix}
  b_1(r_2) & \cdots & b_N(r_2) \\
  \vdots & \ddots & \vdots \\
  b_1(r_{N+1}) & \cdots & b_N(r_{N+1}) \\
\end{vmatrix} = 0.
\]
Now, we have two cases. Let \( \alpha_1(r_2, \ldots, r_{N+1}) \) denote the first determinant in the previous (20). If it is zero for any \( r_2, \ldots, r_{N+1} \in U_{\neq 0} \), then by the induction hypothesis \( (b_2, \ldots, b_{N+1}) \) are linearly dependent, hence the conclusion follows. Otherwise, \( \bar{a}_1 := \alpha_1(\tilde{r}_2, \ldots, \tilde{r}_{N+1}) \neq 0 \) for some \( \tilde{r}_2, \ldots, \tilde{r}_{N+1} \in U_{\neq 0} \). Then, from (20) it follows
\[
b_1(r_1) = b_2(r_1) \cdot \frac{\bar{a}_2}{\bar{a}_1} - \cdots - (-1)^{N+2} \cdot b_{N+1}(r_1) \cdot \frac{\alpha_{N+1}}{\alpha_1}, \quad \forall r_1 \in U_{\neq 0},
\]
where we used obvious notations for the other determinants in (20). From the continuity of \( b_i \) the previous formula is true for \( r_1 = 0 \) too and this proves the conclusion. \( \square \)

**Proof of Theorem 22.** We will define the function \( r : \mathbb{T}U \to \mathbb{T}_{\mathbb{R}} \) patching together smooth functions defined on open subsets covering \( \mathbb{T}U \). Therefore, we have to take a generic point
As usual, since \( x \in \mathcal{U} \), we can write
\[
f | y = \mathcal{U}(p, -) | y, \tag{21}\]
where \( \alpha \in C^\infty(\bar{U} \times \bar{V}, \mathbb{R}) \), \( \mathcal{V} := \mathcal{U} \cap \mathcal{U} = (\bar{V} \cap U) \) is an open neighborhood of \( x \) and \( \mathcal{U} \) is an open neighborhood of \( p \in \mathbb{R}^p \) defined by the open subset \( U \) of \( \mathbb{R}^p \). Because \( \mathcal{U} \) is open in \( \mathbb{R} \times \mathcal{U} = \mathbb{R}^2 \) (Lemma 23), we can find two open subsets \( A \) and \( B \) of \( \mathbb{R} \) such that
\[
(x, h) \in \mathcal{U}(A \times B) \subseteq \mathcal{U}
\]
and such that
\[
a + s \cdot b \in \bar{V}, \quad \forall a \in A, \ b \in B, \ s \in [0, 1] \mathbb{R}. \tag{22}\]
Let us define
\[
\gamma(q, a, b) := \frac{1}{2} \int_0^1 \partial_2 \alpha(q, a + s \cdot b) \, ds, \quad \forall q \in \bar{U}, \ a \in A, \ b \in B. \tag{23}\]
We have that \( \gamma \in C^\infty(\bar{U} \times A \times B, \mathbb{R}) \), so that if we define
\[
r(a, b) := \mathcal{U}(p, a, b), \quad \forall (a, b) \in \mathcal{U}(A \times B),
\]
then we have
\[
r : \mathcal{U}(A \times B) \to \mathbb{R} \text{ is nonstandard smooth.} \tag{24}\]
\[
\mathcal{U}(A \times B) \text{ open neighborhood of } (x, h) \text{ in } \mathcal{U}. \tag{25}\]
For every \( (a, b) \in \mathcal{U}(A \times B) \) we have
\[
b \cdot r(a, b) = \left[ t \mapsto \int_0^1 \partial_2 \alpha(p_t, a_t + s \cdot b_t) \cdot b_t \, ds \right] \sim
\]
\[
= \left[ t \mapsto \int_{a_t}^{a_t + b_t} \partial_2 \alpha(p_t, y) \, dy \right] \sim
\]
\[
= \left[ t \mapsto \alpha(p_t, a_t + b_t) - \alpha(p_t, a_t) \right]. \tag{26}\]
But, from \( (a, b) \in \mathcal{U}(A \times B) = \mathcal{U} \times \mathcal{U} \) and (22) it follows \( \mathcal{U} A \cap \mathcal{U} B \in \bar{V} \), and hence also \( a, a + b \in \bar{V} \). From the definition of thickening we also have that \( a, a + b \in \mathcal{U} \). We can thus use
(21) at the points \(a, b \in \mathcal{V} = \hat{\mathcal{V}} \cap \mathcal{U}\), so that we can write (26) as

\[
\forall(a, b) \in \mathcal{V}(A \times B): \quad b \cdot r(a, b) = f(a + b) - f(a) .
\]

(27)

We have proved that for every \((x, h) \in \hat{\mathcal{V}}\) there exist an open neighborhood \(\mathcal{V}(A \times B)\) of \((x, h)\) and a smooth function \(r \in \mathcal{C}^\infty(\mathcal{V}(A \times B), \mathbb{R})\) such that (27) holds.

If \(\rho \in \mathcal{C}^\infty(\mathcal{V}(C \times D), \mathbb{R})\) is another such functions, then

\[
\forall(x, h) \in \mathcal{V}(C \times D) \cap \mathcal{V}(A \times B): \quad h \cdot [r(x, h) - \rho(x, h)] = 0,
\]

so that for every \(x \in C \cap A\) we have that

\[
\forall h \in \mathcal{V}(D \times B): \quad h \cdot [r(x, h) - \rho(x, h)] = 0.
\]

For Lemma 25 applied with \(g(h) := h\) and \(f(h) := r(x, h) - \rho(x, h)\), we have \(r(x, h) = \rho(x, h)\) for every \((x, h) \in \mathcal{V}(C \times D) \cap \mathcal{V}(A \times B)\), which proves the conclusion for the sheaf property of \(\mathcal{R}\) (see 6 and 4 of Theorem 18). Finally, let us note that from (23) for \(b = 0\) we obtain

\[
r(a, 0) = \partial_2 \alpha(p, a),
\]

which is the last part of the statement.

Using this theorem, we can develop all the differential calculus for quasi-standard smooth functions of type \(f : \mathcal{U} \rightarrow \mathbb{R}^d\). In the present work, we will only study some first properties of derivatives and the Fermat–Reyes method. A more complete development of differential and integral calculus will be presented in future articles.

**Definition 27.** Let \(U\) be an open subset of \(\mathbb{R}\), and \(f : \mathcal{U} \rightarrow \mathbb{R}\) a smooth function. Then the smooth incremental ratio \(f'[-]\) of \(f\) is defined by the following properties:

1. \(f'[-] : \hat{\mathcal{U}} \rightarrow \mathbb{R}\).
2. \(f(x + h) = f(x) + h \cdot f'[x, h], \forall(x, h) \in \hat{\mathcal{U}}\).

Moreover, we will also set \(f'(x) := f'[x, 0]\) for every \(x \in \mathcal{U}\).

Let us note that the notation for the smooth incremental ratio as a function uses square brackets, like in \(f'[-]\). For this reason, there is no way to confuse the smooth incremental ratio \(f'[-]\) and its values \(f'[x, h]\) with the corresponding derivative \(f'\) and its values \(f'(x)\).

First of all, from property 1 in the previous definition, it follows that

\[
f' : \mathcal{U} \rightarrow \mathbb{R}.
\]

The following theorem contains the first expected properties of the derivative.

**Theorem 28.** Let \(U\) be an open subset of \(\mathbb{R}\), and \(f, g : \mathcal{U} \rightarrow \mathbb{R}\) be smooth functions. Finally, let us consider a Fermat real \(r \in \mathbb{R}\). Then:

1. \((f + g)' = f' + g'\).
2. \((r \cdot f)' = r \cdot f'\).
3. \((f \cdot g)' = f' \cdot g + f \cdot g'\).
4. $(1\mathbb{R})' = 1$.
5. $r' = 0$.

**Proof.** We report the proof essentially as a first example to show how to use precisely the Fermat–Reyes method in our context.

The first step is to prove, e.g., that $f + g$ is smooth in $\mathbb{C}^\infty$. Looking at the diagram

\[
\begin{array}{cc}
\mathbb{R} & \mathbb{R}^2 \\
\downarrow f & \downarrow + \\
\mathbb{U} & \mathbb{R} \\
\downarrow g & \downarrow p_1 \\
\mathbb{R} & \mathbb{R} \\
\end{array}
\]

where $+: (r, s) \in \mathbb{R}^2 \mapsto r + s \in \mathbb{R}$ is the sum of Fermat reals, we can see that $f + g = \langle f, g \rangle \cdot +$ and hence it is smooth because it can be expressed as a composition of smooth functions. The proof that the sum $f + g$ is smooth, even if it is almost trivial, can show us why it is very important to work in a cartesian closed category like $\mathbb{C}^\infty$. We have, indeed, the possibility to consider very general set theoretical operations like compositions, evaluations and pairing like $\langle f, g \rangle$ as arrows of $\mathbb{C}^\infty$, i.e. as smooth functions.

Now, we have only to calculate $(f + g)(x + h)$ using the definition of smooth incremental ratio and its uniqueness

\[
(f + g)(x + h) = f(x + h) + g(x + h) = f(x) + h \cdot f'(x, h) + g(x) + h \cdot g'(x, h) = (f + g)(x) + h \cdot \left\{ f'(x, h) + g'(x, h) \right\}, \quad \forall (x, h) \in \tilde{\mathbb{U}}.
\]

From the uniqueness of the smooth incremental ratio of $f + g$ we obtain $(f + g)'[-] = f'[-] + g'[\cdot -]$ and thus the conclusion evaluating these ratios at $h = 0$.

As a further simple example, we consider only the derivative of the product. The smoothness of $f \cdot g$ can be proved analogously to what we have just done for the sum. Now, let us evaluate for every $(x, h) \in \tilde{\mathbb{U}}$

\[
(f \cdot g)(x + h) = f(x + h) \cdot g(x + h) = \left\{ f(x) + h \cdot f'(x, h) \right\} \cdot \left\{ g(x) + h \cdot g'(x, h) \right\} = (f \cdot g)(x) + h \cdot \left\{ f(x) \cdot g'(x, h) + g(x) \cdot f'(x, h) + h^2 \cdot f'(x, h) \cdot g'(x, h) \right\}.
\]

From the uniqueness of the smooth incremental ratio of $f \cdot g$ we have thus

\[
(f \cdot g)'[x, h] = f(x) \cdot g'[x, h] + g(x) \cdot f'[x, h] + h^2 \cdot f'[x, h] \cdot g'[x, h],
\]

which gives the conclusion setting $h = 0$. The other properties can be proved analogously. \qed
The next expected property that permits a deeper understanding of the Fermat–Reyes method is the chain rule.

**Theorem 29.** If \( U \) and \( V \) are open subsets of \( \mathbb{R} \) and
\[
  f : \mathring{U} \rightarrow \mathring{\mathbb{R}},
\]
\[
  g : \mathring{V} \rightarrow \mathring{U}
\]
are smooth functions, then
\[
  (f \circ g)' = (f' \circ g) \cdot g'.
\]

Giving a proof of this theorem, we will explain in a general way the Fermat–Reyes method. We first need the following.

**Lemma 30.** Let \( U \) be an open subset of \( \mathbb{R}^k \), \( x \in \mathring{U} \) and \( v \in \mathring{\mathbb{R}}^k \). Then there exists \( r \in \mathbb{R} > 0 \) such that
\[
  \forall h \in (-r, r) : (x, h) \in \mathring{\tilde{U}}_v.
\]

**Proof.** If \( \circ v = 0 \), then for every \( s \in [0, 1] \) and every \( h \in \mathring{\mathbb{R}} \) we have \( \circ(x + shv) = \circ x \in U \), hence \( x + shv \in \mathring{U} \), that is \( [x, x + hv] \subseteq \mathring{U} \). In this case we have thus \( (x, h) \in \mathring{\tilde{U}}_v \) for every \( h \in \mathring{\mathbb{R}} \).

Otherwise, if \( \circ v \neq 0 \) then from \( \circ x \in U \) we obtain
\[
  \exists \rho > 0 : B_{\rho}(\circ x) \subseteq U,
\]
because \( U \) is open in \( \mathbb{R}^k \). Take as \( r \in \mathbb{R} > 0 \) any real number verifying
\[
  0 < r < \min\left( \rho, \frac{\rho}{\| \circ v \|} \right).
\]
For such an \( r \), if \( s \in [0, 1] \) and \( h \in (-r, r) \), then
\[
  \circ(x + shv) = \circ x + \circ s \cdot \circ h \cdot \circ v \in B_{\rho}(\circ x) \iff \| \circ s \cdot \circ h \cdot \circ v \| < \rho
\]
\[
  \leq \| \circ h \| \cdot \| \circ v \| \leq r < \rho
\]
the last implication is due to the assumption that \( s \in [0, 1] \). But (28) holds because \( |h| < r \) and hence \( \| \circ h \| = |\circ h| < r \) and \( r \cdot \| \circ v \| < \rho \) from the definition of \( r \).

The next result works for the Fermat–Reyes methods like a sort of “compactness principle” analogous to the compactness theorem of mathematical logic. It is the generalization to more than just one open set \( U \) of the previous lemma.
Theorem 31 (Compactness principle). For \( i = 1, \ldots, n \), let \( U^i \) be open sets of \( \mathbb{R}^{k_i} \), \( v \in \mathbb{R}^{k_i} \), \( x_i \in U^i \) and finally \( a_i \in \mathbb{R} \). Then there exists
\[
 r \in \mathbb{R}_{>0}
\]
such that
\[
 \forall i = 1, \ldots, n, \forall h \in (-r, r): (x_i, h \cdot a_i) \in \widetilde{\cdot U^i_v}.
\]

Proof. For every \( x_i \in U^i \) we apply the previous Lemma 30 obtaining the existence of \( r_i \in \mathbb{R}_{>0} \) such that
\[
 \forall k \in (-r_i, r_i): (x_i, k) \in \widetilde{\cdot U^i_v}.
\]
(29)

Now, let us set
\[
 r := \min_{i: {^{{\circ a}_i}} \neq 0} \frac{r_i}{{^{{\circ a}_i}}} \in \mathbb{R}_{>0},
\]
then taking a generic \( h \in (-r, r) \) we have
\[
 -r < {^{{\circ h}}} < r.
\]
(30)

If \( {^{{\circ a}_i}} = 0 \), then trivially \(-r_i < {^{{\circ h}}} \cdot {^{{\circ a}_i}} < r_i \) and hence \(-r_i < h \cdot a_i < r_i \), so that from (29) we get the conclusion for this first case, i.e. \((x_i, h a_i) \in \widetilde{\cdot U^i_v}\).

Otherwise, if \( {^{{\circ a}_i}} \neq 0 \), then \( r \leq \frac{r_i}{{^{{\circ a}_i}}} \) and from (30) we get \( {^{{\circ h}}} < r \leq \frac{r_i}{{^{{\circ a}_i}}} \) and hence \(-r_i < h a_i < r \), and once again the conclusion follows from (29). \( \square \)

We can use this theorem in the following way:

1. Every time in a proof we need a property of the form
\[
 (x_i, h a_i) \in \widetilde{\cdot U^i}
\]
(31)
we will assume “to have chosen \( h \) so little that (31) is verified”.

2. We derive the conclusion \( A(h) \) under \( n \) of such hypothesis, so that we have concretely deduced that
\[
 (\forall i = 1, \ldots, n: (x_i, h a_i) \in \widetilde{\cdot U^i}) \implies A(h).
\]

3. At this point we can apply the compactness principle obtaining
\[
 \exists r \in \mathbb{R}_{>0}, \forall h \in (-r, r): A(h).
\]
4. Usually the property $A(h)$ is of the form

$$A(h) \iff h \cdot \tau(h) = h \cdot \sigma(h),$$

and hence we can deduce $\tau(h) = \sigma(h)$ for every $h \in (-r, r)$ from the cancellation law of non-infinitesimal functions, and in particular $\tau(0) = \sigma(0)$. If the property $A$ has the form (32), then we can also suppose that $h$ is invertible because the cancellation law can be applied also with this additional hypotheses. But at the end, we will anyway set $h = 0$, in perfect agreement with the classical description of the Fermat method (see e.g. Bottazzini et al. [10], Bell [5], Edwards [18]).

Let us note that, as mentioned above, conceptually this way to proceed reflects the same idea of the compactness theorem of mathematical logic, because in every proof we can only have a finite number of hypothesis of type (31). Even if this method does not involve explicitly infinitesimal methods, using it the final proofs are very similar to those we would have if $h$ were an actual infinitesimal, i.e. $h \in D_\infty$.

In the following proof we will concretely use this method.

**Proof of Theorem 29.** First of all the composition

$$(-) \circ (-): \bullet U^* V \times \bullet \mathbb{R}^* U \to \bullet \mathbb{R}^* V$$

is a smooth map of $\bullet \mathcal{C}^\infty$, and hence $f \circ g$ is smooth because it can be written as a composition of smooth maps.

For a generic

$$(x, h) \in \tilde{\mathcal{V}}$$

we can always write

$$(f \circ g)(x + h) = f[g(x + h)] = f[g(x) + h \cdot g'[x, h]]$$

because $x + h \in \bullet V$ and hence $f \circ g$ is defined at $x + h$. Now, we would like to use the smooth incremental ratio of $f$ at the point $g(x)$ with increment $h \cdot g'[x, h]$. For this end we assume

$$(g(x), h \cdot g'[x, h]) \in \tilde{\mathcal{U}}$$

so that we can write

$$(f \circ g)(x + h) = f[g(x)] + h \cdot g'[x, h] \cdot f'[g(x), h \cdot g'[x, h]]$$

Using the compactness principle and the cancellation law of non-infinitesimal functions we get

$$\exists r \in \mathbb{R}_{>0}: \forall h \in (-r, r): \quad g'[x, h] \cdot f'[g(x, h \cdot g'[x, h])] = (f \circ g)'[x, h],$$

and thus the conclusion for $h = 0$. □
Let us note that these ideas, that do not use infinitesimal methods, can be repeated in a standard context, with only slight modifications, so that they represent an interesting alternative way to teach a significant part of the calculus with strongly simpler proofs.

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References