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Decomposition of sparse graphs, with application to game coloring number

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ABSTRACT

Let k be a nonnegative integer, and let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$. We prove that every simple graph with maximum average degree less than m_k decomposes into a forest and a subgraph with maximum degree at most k (furthermore, when $k \le 3$ both subgraphs can be required to be forests). It follows that every simple graph with maximum average degree less than m_k has game coloring number at most $4 + k$.

1. Introduction

The *game coloring number* of a graph G is defined using a two-person game to produce an ordering of the vertices of G. In the *ordering game* on G, Alice and Bob take turns choosing vertices from the set of unchosen vertices of G. This places the vertices in a linear order L, with $x < y$ if x is chosen before y. The *back degree* of a vertex x with respect to L, written $b_L(x)$, is the number of neighbors of x that precede x in L. The *back degree* of L, written $b(L)$, is $\max_{x\in V(G)} b_L(x)$. Alice's goal is to minimize $b(L)$, and Bob's goal is to maximize it.

The *game coloring number* $\mathrm{col}_g(G)$ of G is defined to be $1 + k$, where k is the least integer such that Alice can guarantee $b(L) \le k$. Equivalently, k is the greatest integer such that Bob can guarantee $b(L) \ge k$. The game coloring number was first formally defined in [9] as a tool for proving upper bounds on the game chromatic number [2]. It is the game version of the *coloring number*, which is defined to be $1 + \min_L b(L)$ and received its somewhat unfortunate name because it is an upper bound on the chromatic number. A more accurate and less confusable term might be "(game) coloring bound", but we will use the traditional term and notation. The definition of back degree makes multi-edges and loops irrelevant in the game, so we use the model of "graph" that forbids these.

Recently, Zhu [10] proved that $\mathrm{col}_g(G) \le 17$ when G is planar. Borodin et al. [3], He et al. [6], and Kleitman [7] improved this for planar graphs with large girth by proving structural properties of planar graphs with large girth. A *decomposition* of a graph G is a set of edge-disjoint subgraphs whose union is G.

Theorem 1. *Let G be a planar graph with girth at least g.*

1. [3] *If $g \ge 9$, then G decomposes into a forest and a matching.*

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2. [7] *If* $g \geq 6$, *then G decomposes into a forest and a graph with maximum degree* 2.
3. [6] *If* $g \geq 5$, *then G decomposes into a forest and a graph with maximum degree* 4.

Nash-Williams [8] proved that every planar graph decomposes into three forests. Balogh et al. [1] conjectured that one of the three forests can be required to have maximum degree at most 4, which is sharp infinitely often. They proved several results in this direction, and Gonçalves [5] proved the full conjecture. In addition, he showed that planar graphs with girth at least 6 (at least 7) decompose into two forests with one having maximum degree at most 4 (at most 2).

Two lemmas show the importance, for game coloring number, of decomposing a graph into a forest and a graph with small maximum degree.

Lemma 1 (*Zhu [9]*). *If a graph G decomposes into subgraphs* G_1 *and* G_2, *then* $\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$.

Lemma 2 (*Faigle et al. [4]*). *If T is a forest, then* $col_g(T) \leq 4$.

Combining these two lemmas with Theorem 1 yields:

Corollary 1 (*[3,6,7]*). *If G is a planar graph with girth at least* 5, *then* $\mathrm{col}_g(G) \leq 8$. *The upper bound decreases to* 6 *for girth at least* 6 *and to* 5 *for girth at least* 9.

In this note, we bound the game coloring number of sparse graphs using this decomposition approach. We measure sparseness by avoidance of dense subgraphs. The *maximum average degree* of a graph *G*, written $\mathrm{Mad}(G)$, is the largest average degree among the subgraphs of *G*. That is,

$$\mathrm{Mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}.$$

We can now state our main result.

Theorem 2. *Every graph G satisfying* $\mathrm{Mad}(G) < \frac{4(k+1)(k+3)}{k^2+6k+6}$ *decomposes into a forest and a subgraph with maximum degree at most k. When* $k \leq 3$, *both subgraphs can be required to be forests.*

Theorem 2 combines with Lemmas 1 and 2 to yield:

Corollary 2. *If a graph G satisfies* $\mathrm{Mad}(G) < \frac{4(k+1)(k+3)}{k^2+6k+6}$, *then* $\mathrm{col}_g(G) \leq 4 + k$.

Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$. The value m_k is the largest bound our approach can prove. However, we do not know whether the result is sharp. Let $f(k)$ be the infimum of $\mathrm{Mad}(H)$ over graphs *H* that do not decompose into a forest and a graph with maximum degree at most *k*. For an upper bound, the complete bipartite graph $K_{2,2k+2}$ has average degree $\frac{4k+4}{k+2}$ but has no such decomposition. The graph obtained from any $(2k+2)$-regular multigraph by subdividing each edge is another such example, with the *same* average degree. Thus

$$4 - \frac{8k+12}{k^2+6k+6} \leq f(k) \leq 4 - \frac{4}{k+2}.$$

Although Theorem 2 holds for both planar and nonplanar graphs, it does not imply Theorem 1 by using the usual inequality $\mathrm{Mad}(G) \leq 2g/(g-2)$ that holds for every planar graph *G* having girth at least *g*. Those results would follow from $f(k) = \frac{4k+4}{k+2}$, and to imply the result for girth 5 no smaller $f(k)$ suffices. For girth at least 7, the weaker threshold $f(2) \geq 14/5$ would imply the result of Gonçalves [5].

Answering the following question would solve the problem completely.

Question 1. *For every k, what are the graphs with smallest maximum average degree that do not decompose into a forest and a subgraph with maximum degree at most k?*

The proof of Theorem 2 uses reducible configurations (discussed in Section 2) and a discharging procedure (discussed in Section 3). The key structures in the proof are "banks" and "cores" that allow the transfer of charge over unlimited distances.

2. Reducible configurations and special subgraphs

Let $d(x)$ denote the degree of a vertex *x* in a graph *G*. A *k-vertex* is a vertex of degree *k*. A $^{\geq}k$*-vertex* or $^{\leq}k$*-vertex* is a vertex of degree at least *k* or at most *k*, respectively. An (a, b)*-alternating cycle* is an even cycle that alternates between *a*-vertices and *b*-vertices.

We prove Theorem 2 by considering a counterexample such that $|V(G)| + |E(G)|$ is smallest. Since $\mathrm{Mad}(H) \leq \mathrm{Mad}(G)$ when *H* is a subgraph of *G*, every proper subgraph of *G* decomposes into a forest and a graph with maximum degree at most *k*, but *G* has no such decomposition. We use this to exclude various configurations from *G*. Since $m_0 = 2$, and $\mathrm{Mad}(G) < 2$ implies that *G* is a forest, we may assume that $k \geq 1$.

Lemma 3. *A minimal counterexample G to Theorem 2 has* (a) *no* 1*-vertex,* (b) *no edge uv with $d(v) \le k+1$ and $d(u) = 2$ (or $d(u) = 3$ when $k > 3$), and* (c) *no $(k+2, 2)$-alternating cycle.*

Proof. When G contains any such configuration, we decompose an appropriate subgraph of G into a forest F and a subgraph D with maximum degree at most k (when $k \le 3$, also D is a forest) and use it to obtain such a decomposition of G, contradicting that G is a counterexample.

(a) When $d(u) = 1$, the decomposition of $G - u$ extends by adding the extra edge to F.

(b) If $k > 3$, then D need not be a forest; we prove the stronger statement that G has no adjacent $^{\le}(k+1)$-vertices. If u and v are such, then consider the decomposition of $G - uv$. If u or v has k neighbors in D, then add uv to F; otherwise, add uv to D.
If $k \le 3$, then D in the decomposition of $G - uv$ is a forest, and we consider only $d(u) = 2$. If v has k neighbors in D, then add uv to F; otherwise, add uv to whichever of D and F does not contain the edge incident to u in $G - uv$.

(c) Let C be a $(k+2, 2)$-alternating cycle in G. In the decomposition of $G - E(C)$ into F and D, we enforce that each $(k+2)$-vertex on C has an incident edge in F, by moving an incident edge from D to F if not. Now adding one perfect matching in C to D and the other to F extends the decomposition to G without creating cycles in either subgraph. □

We use discharging to show that every graph satisfying (a), (b), and (c) of Lemma 3 has average degree at least m_k, and hence there is no counterexample to Theorem 2. To apply the discharging method, we first give each vertex a "charge" equal to its degree. We then redistribute the charge (without changing the total charge) to obtain charge at least m_k on each vertex. To facilitate the discharging argument, we also move some charge to special subgraphs. They start with charge 0 and will end with nonnegative charge, so the initial average degree is at least m_k.

Given such a graph G, let X be the set of all $(k+2)$-vertices in G that are adjacent to at least $k+1$ vertices of degree 2, and let Y be the set of all 2-vertices adjacent to at least one vertex of X. Define the *bank* of G to be the maximal bipartite subgraph of G with partite sets X and Y. When $k \ge 4$, we modify this slightly by restricting X to use only the $(k+2)$-vertices whose neighbors all have degree 2.

A cycle in the bank would be a $(k+2, 2)$-alternating cycle in G, which is forbidden. Hence the bank is a forest. We call each component of the bank a *core*. By construction, each vertex of X has at least $k+1$ neighbors in the bank ($k+2$ when $k \ge 4$); hence each leaf in the bank belongs to Y.

3. The discharging argument

The initial charge at each vertex of G is its degree, and also each core has initial charge 0. We use three discharging rules (plus a special rule when $k \ge 4$) to redistribute charges. In most discharging arguments, movement of charge is local. Assigning charge to cores permits charge to move long distances.

For the computations, recall that $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$. Each discharging rule Ri moves a constant amount r_i of charge. These constants r_1, r_2, r_3, r_4 are defined in terms of m_k by

$$r_1 = \frac{m_k - 2}{2}, \qquad r_2 = 1 - r_1 - \frac{m_k}{k+3}, \qquad r_3 = m_k - (k+2)(1 - r_1), \qquad r_4 = \frac{m_k - 3}{3}.$$

The discharging rules are as follows, with R4 used only when $k \ge 4$. We add R4 because $m_k > 3$ if and only if $k \ge 4$, so when $k \ge 4$ the 3-vertices need to gain charge. A vertex belonging to no core is *adjacent to a core C* if it is adjacent in G to a leaf of C.

R1 Every $^{\ge}(k+2)$-vertex gives r_1 to each neighbor that is a 2-vertex.
R2 If C is a core, v is a $^{\ge}(k+2)$-vertex belonging to no core, and v is adjacent to l leaves of C, then v gives lr_2 to C.
R3 Every core gives r_3 to each of its $(k+2)$-vertices whose neighbors all have degree 2.
R4 (*For $k \ge 4$ only.*) Every $^{\ge}(k+2)$-vertex gives r_4 to each neighboring 3-vertex.

The proof of Theorem 2 is now completed by proving the following lemma.

Lemma 4. *If a graph G satisfies* (a), (b), *and* (c) *of Lemma 3, then* $\mathrm{Mad}(G) \ge m_k$.

Proof. As described above, we give initial charge $d(v)$ to each vertex v and initial charge 0 to each core C. After applying the discharging rules, let $\omega(v)$ and $\omega(C)$ denote the final charges. We prove that $\omega(v) \ge m_k$ for each vertex v and $\omega(C) \ge 0$ for each core C.

By (b), the neighbors of 2-vertices are $^{\ge}(k+2)$-vertices. Using R1, the final charge of each 2-vertex is $2 + 2r_1$, which equals m_k, as desired.

If $3 \le d(v) \le k+1$, then v does not give or receive charge, unless $d(v) = 3 < k$. Thus $\omega(v) = d(v) > m_k$ except in that case. If $d(v) = 3 < k$, then by (b) its neighbors are all $^{\ge}(k+2)$-vertices. Via R4 it receives $3r_4$, and hence $\omega(v) = 3 + 3r_4 = m_k$.

Now suppose that $d(v) \ge k+2$. Vertex v may lose charge to each neighbor, and v may lose additional charge when v is not in a core and its neighbors are. Since always $r_1 > r_4$, we may assume that each neighbor getting charge from v is a

2-vertex. Hence the maximum charge lost from v, via {R1, R2, R4}, is $d(v)(r_1+r_2)$. Hence $\omega(v) \geq d(v)(1-r_1-r_2) = \frac{d(v)m_k}{k+3}$. If $d(v) \geq k+3$, then $\omega(v) \geq m_k$.

The case $d(v) = k+2$ is more delicate. If v is not in a core, then the definition of the bank limits the number of 2-neighbors of v (to k if $k \leq 3$, to $k+1$ if $k \geq 4$). If $k \leq 3$, then $\omega(v) \geq 2 + k(1-r_1-r_2) = m_k + 2 - \frac{3m_k}{k+3}$. The formula for m_k yields $2 - \frac{3m_k}{k+3} = \frac{2k^2}{k^2+6k+6} > 0$, and hence $\omega(v) > m_k$. If $k \geq 4$, then v may have one 3-neighbor in addition to the maximum number of 2-neighbors. Hence $\omega(v) \geq 1 + (k+1)(1-r_1-r_2) - r_4 = m_k + 2 - \frac{2m_k}{k+3} - \frac{m_k}{3}$. The formula for m_k converts the last expression to $m_k + \frac{2k(k-2)}{3(k^2+6k+6)}$, and hence $\omega(v) > m_k$.

Suppose now that $d(v) = k+2$ and v is in a core. If every neighbor of v has degree 2, then v loses r_1 exactly $k+2$ times, but it loses nothing by R2 and gains r_3 by R3. Hence $\omega(v) = (k+2)(1-r_1)+r_3 = m_k$. If v has a neighbor with degree more than 2, then $k \leq 3$. Now v loses r_1 exactly $k+1$ times by R1 and is unaffected by {R2, R3, R4}. Hence $\omega(v) = (k+2) - (k+1)r_1$. Using $r_1 = m_k/2 - 1$ and the formula for m_k, we compute $(k+2) - (k+1)r_1 - m_k = \frac{k^2}{k^2+6k+6} > 0$, and hence $\omega(v) > m_k$.

Finally, we check that $\omega(C) \geq 0$ when C is a core. We have observed (using (c)) that C is a tree whose leaves are 2-vertices in G. The non-leaves in X have degree $k+1$ or $k+2$ in C, while the non-leaves in Y have degree 2 in C. Let there be n_1 non-leaves of the first type, n_2 of the second, and n' of the third, and let n_0 be the number of leaves. Since C is a tree, its vertex degrees must sum to $2(n_0+n_1+n_2+n')-2$, so we obtain $n_0 = (k-1)n_1 + kn_2 + 2$. By (b), the neighbor outside C of a leaf of C is a $^{\geq}(k+2)$-vertex. Also, those vertices are not in cores, so C receives n_0r_2 via R2. Via R3, C distributes n_2r_3. Since $n_0 > kn_2$, it suffices to have $kr_2 \geq r_3$. Using the definitions of r_3, r_2, and then r_1 in terms of m_k, we compute

$$kr_2 - r_3 = (2k+2)(1-r_1) - \frac{2k+3}{k+3}m_k = 4(k+1) - \frac{k^2+6k+6}{k+3}m_k = 0.$$

We have shown that all vertices and cores have sufficient final charge. □

Note that r_1 is defined in terms of m_k so that 2-vertices have final charge m_k, and then r_2 is defined in terms of m_k and r_1 to give $(k+3)$-vertices final charge m_k. Next r_3 is defined in terms of these so that $(k+2)$-vertices in cores whose neighbors all have degree 2 have final charge m_k, and r_4 is defined so that 3-vertices have final charge m_k when $k \geq 4$. Given all this, and the fact that n_1 may equal 0 in a core, m_k has been chosen as the largest value allowing us to guarantee nonnegative final charge for cores. In this sense the theorem cannot be improved using the present argument.

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