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Compactness properties of operator multipliers

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Abstract

We continue the study of multidimensional operator multipliers initiated in [K. Juschenko, I.G. Todorov, L. Turowska, Multidimensional operator multipliers, Trans. Amer. Math. Soc., in press]. We introduce the notion of the symbol of an operator multiplier. We characterise completely compact operator multipliers in terms of their symbol as well as in terms of approximation by finite rank multipliers. We give sufficient conditions for the sets of compact and completely compact multipliers to coincide and characterise the cases where an operator multiplier in the minimal tensor product of two C^*-algebras is automatically compact. We give a description of multilinear modular completely compact completely bounded maps defined on the direct product of finitely many copies of the C^*-algebra of compact operators in terms of tensor products, generalising results of Saar [H. Saar, Kompakte, vollständig beschränkte Abbildungen mit Werten in einer nuklearen C^*-Algebra, Diplomarbeit, Universität des Saarlandes, Saarbrücken, 1982].
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1. Introduction

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i, j)a_{ij})$ is the matrix of a bounded linear operator on ℓ_2 whenever (a_{ij}) is such. The study of Schur multipliers was initiated by Schur in the early 20th century and since then has attracted considerable attention, much of

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which was inspired by A. Grothendieck's characterisation of these objects in his *Résumé* [9]. Grothendieck showed that a function φ is a Schur multiplier precisely when it has the form $\varphi(i,j)=\sum_{k=1}^{\infty} a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ satisfy the conditions $\sup_i \sum_{k=1}^{\infty} |a_k(i)|^2 < \infty$ and $\sup_j \sum_{k=1}^{\infty} |b_k(j)|^2 < \infty$. In modern terminology, this characterisation can be expressed by saying that φ is a Schur multiplier precisely when it belongs to the extended Haagerup tensor product $\ell_\infty \otimes_{\mathrm{eh}} \ell_\infty$ of two copies of ℓ_∞.

Special classes of Schur multipliers, e.g. Toeplitz and Hankel Schur multipliers, have played an important role in analysis and have been studied extensively (see [19]). Compact Schur multipliers, that is, the functions φ for which the mapping $(a_{ij}) \to (\varphi(i,j)a_{ij})$ on $\mathcal{B}(\ell_2)$ is compact, were characterised by Hladnik [11], who identified them with the elements of the Haagerup tensor product $c_0 \otimes_{\mathrm{h}} c_0$.

A non-commutative version of Schur multipliers was introduced by Kissin and Shulman [14] as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let π and ρ be representations of $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces H and K, respectively. Identifying $H \otimes K$ with the Hilbert space $\mathcal{C}_2(H^{\mathrm{d}}, K)$ of all Hilbert–Schmidt operators from the dual space H^{d} of H into K, we obtain a representation $\sigma_{\pi,\rho}$ of the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$ acting on $\mathcal{C}_2(H^{\mathrm{d}}, K)$. An element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is called a π,ρ-multiplier if $\sigma_{\pi,\rho}(\varphi)$ is bounded in the operator norm of $\mathcal{C}_2(H^{\mathrm{d}}, K)$. If φ is a π,ρ-multiplier for any pair of representations (π,ρ) then φ is called a universal (operator) multiplier.

Multidimensional Schur multipliers and their non-commutative counterparts were introduced and studied in [12], where the authors gave, in particular, a characterisation of universal multipliers as certain weak limits of elements of the algebraic tensor product of the corresponding C^*-algebras, generalising the corresponding results of Grothendieck and Peller [9,18] as previously conjectured by Kissin and Shulman in [14]. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras. Like Schur multipliers, elements of the set $M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ of (multidimensional) universal multipliers give rise to completely bounded (multilinear) maps. Requiring these maps to be compact or completely compact, we define the sets of compact and completely compact operator multipliers denoted by $M_c(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and $M_{cc}(\mathcal{A}_1, \dots, \mathcal{A}_n)$, respectively. The notion of complete compactness we use is an operator space version of compactness which was introduced by Saar [21] and subsequently studied by Oikhberg [15] and Webster [27]. Our results on operator multipliers rely on the main result of Section 3 where we prove a representation theorem for completely compact completely bounded multilinear maps. In [3] Christensen and Sinclair established a representation result for completely bounded multilinear maps which implies that every such map $\Phi : \mathcal{K}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_n, H_{n-1}) \to \mathcal{K}(H_n, H_1)$ (where, for Hilbert spaces H' and H'', we denote by $\mathcal{K}(H', H'')$ the space of all compact operators from H' into H'') has the form

$$\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes I)A_2 \dots (x_{n-1} \otimes I)A_n, \tag{1}$$

for some index set J and bounded block operator matrices $A_1 \in M_{1,J}(\mathcal{B}(H_1))$, $A_2 \in M_J(\mathcal{B}(H_2)), \dots, A_n \in M_{J,1}(\mathcal{B}(H_n))$. In other words, Φ arises from an element

$$u = A_1 \odot \cdots \odot A_n \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$$

of the extended Haagerup tensor product of $\mathcal{B}(H_1), \dots, \mathcal{B}(H_n)$. Moreover, if Φ is $\mathcal{A}_1', \dots, \mathcal{A}_n'$-modular for some von Neumann algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$, then the entries of A_i can be chosen from $\mathcal{A}_i$. We show in Section 3 that a map Φ as above is completely compact precisely when it has a representation of the form (1) where

$$u = A_1 \odot \cdots \odot A_n \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} \big(\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})\big) \otimes_{\mathrm{h}} \mathcal{K}(H_n).$$

This extends a result of Saar [21] in the two-dimensional case. If, additionally, $\mathcal{A}_1, \dots, \mathcal{A}_n$ are von Neumann algebras and Φ is $\mathcal{A}'_1, \dots, \mathcal{A}'_n$-modular then u can be chosen from $\mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)$, where $\mathcal{K}'(\mathcal{A})$ denotes the ideal of compact operators contained in $\mathcal{A}$ in its identity representation. As a consequence of this and a result of Effros and Kishimoto [4] we point out the completely isometric identifications

$$CC\big(\mathcal{K}(H_2, H_1)\big)^{**} \simeq \big(\mathcal{K}(H_1) \otimes_{\mathrm{h}} \mathcal{K}(H_2)\big)^{**} \simeq CB\big(\mathcal{B}(H_2, H_1)\big),$$

where $CC(\mathcal{X})$ and $CB(\mathcal{X})$ are the spaces of completely compact and completely bounded maps on an operator space $\mathcal{X}$, respectively.

In Section 4 we pinpoint the connection between universal operator multipliers and completely bounded maps. This technical result is used in Section 5 to define the symbol u_φ of an operator multiplier $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ which, in the case n is even (resp. odd) is an element of $\mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}^o_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}^o_1$ (resp. $\mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}^o_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}^o_2 \otimes_{\mathrm{eh}} \mathcal{A}_1$). Here $\mathcal{A}^o$ is the opposite C^*-algebra of a C^*-algebra $\mathcal{A}$. This notion extends a similar notion that was given in the case of completely bounded masa-bimodule maps by Katavolos and Paulsen in [13]. We give a symbolic calculus for universal multipliers which is used to establish a universal property of the symbol related to the representation theory of the C^*-algebras under consideration.

The symbol of a universal multiplier is used in Section 6 to single out the completely compact multipliers within the set of all operator multipliers. In fact, we show that $\varphi \in M_{cc}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ if and only if

$$u_\varphi \in \begin{cases} \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}^o_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}^o_3 \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}^o_1) & \text{if } n \text{ is even,} \\ \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}^o_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}^o_2) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1) & \text{if } n \text{ is odd,} \end{cases}$$

which is equivalent to the approximability of φ in the multiplier norm by operator multipliers of finite rank whose range consists of finite rank operators. It follows that a multidimensional Schur multiplier $\varphi \in \ell_\infty(\mathbb{N}^n)$ is compact if and only if $\varphi \in c_0 \otimes_{\mathrm{h}} (\ell_\infty \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \ell_\infty) \otimes_{\mathrm{h}} c_0$.

In Section 7 we use Saar's construction [21] of a completely bounded compact mapping which is not completely compact to show that the inclusion $M_{cc}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq M_c(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is proper if both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes. However, if both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ are isomorphic to a c_0-sum of matrix algebras of uniformly bounded sizes then the sets of compact and completely compact multipliers coincide. The case when only one of $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contains matrix algebras of arbitrary large size remains, however, unsettled. Finally, for $n = 2$, we characterise the cases where every universal multiplier is automatically compact: this happens precisely when one of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is finite dimensional and the other one coincides with its algebra of compact elements.

2. Preliminaries

We start by recalling standard notation and notions from operator space theory. We refer the reader to [1,6,16,20] for more details.

If H and K are Hilbert spaces we let $\mathcal{B}(H,K)$ (resp. $\mathcal{K}(H,K)$) denote the set of all bounded linear (resp. compact) operators from H into K. If I is a set we let H^I be the direct sum of $|I|$ copies of H and set $H^\infty = H^{\mathbb{N}}$. Writing $H \otimes K$ for the Hilbertian tensor product of two Hilbert spaces, we observe that $H^I = H \otimes \ell_2(I)$ as Hilbert spaces.

An operator space $\mathcal{E}$ is a closed subspace of $\mathcal{B}(H,K)$, for some Hilbert spaces H and K. The opposite operator space $\mathcal{E}^o$ associated with $\mathcal{E}$ is the space $\mathcal{E}^o = \{x^{\mathrm{d}}\colon x \in \mathcal{E}\} \subseteq \mathcal{B}(K^{\mathrm{d}}, H^{\mathrm{d}})$. Here, and in the sequel, $H^{\mathrm{d}} = \{\xi^{\mathrm{d}}\colon \xi \in H\}$ denotes the dual of the Hilbert space H, where $\xi^{\mathrm{d}}(\eta) = (\eta, \xi)$ for $\eta \in H$. Note that H^{d} is canonically conjugate-linearly isometric to H. We also adopt the notation $x^{\mathrm{d}} \in \mathcal{B}(K^{\mathrm{d}}, H^{\mathrm{d}})$ for the Banach space adjoint of $x \in \mathcal{B}(H,K)$, so that $x^{\mathrm{d}}\xi^{\mathrm{d}} = (x^*\xi)^{\mathrm{d}}$ for $\xi \in K$. As usual, $\mathcal{E}^*$ will denote the operator space dual of $\mathcal{E}$. If $n, m \in \mathbb{N}$, by $M_{n,m}(\mathcal{E})$ we denote the space of all n by m matrices with entries in $\mathcal{E}$ and let $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$. The space $M_{n,m}(\mathcal{E})$ carries a natural norm arising from the embedding $M_{n,m}(\mathcal{E}) \subseteq \mathcal{B}(H^m, K^n)$. Let I and J be arbitrary index sets. If v is a matrix with entries in $\mathcal{E}$ and indexed by $I \times J$, and $I_0 \subseteq I$ and $J_0 \subseteq J$ are finite sets, we let $v_{I_0,J_0} \in M_{I_0,J_0}(\mathcal{E})$ be the matrix obtained by restricting v to the indices from $I_0 \times J_0$. We define $M_{I,J}(\mathcal{E})$ to be the space of all such v for which

$$\|v\| \stackrel{\mathrm{def}}{=} \sup\bigl\{\|v_{I_0,J_0}\|\colon I_0 \subseteq I,\ J_0 \subseteq J \text{ finite}\bigr\} < \infty.$$

Then $M_{I,J}(\mathcal{E})$ is an operator space [6, §10.1]. Note that $M_{I,J}(\mathcal{B}(H,K))$ can be naturally identified with $\mathcal{B}(H^J, K^I)$ and every $v \in M_{I,J}(\mathcal{B}(H,K))$ is the weak limit of $\{v_{I_0,J_0}\}$ along the net $\{(I_0, J_0)\colon I_0 \subseteq I,\ J_0 \subseteq J \text{ finite}\}$. We set $M_I(\mathcal{E}) = M_{I,I}(\mathcal{E})$. For $A = (a_{ij}) \in M_I(\mathcal{E})$, we write $A^{\mathrm{d}} = (a_{ij}^{\mathrm{d}}) \in M_I(\mathcal{E}^o)$.

2.1. *Completely bounded maps and Haagerup tensor products*

If $\mathcal{E}$ and $\mathcal{F}$ are operator spaces, a linear map $\Phi : \mathcal{E} \to \mathcal{F}$ is called completely bounded if the maps $\Phi^{(k)} : M_k(\mathcal{E}) \to M_k(\mathcal{F})$ given by $\Phi^{(k)}((a_{ij})) = (\Phi(a_{ij}))$ are bounded for every $k \in \mathbb{N}$ and $\|\Phi\|_{\mathrm{cb}} \stackrel{\mathrm{def}}{=} \sup_k \|\Phi^{(k)}\| < \infty$.

Given linear spaces $\mathcal{E}_1, \ldots, \mathcal{E}_n$, we denote by $\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n$ their algebraic tensor product. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are operator spaces and $a^k = (a_{ij}^k) \in M_{m_k, m_{k+1}}(\mathcal{E}_k)$, $m_k \in \mathbb{N}$, $k = 1, \ldots, n$, we define the multiplicative product

$$a^1 \odot \cdots \odot a^n \in M_{m_1, m_{n+1}}(\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n)$$

by letting its (i,j)-entry $(a^1 \odot \cdots \odot a^n)_{ij}$ be $\sum_{i_2,\ldots,i_n} a^1_{i,i_2} \otimes a^2_{i_2,i_3} \otimes \cdots \otimes a^n_{i_n,j}$. If $\mathcal{E}$ is another operator space and $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ is a multilinear map we let

$$\Phi^{(m)} : M_m(\mathcal{E}_1) \times \cdots \times M_m(\mathcal{E}_n) \to M_m(\mathcal{E})$$

be the map given by

$$\bigl(\Phi^{(m)}\bigl(a^1, \ldots, a^n\bigr)\bigr)_{ij} = \sum_{i_2,\ldots,i_n} \Phi\bigl(a^1_{i,i_2}, a^2_{i_2,i_3}, \ldots, a^n_{i_n,j}\bigr),$$

where $a^k = (a^k_{s,t}) \in M_m(\mathcal{E}_k)$, $k = 1, \ldots, n$. The multilinear map Φ is called completely bounded if there exists a constant $C > 0$ such that, for all $m \in \mathbb{N}$,

$$\bigl\|\Phi^{(m)}\bigl(a^1, \ldots, a^n\bigr)\bigr\| \leqslant C\bigl\|a^1\bigr\| \ldots \bigl\|a^n\bigr\|, \quad a^k \in M_m(\mathcal{E}_k),\ k = 1, \ldots, n.$$

Set $\|\Phi\|_{\mathrm{cb}} \stackrel{\mathrm{def}}{=} \sup\{\|\Phi^{(m)}(a^1,\dots,a^n)\|:\ m\in\mathbb{N}, \|a^1\|,\dots,\|a^n\|\leqslant 1\}$. It is well known (see [6,17]) that a completely bounded multilinear map Φ gives rise to a completely bounded map on the Haagerup tensor product $\mathcal{E}_1\otimes_{\mathrm{h}}\cdots\otimes_{\mathrm{h}}\mathcal{E}_n$ (see [6] and [20] for its definition and basic properties).

The set of all completely bounded multilinear maps from $\mathcal{E}_1\times\cdots\times\mathcal{E}_n$ into $\mathcal{E}$ will be denoted by $CB(\mathcal{E}_1\times\cdots\times\mathcal{E}_n,\mathcal{E})$. If $\mathcal{E}_1,\dots,\mathcal{E}_n$ and $\mathcal{E}$ are dual operator spaces we say that a map $\Phi\in CB(\mathcal{E}_1\times\cdots\times\mathcal{E}_n,\mathcal{E})$ is normal [3] if it is weak* continuous in each variable. We write $CB^{\sigma}(\mathcal{E}_1\times\cdots\times\mathcal{E}_n,\mathcal{E})$ for the space of all normal maps in $CB(\mathcal{E}_1\times\cdots\times\mathcal{E}_n,\mathcal{E})$.

The *extended Haagerup tensor product* $\mathcal{E}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{E}_n$ is defined [5] as the space of all normal completely bounded maps $u:\mathcal{E}_1^*\times\cdots\times\mathcal{E}_n^*\to\mathbb{C}$. It was shown in [5] that if $u\in\mathcal{E}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{E}_n$ then there exist index sets $J_1,J_2,\dots,J_{n-1}$ and matrices $a^1=(a^1_{1,s})\in M_{1,J_1}(\mathcal{E}_1)$, $a^2=(a^2_{s,t})\in M_{J_1,J_2}(\mathcal{E}_2)$, …, $a^n=(a^n_{t,1})\in M_{J_{n-1},1}(\mathcal{E}_n)$ such that if $f_i\in\mathcal{E}_i^*$, $i=1,\dots,n$, then

$$\langle u, f_1\otimes\cdots\otimes f_n\rangle \stackrel{\mathrm{def}}{=} u(f_1,\dots,f_n)=\langle a^1,f_1\rangle\dots\langle a^n,f_n\rangle, \tag{2}$$

where $\langle a^k,f_k\rangle=(f_k(a^k_{s,t}))$ and the product of the (possibly infinite) matrices in (2) is defined to be the limit of the sums

$$\sum_{i_1\in F_1,\dots,i_{n-1}\in F_{n-1}} f_1\big(a^1_{1,i_1}\big)f_2\big(a^2_{i_1,i_2}\big)\dots f_n\big(a^n_{i_{n-1},1}\big)$$

along the net $\{(F_1\times\cdots\times F_{n-1}):\ F_j\subseteq J_j \text{ finite}, 1\leqslant j\leqslant n-1\}$. We may thus identify u with the matrix product $a^1\odot\cdots\odot a^n$; two elements $a^1\odot\cdots\odot a^n$ and $\tilde a^1\odot\cdots\odot\tilde a^n$ coincide if $\langle a^1,f_1\rangle\dots\langle a^n,f_n\rangle=\langle\tilde a^1,f_1\rangle\dots\langle\tilde a^n,f_n\rangle$ for all $f_i\in\mathcal{E}_i^*$. Moreover,

$$\|u\|_{\mathrm{eh}}=\inf\big\{\|a^1\|\dots\|a^n\|:\ u=a^1\odot\cdots\odot a^n\big\}.$$

The space $\mathcal{E}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{E}_n$ has a natural operator space structure [5]. If $\mathcal{E}_1,\dots,\mathcal{E}_n$ are dual operator spaces then by [5, Theorem 5.3] $\mathcal{E}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{E}_n$ coincides with the weak* Haagerup tensor product $\mathcal{E}_1\otimes_{\mathrm{w*h}}\cdots\otimes_{\mathrm{w*h}}\mathcal{E}_n$ of Blecher and Smith [2]. Given operator spaces $\mathcal{F}_i$ and completely bounded maps $g_i:\mathcal{E}_i\to\mathcal{F}_i$, $i=1,\dots,n$, Effros and Ruan [5] define a completely bounded map

$$g=g_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}} g_n:\mathcal{E}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{E}_n\to\mathcal{F}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{F}_n,$$
$$a^1\odot\cdots\odot a^n\mapsto\langle a^1,g_1\rangle\odot\cdots\odot\langle a^n,g_n\rangle$$

where $\langle a^k,g_k\rangle=(g_k(a^k_{ij}))$. Thus

$$\big\langle g(u), f_1\otimes\cdots\otimes f_n\big\rangle=\big\langle u,(f_1\circ g_1)\otimes\cdots\otimes(f_n\circ g_n)\big\rangle \tag{3}$$

for $u\in\mathcal{E}_1\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{E}_n$ and $f_i\in\mathcal{F}_i^*$, $i=1,\dots,n$.

The following fact is a straightforward consequence of a well-known theorem due to Christensen and Sinclair [3], and it will be used throughout the exposition.

Theorem 2.1. *Let H_i be a Hilbert space and $\mathcal{R}_i \subseteq \mathcal{B}(H_i)$ be a von Neumann algebra, $i = 1, \dots, n$. There exists an isometry γ from $\mathcal{R}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{R}_n$ onto the space of all $\mathcal{R}'_1, \dots, \mathcal{R}'_n$-modular maps in $CB^{\sigma}(\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}), \mathcal{B}(H_n, H_1))$, given as follows: if $u \in \mathcal{R}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{R}_n$ has a representation $u = A_1 \odot \cdots \odot A_n$ where $A_i \in M_J(\mathcal{R}_i) \subseteq \mathcal{B}(H_i \otimes \ell_2(J))$ for some index set J, then*

$$\gamma(u)(T_1, \dots, T_{n-1}) = A_1(T_1 \otimes I)A_2 \dots A_{n-1}(T_{n-1} \otimes I)A_n,$$

for all $T_i \in \mathcal{B}(H_{i+1}, H_i)$, $i = 1, \dots, n-1$, where I is the identity operator on $\ell_2(J)$.

We now turn to the definition of slice maps which will play an important role in our proofs. Given $\omega_1 \in \mathcal{B}(H_1)^*$ we set $L_{\omega_1} = \omega_1 \otimes_{\mathrm{eh}} \mathrm{id}_{\mathcal{B}(H_2)}$. After identifying $\mathbb{C} \otimes \mathcal{B}(H_2)$ with $\mathcal{B}(H_2)$ we obtain a mapping $L_{\omega_1} : \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \to \mathcal{B}(H_2)$ called a left slice map. Similarly, for $\omega_2 \in \mathcal{B}(H_2)^*$ we obtain a right slice map $R_{\omega_2} : \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \to \mathcal{B}(H_1)$. If $u = \sum_{i \in I} v_i \otimes w_i \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2)$ where $v = (v_i)_{i \in I} \in M_{1,I}(\mathcal{B}(H_1))$ and $w = (w_i)_{i \in I} \in M_{I,1}(\mathcal{B}(H_2))$, then

$$L_{\omega_1}(u) = \sum_{i \in I} \omega_1(v_i) w_i \quad \text{and} \quad R_{\omega_2}(u) = \sum_{i \in I} \omega_2(w_i) v_i.$$

Moreover,

$$\langle R_{\omega_2}(u), \omega_1 \rangle = \langle u, \omega_1 \otimes \omega_2 \rangle = \langle L_{\omega_1}(u), \omega_2 \rangle = \sum_{i \in I} \omega_1(v_i)\omega_2(w_i). \tag{4}$$

It was shown in [24] that if $\mathcal{E} \subseteq \mathcal{B}(H_1)$ and $\mathcal{F} \subseteq \mathcal{B}(H_2)$ are closed subspaces then, up to a complete isometry,

$$\begin{aligned} \mathcal{E} \otimes_{\mathrm{eh}} \mathcal{F} = \{&u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \colon\ L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \\ &\text{for all } \omega_1 \in \mathcal{B}(H_1)_* \text{ and } \omega_2 \in \mathcal{B}(H_2)_*\} \\ = \{&u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \colon\ L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \\ &\text{for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^*\}. \end{aligned} \tag{5}$$

Moreover [23],

$$\begin{aligned} \mathcal{E} \otimes_{\mathrm{h}} \mathcal{F} = \{&u \in \mathcal{B}(H_1) \otimes_{\mathrm{h}} \mathcal{B}(H_2) \colon\ L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \\ &\text{for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^*\}. \end{aligned} \tag{6}$$

Thus, $\mathcal{E} \otimes_{\mathrm{h}} \mathcal{F}$ can be canonically identified with a subspace of $\mathcal{B}(H_1) \otimes_{\mathrm{h}} \mathcal{B}(H_2)$ which, on the other hand, sits completely isometrically in $\mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2)$. These identifications are made in the statement of the following lemma which will be useful for us later.

Lemma 2.2. *If H_1, H_2, H_3 are Hilbert spaces and $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{B}(H_1)$, $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{B}(H_2)$ and $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{B}(H_3)$ are operator spaces, then*

$$(\mathcal{E}_1 \otimes_{\mathrm{eh}} \mathcal{F}_1) \cap (\mathcal{E}_2 \otimes_{\mathrm{h}} \mathcal{F}_2) = (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_{\mathrm{h}} (\mathcal{F}_1 \cap \mathcal{F}_2) \quad \text{and}$$
$$(\mathcal{E}_1 \otimes_{\mathrm{eh}} \mathcal{F}_1 \otimes_{\mathrm{eh}} \mathcal{G}_1) \cap (\mathcal{E}_2 \otimes_{\mathrm{h}} \mathcal{F}_2 \otimes_{\mathrm{h}} \mathcal{G}_2) = (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_{\mathrm{h}} (\mathcal{F}_1 \cap \mathcal{F}_2) \otimes_{\mathrm{h}} (\mathcal{G}_1 \cap \mathcal{G}_2).$$

Proof. Since $\otimes_{\mathrm{eh}}$ and $\otimes_{\mathrm{h}}$ are both associative, the second equation follows from the first. If $u \in (\mathcal{E}_1 \otimes_{\mathrm{eh}} \mathcal{F}_1) \cap (\mathcal{E}_2 \otimes_{\mathrm{h}} \mathcal{F}_2) \subseteq \mathcal{B}(H_1) \otimes_{\mathrm{h}} \mathcal{B}(H_2)$ then $L_\varphi(u) \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $R_\psi(u) \in \mathcal{E}_1 \cap \mathcal{E}_2$ whenever $\varphi \in \mathcal{B}(H_1)^*$ and $\psi \in \mathcal{B}(H_2)^*$. By (6), $u \in (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_{\mathrm{h}} (\mathcal{F}_1 \cap \mathcal{F}_2)$. The converse inclusion follows immediately in light of the injectivity of the Haagerup tensor product. $\square$

2.2. *Operator multipliers*

We now recall some definitions and results from [14] and [12] that will be needed later. Let $H_1, \ldots, H_n$ be Hilbert spaces and $H = H_1 \otimes \cdots \otimes H_n$ be their Hilbertian tensor product. Set $HS(H_1, H_2) = \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$ and let $\theta_{H_1,H_2} : H_1 \otimes H_2 \to HS(H_1, H_2)$ be the canonical isometry given by $\theta(\xi_1 \otimes \xi_2)(\eta^{\mathrm{d}}) = (\xi_1, \eta)\xi_2$ for $\xi_1, \eta \in H_1$ and $\xi_2 \in H_2$. When n is even, we inductively define

$$HS(H_1, \ldots, H_n) \stackrel{\mathrm{def}}{=} \mathcal{C}_2\big(HS(H_2, H_3)^{\mathrm{d}}, HS(H_1, H_4, \ldots, H_n)\big),$$

and let $\theta_{H_1,\ldots,H_n} : H \to HS(H_1, \ldots, H_n)$ be given by

$$\theta_{H_1,\ldots,H_n}(\xi_{2,3} \otimes \xi) = \theta_{HS(H_2,H_3),HS(H_1,H_4,\ldots,H_n)}\big(\theta_{H_2,H_3}(\xi_{2,3}) \otimes \theta_{H_1,H_4,\ldots,H_n}(\xi)\big),$$

where $\xi_{2,3} \in H_2 \otimes H_3$ and $\xi \in H_1 \otimes H_4 \otimes \cdots \otimes H_n$. When n is odd, we let

$$HS(H_1, \ldots, H_n) \stackrel{\mathrm{def}}{=} HS(\mathbb{C}, H_1, \ldots, H_n).$$

If K is a Hilbert space, we will identify $\mathcal{C}_2(\mathbb{C}^{\mathrm{d}}, K)$ with K via the map $S \to S(1^{\mathrm{d}})$. The isomorphism $\theta_{H_1,\ldots,H_n}$ in the odd case is given by

$$\theta_{H_1,\ldots,H_n}(\xi) = \theta_{\mathbb{C},H_1,\ldots,H_n}(1 \otimes \xi).$$

We will omit the subscripts when they are clear from the context and simply write θ.

If $\xi \in H_1 \otimes H_2$ we let $\|\xi\|_{\mathrm{op}}$ denote the operator norm of $\theta(\xi)$. By $\|\cdot\|_2$ we will denote the Hilbert–Schmidt norm.

Let

$$\Gamma(H_1, \ldots, H_n) = \begin{cases} (H_1 \otimes H_2) \odot (H_2 \otimes H_3)^{\mathrm{d}} \odot \cdots \odot (H_{n-1} \otimes H_n) & \text{if } n \text{ is even}, \\ (H_1 \otimes H_2)^{\mathrm{d}} \odot (H_2 \otimes H_3) \odot \cdots \odot (H_{n-1} \otimes H_n) & \text{if } n \text{ is odd}. \end{cases}$$

We equip $\Gamma(H_1, \ldots, H_n)$ with the Haagerup norm $\|\cdot\|_{\mathrm{h}}$ where each of the terms of the algebraic tensor product is given the opposite operator space structure to the one arising from the embedding $H \otimes K \hookrightarrow (\mathcal{C}_2(H^{\mathrm{d}}, K), \|\cdot\|_{\mathrm{op}})$. We denote by $\|\cdot\|_{2,\wedge}$ the projective norm on $\Gamma(H_1, \ldots, H_n)$ where each of the terms is given its Hilbert space norm.

Suppose n is even. For each $\varphi \in \mathcal{B}(H)$ we let $S_\varphi : \Gamma(H_1, \dots, H_n) \to \mathcal{B}(H_1^{\mathrm{d}}, H_n)$ be the map given by

$$S_\varphi(\zeta) = \theta\big(\varphi(\xi_{1,2} \otimes \xi_{3,4} \otimes \cdots \otimes \xi_{n-1,n})\big)\big(\theta\big(\eta_{2,3}^{\mathrm{d}}\big)\big)\big(\theta\big(\eta_{4,5}^{\mathrm{d}}\big)\big) \dots \big(\theta\big(\eta_{n-2,n-1}^{\mathrm{d}}\big)\big)$$

where $\zeta = \xi_{1,2} \odot \eta_{2,3}^{\mathrm{d}} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n)$ is an elementary tensor. In particular, if $A_i \in \mathcal{B}(H_i)$, $i = 1, \dots, n$, and $\varphi = A_1 \otimes \cdots \otimes A_n$ then

$$S_\varphi(\zeta) = A_n \theta(\xi_{n-1,n}) \dots A_3^{\mathrm{d}} \theta\big(\eta_{2,3}^{\mathrm{d}}\big) A_2 \theta\big(\xi_{1,2}\big) A_1^{\mathrm{d}}.$$

Now suppose that n is odd and let $\zeta \in \Gamma(H_1, \dots, H_n)$ and $\xi_1 \in H_1$. Then

$$\xi_1 \otimes \zeta \in H_1 \odot \Gamma(H_1, \dots, H_n) = \Gamma(\mathbb{C}, H_1, \dots, H_n).$$

For $\varphi \in \mathcal{B}(H)$ we let $S_\varphi(\zeta)$ be the operator defined on H_1 by

$$S_\varphi(\zeta)(\xi_1) = S_{1 \otimes \varphi}(\xi_1 \otimes \zeta).$$

Note that $S_{1 \otimes \varphi}(\xi_1 \otimes \zeta) \in \mathcal{C}_2(\mathbb{C}^{\mathrm{d}}, H_n)$; thus, $S_\varphi(\zeta)(\xi_1)$ can be viewed as an element of H_n. It was shown in [12] that $S_\varphi(\zeta) \in \mathcal{B}(H_1, H_n)$. If $\zeta = \eta_{1,2}^{\mathrm{d}} \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}$ and $\varphi = A_1 \otimes \cdots \otimes A_n$ for $A_i \in \mathcal{B}(H_i)$, $i = 1, \dots, n$, then

$$S_\varphi(\zeta) = A_n \theta(\xi_{n-1,n}) \dots A_3 \theta(\xi_{2,3}) A_2^{\mathrm{d}} \theta\big(\eta_{1,2}^{\mathrm{d}}\big) A_1.$$

As observed in [12, Remark 4.3], for any $\varphi \in \mathcal{B}(H)$ and $\zeta \in \Gamma(H_1, \dots, H_n)$,

$$\big\|S_\varphi(\zeta)\big\|_{\mathrm{op}} \leqslant \|\varphi\| \, \|\zeta\|_{2,\wedge}. \tag{7}$$

On the other hand, an element $\varphi \in \mathcal{B}(H)$ is called a *concrete operator multiplier* if there exists $C > 0$ such that $\|S_\varphi(\zeta)\|_{\mathrm{op}} \leqslant C\|\zeta\|_{\mathrm{h}}$ for each $\zeta \in \Gamma(H_1, \dots, H_n)$. When $n = 2$, this is equivalent to $\|S_\varphi(\zeta)\|_{\mathrm{op}} \leqslant C\|\theta(\zeta)\|_{\mathrm{op}}$ for each $\zeta \in H_1 \otimes H_2$. We call the smallest constant C with this property the concrete multiplier norm of φ.

Now let $\mathcal{A}_i$ be a C^*-algebra and $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ be a representation, $i = 1, \dots, n$. Set $\pi = \pi_1 \otimes \cdots \otimes \pi_n : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{B}(H_1 \otimes \cdots \otimes H_n)$ (here, and in the sequel, by $\mathcal{A} \otimes \mathcal{B}$ we will denote the minimal tensor product of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$). An element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is called a $\pi_1, \dots, \pi_n$*-multiplier* if $\pi(\varphi)$ is a concrete operator multiplier. We denote by $\|\varphi\|_{\pi_1,\dots,\pi_n}$ the concrete multiplier norm of $\pi(\varphi)$. We call φ a *universal multiplier* if it is a $\pi_1, \dots, \pi_n$-multiplier for all representations π_i of $\mathcal{A}_i$, $i = 1, \dots, n$. We denote the collection of all universal multipliers by $M(\mathcal{A}_1, \dots, \mathcal{A}_n)$; from this definition, it immediately follows that

$$\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n \subseteq M(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n.$$

It was observed in [12] that if $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ then

$$\|\varphi\|_{\mathrm{m}} \stackrel{\mathrm{def}}{=} \sup\big\{\|\varphi\|_{\pi_1,\dots,\pi_n}\colon \pi_i \text{ is a representation of } \mathcal{A}_i,\ i = 1, \dots, n\big\} < \infty.$$

It is obvious that if $\mathcal{A}_i$ and $\mathcal{B}_i$ are C^*-algebras and $\rho_i : \mathcal{A}_i \to \mathcal{B}_i$ is a $*$-isomorphism, $i = 1, \ldots, n$, then

$$(\rho_1 \otimes \cdots \otimes \rho_n)\big(M(\mathcal{A}_1, \ldots, \mathcal{A}_n)\big) = M(\mathcal{B}_1, \ldots, \mathcal{B}_n).$$

If φ is an operator, and $\{\varphi_\nu\}$ a net of operators, acting on $H_1 \otimes \cdots \otimes H_n$ we say that $\{\varphi_\nu\}$ converges semi-weakly to φ if $(\varphi_\nu \xi, \eta) \to_\nu (\varphi\xi, \eta)$ for all $\xi, \eta \in H_1 \odot \cdots \odot H_n$. The following characterisation of universal multipliers was established in [12] (see Theorem 6.5, the subsequent remark and the proof of Proposition 6.2) and will be used extensively in the sequel.

Theorem 2.3. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $i = 1, \ldots, n$, and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. Suppose that n is even. The following are equivalent*:

(i) $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$;
(ii) *there exists a net $\{\varphi_\nu\}$ where $\varphi_\nu = A_1^\nu \odot A_2^\nu \odot \cdots \odot A_n^\nu$ and A_i^ν is a finite block operator matrix with entries in $\mathcal{A}_i$ such that $\varphi_\nu \to \varphi$ semi-weakly, $\|\varphi_\nu\|_{\mathrm{m}} \leqslant \prod_{i=1}^n \|A_{2i}^\nu\| \prod_{i=1}^n \|A_{2i-1}^{\nu \mathrm{d}}\|$ and the operator norms $\|A_i^\nu\|$ for i even and $\|A_i^{\nu \mathrm{d}}\|$ for i odd, are bounded by a constant depending only on n.*

For every net $\{\varphi_\nu\}$ satisfying (ii) *we have that $S_{\varphi_\nu}(\zeta) \to S_\varphi(\zeta)$ weakly for all $\zeta = \xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n)$ and that* $\sup_\nu \|\varphi_\nu\|_{\mathrm{m}}$ *is finite.*

Moreover, the net φ_ν can be chosen in (ii) *so that $A_i^\nu \to A_i$* (*resp.* $A_i^{\nu \mathrm{d}} \to A_i^{\mathrm{d}}$) *strongly for i even* (*resp. for i odd*) *for some bounded block operator matrix A_i with entries in $\mathcal{A}_i''$* (*resp.* $(\mathcal{A}_i^{\mathrm{d}})''$) *such that*

$$S_{\mathrm{id} \otimes \cdots \otimes \mathrm{id}(\varphi)}(\zeta) = A_n\big(\theta(\xi_{n-1,n}) \otimes I\big) \ldots \big(\theta(\xi_{1,2}) \otimes I\big) A_1^{\mathrm{d}},$$

for all $\zeta = \xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n)$.

A similar statement holds if n is odd.

Finally, recall that an element a of a C^*-algebra $\mathcal{A}$ is called *compact* if the operator $x \mapsto axa$ on $\mathcal{A}$ is compact. Let $\mathcal{K}(\mathcal{A})$ be the collection of all compact elements of $\mathcal{A}$. It is well known [7,29] that $a \in \mathcal{K}(\mathcal{A})$ if and only if there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator. Moreover, π can be taken to be the reduced atomic representation of $\mathcal{A}$. The notion of a compact element of a C^*-algebra will play a central role in Sections 6 and 7 of the paper.

3. Completely compact maps

We start by recalling the notion of a completely compact map introduced in [21] and studied further in [27] and [15]. By way of motivation, recall that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces then a bounded linear map $\Phi : \mathcal{X} \to \mathcal{Y}$ is compact if and only if for every $\varepsilon > 0$, there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that $\operatorname{dist}(\Phi(x), F) < \varepsilon$ for every x in the unit ball of $\mathcal{X}$.

Now let $\mathcal{X}$ and $\mathcal{Y}$ be operator spaces. A completely bounded map $\Phi : \mathcal{X} \to \mathcal{Y}$ is called *completely compact* if for each $\varepsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that

$$\operatorname{dist}\big(\Phi^{(m)}(x), M_m(F)\big) < \varepsilon,$$

for every $x \in M_m(\mathcal{X})$ with $\|x\| \leqslant 1$ and every $m \in \mathbb{N}$. We extend this definition to multilinear maps: if $\mathcal{Y}, \mathcal{X}_1, \dots, \mathcal{X}_n$ are operator spaces and $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is a completely bounded multilinear map, we call Φ *completely compact* if for each $\varepsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that

$$\operatorname{dist}\bigl(\Phi^{(m)}(x_1, \dots, x_n), M_m(F)\bigr) < \varepsilon,$$

for all $x_i \in M_m(\mathcal{X}_i)$, $\|x_i\| \leqslant 1$, $i = 1, \dots, n$, and all $m \in \mathbb{N}$. We denote by $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ the space of all completely bounded completely compact multilinear maps from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ into $\mathcal{Y}$. A straightforward verification shows the following:

Remark 3.1. A completely bounded map $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is completely compact if and only if its linearisation $\tilde{\Phi} : \mathcal{X}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{X}_n \to \mathcal{Y}$ is completely compact.

In view of this remark, we frequently identify the spaces $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ and $CC(\mathcal{X}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{X}_n, \mathcal{Y})$. The next result is essentially due to Saar (see Lemmas 1 and 2 of [21]).

Proposition 3.2.

(i) *$CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ is closed in $CB(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$.*
(ii) *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be operator spaces. If $\Phi \in CC(\mathcal{E}, \mathcal{F})$ and $\Psi \in CB(\mathcal{F}, \mathcal{G})$ then $\Psi \circ \Phi \in CC(\mathcal{E}, \mathcal{G})$. If $\Phi \in CC(\mathcal{F}, \mathcal{G})$ and $\Psi \in CB(\mathcal{E}, \mathcal{F})$ then $\Phi \circ \Psi \in CC(\mathcal{E}, \mathcal{G})$.*

Let $H_1, \dots, H_n$ be Hilbert spaces. Recall the isometry

$$\gamma : \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n) \to CB^{\sigma}\bigl(\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}), \mathcal{B}(H_n, H_1)\bigr)$$

from Theorem 2.1. Let us identify a completely bounded map defined on $\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1})$ with the corresponding completely bounded map defined on

$$\mathcal{B}_{\mathrm{h}} \stackrel{\mathrm{def}}{=} \mathcal{B}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{B}(H_n, H_{n-1}).$$

For $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ we let $\gamma_0(u)$ be the restriction of $\gamma(u)$ to

$$\mathcal{K}_{\mathrm{h}} \stackrel{\mathrm{def}}{=} \mathcal{K}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_n, H_{n-1}).$$

Proposition 3.3. *The map γ_0 is an isometry from $\mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ onto $CB(\mathcal{K}_{\mathrm{h}}, \mathcal{B}(H_n, H_1))$.*

Proof. Let $\Phi \in CB(\mathcal{K}_{\mathrm{h}}, \mathcal{B}(H_n, H_1))$. Since Φ is completely bounded, its second dual

$$\Phi^{**} : \mathcal{B}(H_2, H_1) \otimes_{\sigma\mathrm{h}} \cdots \otimes_{\sigma\mathrm{h}} \mathcal{B}(H_n, H_{n-1}) \to \mathcal{B}(H_n, H_1)^{**}$$

is completely bounded (here $\otimes_{\sigma\mathrm{h}}$ denotes the normal Haagerup tensor product [5]). Let $Q : \mathcal{B}(H_n, H_1)^{**} \to \mathcal{B}(H_n, H_1)$ be the canonical projection. The multilinear map

$$\tilde{\Phi} : \mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}) \to \mathcal{B}(H_n, H_1)$$

corresponding to $Q \circ \Phi^{**}$ is completely bounded and, by (5.22) of [5], weak* continuous in each variable. By Theorem 2.1, there exists an element $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ such that $\tilde{\Phi} = \gamma(u)$. Hence $\gamma_0(u) = \gamma(u)|_{\mathcal{K}_{\mathrm{h}}} = \tilde{\Phi}|_{\mathcal{K}_{\mathrm{h}}} = \Phi$. Thus γ_0 is surjective.

Fix $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$. From the definition of γ_0 we have $\|\gamma_0(u)\|_{\mathrm{cb}} \leqslant \|\gamma(u)\|_{\mathrm{cb}} = \|u\|_{\mathrm{eh}}$. On the other hand, the restrictions of the maps $Q \circ \gamma_0(u)^{**}$ and $\gamma(u)$ to $\mathcal{K}_{\mathrm{h}}$ coincide, and since both maps are weak* continuous, $\gamma(u) = Q \circ \gamma_0(u)^{**}|_{\mathcal{B}_{\mathrm{h}}}$. Hence,

$$\|u\|_{\mathrm{eh}} \leqslant \left\| Q \circ \gamma_0(u)^{**} \right\|_{\mathrm{cb}} \leqslant \left\| \gamma_0(u)^{**} \right\|_{\mathrm{cb}} = \left\| \gamma_0(u) \right\|_{\mathrm{cb}}.$$

Thus, γ_0 is an isometry. □

Theorem 3.4. *Let $H_1, \ldots, H_n$ be Hilbert spaces. The image under γ_0 of the operator space $\mathcal{E} \stackrel{\mathrm{def}}{=} \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$ is $\mathcal{F} \stackrel{\mathrm{def}}{=} CC(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$.*

Proof. We first establish the inclusion $\gamma_0(\mathcal{E}) \subseteq \mathcal{F}$. If $\Phi = \gamma_0(u)$ where $u \in \mathcal{E}$ then, by Proposition 3.3, Φ is the limit in the cb norm of maps of the form $\gamma_0(v)$, where

$$v = a \odot B \odot b \in \mathcal{K}(H_1) \odot \big(\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})\big) \odot \mathcal{K}(H_n),$$

a and b have finite rank and B is a finite matrix with entries in $\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})$. But each such map has finite rank and hence is completely compact. Moreover, every operator in the image of $\gamma_0(v)$ has range contained in the range of a, which is finite dimensional. It follows that Φ takes compact values; it is completely compact by Proposition 3.2.

To see that $\mathcal{F} \subseteq \gamma_0(\mathcal{E})$, let $\Phi \in \mathcal{F}$. We will assume for technical simplicity that $H_1, \ldots, H_n$ are separable. Let $\{p_k\}_k$ (resp. $\{q_k\}_k$) be a sequence of projections of finite rank on H_1 (resp. H_n) such that $p_k \to I$ (resp. $q_k \to I$) in the strong operator topology. Let $\Psi_k : \mathcal{K}(H_n, H_1) \to \mathcal{K}(H_n, H_1)$ be the complete contraction given by $\Psi_k(x) = p_k x q_k$.

Let $\varepsilon > 0$. Since Φ is completely compact there exists a subspace $F \subseteq \mathcal{K}(H_n, H_1)$ of dimension $\ell < \infty$ such that $\operatorname{dist}(\Phi^{(m)}(x), M_m(F)) < \varepsilon$ whenever $x \in M_m(\mathcal{K}_{\mathrm{h}})$ has norm at most one. Denote the restriction of Ψ_k to F by $\Psi_{k,F}$ and let ι be the inclusion map $\iota : F \hookrightarrow \mathcal{K}(H_n, H_1)$. By [6, Corollary 2.2.4], $\|\Psi_{k,F} - \iota\|_{\mathrm{cb}} \leqslant \ell \|\Psi_{k,F} - \iota\|$. Since $F \subseteq \mathcal{K}(H_n, H_1)$, we have that $\Psi_{k,F}(x) \to x$ in norm for each $x \in F$. It follows easily that there exists k_0 such that $\|\Psi_{k,F} - \iota\|_{\mathrm{cb}} < \varepsilon$ whenever $k \geqslant k_0$.

Let $x \in M_m(\mathcal{K}_{\mathrm{h}})$ be of norm at most one. Then there exists $y \in M_m(F)$ such that $\|\Phi^{(m)}(x) - y\| < \varepsilon$. Note that

$$\|y\| \leqslant \left\|\Phi^{(m)}(x) - y\right\| + \left\|\Phi^{(m)}(x)\right\| \leqslant \varepsilon + \|\Phi\|_{\mathrm{cb}}.$$

Let $\Phi_k = \Psi_k \circ \Phi$. If $k \geqslant k_0$ then

$$\begin{aligned}
\left\|\big(\Phi_k^{(m)} - \Phi^{(m)}\big)(x)\right\| &\leqslant \left\|\Phi_k^{(m)}(x) - \Psi_k^{(m)}(y)\right\| + \left\|\Psi_k^{(m)}(y) - y\right\| + \left\|y - \Phi^{(m)}(x)\right\| \\
&= \left\|\Psi_k^{(m)}\big(\Phi^{(m)}(x) - y\big)\right\| + \left\|(\Psi_{k,F} - \iota)^{(m)}(y)\right\| + \left\|y - \Phi^{(m)}(x)\right\| \\
&\leqslant 2\varepsilon + \varepsilon\big(\varepsilon + \|\Phi\|_{\mathrm{cb}}\big).
\end{aligned}$$

This shows that $\|\Phi_k - \Phi\|_{\mathrm{cb}} \to 0$.

By Proposition 3.2, it only remains to prove that each Φ_k lies in $\gamma_0(\mathcal{E})$. By Proposition 3.3, there exists an element

$$u = A_1 \odot A_2 \odot \cdots \odot A_{n-1} \odot A_n \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$$

where $A_1 : H_1^\infty \to H_1$, $A_i : H_i^\infty \to H_i^\infty$, $i = 2, \ldots, n-1$ and $A_n : H_n \to H_n^\infty$ are bounded operators, such that $\Phi = \gamma_0(u)$. Observe that $\Phi_k = \gamma_0(u_k)$ where $u_k = (p_k A_1) \odot A_2 \odot \cdots \odot A_{n-1} \odot (A_n q_k)$. It therefore suffices to show that $u_k \in \mathcal{E}$ for each k. Fix k and let $p = p_k$, $q = q_k$. The operator $pA_1 : H_1^\infty \to H_1$ has finite dimensional range and is hence compact. For $i = 1, \ldots, n$, let $Q_{i,r} : H_i^\infty \to H_i^\infty$ be a projection with block matrix whose first r diagonal entries are equal to the identity operator while the rest are zero. Then by compactness, $(pA_1)Q_{1,r} \to pA_1$ and $Q_{n,r}(A_n q) \to A_n q$ in norm as $r \to \infty$. Let $B = A_2 \odot \cdots \odot A_{n-1}$, $C_r = (pA_1)Q_{1,r} \odot B \odot Q_{n,r}(A_n q)$, $r \in \mathbb{N}$, and $C = (pA_1) \odot B \odot (A_n q)$. Then

$$\begin{aligned}
\|C_r - C\|_{\mathrm{eh}} &\leqslant \big\|C_r - (pA_1)Q_{1,r} \odot B \odot (A_n q)\big\|_{\mathrm{eh}} + \big\|(pA_1)Q_{1,r} \odot B \odot (A_n q) - C\big\|_{\mathrm{eh}} \\
&\leqslant \big\|(pA_1)Q_{1,r}\big\| \|B\| \big\|Q_{n,r}(A_n q) - A_n q\big\| + \big\|(pA_1)Q_{1,r} - pA_1\big\| \|B\| \|A_n q\|.
\end{aligned}$$

It follows that $\|C_r - C\|_{\mathrm{eh}} \to 0$ as $r \to \infty$. Our claim will follow if we show that $C_r \in \mathcal{E}$. To this end, it suffices to show that if $A_1 = [a_1, \ldots, a_r, 0, \ldots]$ and $A_n = [b_1, \ldots, b_r, 0, \ldots]^t$, where a_i, b_i are operators of finite rank, then $A_1 \odot B \odot A_n \in \mathcal{E}$. Let A_1 and A_n be as stated and let $B' = (Q_{2,r} A_2) \odot A_3 \odot \cdots \odot A_{n-2} \odot (A_{n-1} Q_{n,r})$. Then $A_1 \odot B \odot A_n = A_1 \odot B' \odot A_{n+1}$ belongs to the algebraic tensor product $\mathcal{K}(H_1) \odot (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \odot \mathcal{K}(H_n)$ and hence to $\mathcal{E} = \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$. □

Remarks 3.5.

(i) It follows from Theorem 3.4 that if $\Phi : \mathcal{K}_{\mathrm{h}} \to \mathcal{K}(H_n, H_1)$ is a mapping of finite rank whose image consists of finite rank operators then there exist finite rank projections p and q on H_1 and H_n, respectively, and $u \in (p\mathcal{K}(H_1)) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} (\mathcal{K}(H_n)q)$ such that $\Phi = \gamma_0(u)$.

(ii) The identity $\mathcal{E}_1 \otimes_{\mathrm{h}} (\mathcal{E}_2 \otimes_{\mathrm{eh}} \mathcal{E}_3) = (\mathcal{E}_1 \otimes_{\mathrm{h}} \mathcal{E}_2) \otimes_{\mathrm{eh}} \mathcal{E}_3$ does not hold in general; for an example, take $\mathcal{E}_1 = \mathcal{E}_3 = \mathcal{B}(H)$ and $\mathcal{E}_2 = \mathbb{C}$.

(iii) For every $\Phi \in CC(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$ there exist $A_1 \in \mathcal{K}(H_1^{J_1}, H_1)$, $A_i \in \mathcal{B}(H_i^{J_i}, H_i^{J_{i-1}})$, $i = 2, \ldots, n-1$ and $A_n \in \mathcal{K}(H_n, H_n^{J_{n-1}})$ such that

$$\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes I)A_2 \ldots (x_{n-1} \otimes I)A_n,$$

whenever $x_i \in \mathcal{K}(H_{i+1}, H_i)$, $i = 1, \ldots, n-1$. Indeed, by Proposition 3.4 we have $\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes I)A_2 \ldots (x_{n-1} \otimes I)A_n$ for some $A_1 \odot A_2 \odot \cdots \odot A_n \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$. Using an idea of Blecher and Smith [2, Theorem 3.1], we can choose $A_1 = [t_j]_{j \in J_1} \in M_{J_1,1}(\mathcal{K}(H_1)) \subseteq \mathcal{B}(H_1^{J_1}, H_1)$ and $A_n = [s_i]_{i \in J_{n-1}} \in M_{1,J_{n-1}}(\mathcal{K}(H_n)) \subseteq \mathcal{B}(H_n, H_n^{J_{n-1}})$ such that the sums $\sum_i s_i s_i^*$ and $\sum_j t_j^* t_j$ converge uniformly. Then A_1 is the norm limit of $A_1^{\mathcal{F}} = [t_j^{\mathcal{F}}]_{i \in J_1}$, where $\mathcal{F}$ is a finite subset of J_1 and $t_j^{\mathcal{F}} = t_j$ if $j \in \mathcal{F}$ and $t_j^{\mathcal{F}} = 0$ otherwise. Therefore $A_1 \in \mathcal{K}(H_1^{J_1}, H)$. Similarly, $A_n \in \mathcal{K}(H_n, H_n^{J_n - 1})$.

In the case $n = 2$, Theorem 3.4 reduces to the following result which was established by Saar (Satz 6 of [21]) using the fact that every completely compact completely bounded map on $\mathcal{K}(H_1, H_2)$ is a linear combination of completely compact completely positive maps.

Corollary 3.6. *A completely bounded map $\Phi : \mathcal{K}(H_1, H_2) \to \mathcal{K}(H_1, H_2)$ is completely compact if and only if there exist an index set J and families $\{a_i\}_{i\in J} \subseteq \mathcal{K}(H_1)$ and $\{b_i\}_{i\in J} \subseteq \mathcal{K}(H_2)$ such that the series $\sum_{i\in J} b_i b_i^*$ and $\sum_{i\in J} a_i^* a_i$ converge uniformly and*

$$\Phi(x) = \sum_{i\in J} b_i x a_i, \quad x \in \mathcal{K}(H_1, H_2).$$

We note in passing that Theorem 3.4 together with a result of Effros and Kishimoto [4] yields the following completely isometric identification:

Corollary 3.7. $CC(\mathcal{K}(H_2, H_1))^{**} \simeq (\mathcal{K}(H_1) \otimes_{\mathrm{h}} \mathcal{K}(H_2))^{**} \simeq CB(\mathcal{B}(H_2, H_1))$.

Saar [21] constructed an example of a compact map $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ which is not completely compact (see Section 7 where we give a detailed account of this construction). We note that a compact completely positive map $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ is automatically completely compact. Indeed, the Stinespring Theorem implies that there exist an index set J and a row operator $A = [a_i]_{i\in J} \in \mathcal{B}(H^J, H)$ such that $\Phi(x) = \sum_{i\in J} a_i x a_i^*$, $x \in \mathcal{K}(H)$. The second dual $\Phi^{**} : \mathcal{B}(H) \to \mathcal{B}(H)$ of Φ is a compact map given by the same formula. A standard Banach space argument shows that Φ^{**} takes values in $\mathcal{K}(H)$, and hence $\Phi^{**}(I) \in \mathcal{K}(H)$. This means that $AA^* \in \mathcal{K}(H)$ and so $A \in \mathcal{K}(H^J, H)$ which easily implies that Φ is completely compact.

The previous paragraph shows that there exists a compact completely bounded map on $\mathcal{K}(H)$ which cannot be written as a linear combination of compact completely positive maps.

We finish this section with a modular version of Theorem 3.4. Given von Neumann algebras $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, we let $CC_{\mathcal{A}'_1,\dots,\mathcal{A}'_n}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$ denote the space of all completely compact multilinear maps from $\mathcal{K}_{\mathrm{h}}$ into $\mathcal{K}(H_n, H_1)$ such that the corresponding multilinear map from $\mathcal{K}(H_2, H_1) \times \cdots \times \mathcal{K}(H_n, H_{n-1})$ into $\mathcal{K}(H_n, H_1)$ is $\mathcal{A}'_1, \dots, \mathcal{A}'_n$-modular.

Corollary 3.8. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$, be von Neumann algebras. Set $\mathcal{K}'(\mathcal{A}_i) = \mathcal{K}(H_i) \cap \mathcal{A}_i$, for $i = 1$ and $i = n$. Then*

$$\gamma_0\big(\mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)\big) = CC_{\mathcal{A}'_1,\dots,\mathcal{A}'_n}\big(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1)\big).$$

Proof. By Theorems 2.1 and 3.4, the image of $\mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)$ under γ_0 is contained in $CC_{\mathcal{A}'_1,\dots,\mathcal{A}'_n}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$. For the converse, fix an element $\Phi \in CC_{\mathcal{A}'_1,\dots,\mathcal{A}'_n}(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$. By Theorem 3.4, there exists a unique $u \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$ such that $\gamma_0(u) = \Phi$. By Theorem 2.1, $u \in \mathcal{A}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_n$. Lemma 2.2 now shows that $u \in \mathcal{K}'(\mathcal{A}_1) \otimes_{\mathrm{h}} (\mathcal{A}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}) \otimes_{\mathrm{h}} \mathcal{K}'(\mathcal{A}_n)$. $\square$

4. Complete boundedness of multipliers

Our aim in this section is to clarify the relationship between universal operator multipliers and completely bounded maps, extending results of [12]. We begin with an observation which will allow us to deal with the cases of even and odd numbers of variables in the same manner. We use the notation established in Section 2.

Proposition 4.1. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Let π_i be a representation of $\mathcal{A}_i$ on a Hilbert space H_i, $i = 1, \dots, n$, and $\pi = \pi_1 \otimes \cdots \otimes \pi_n$. The map $S_{\pi(\varphi)}$ takes values in $\mathcal{K}(H_1, H_n)$ if n is odd, and in $\mathcal{K}(H_1^{\mathrm{d}}, H_n)$ if n is even.*

Proof. For even n, this is immediate as observed in [12]. Let n be odd. Assume without loss of generality that $\mathcal{A}_i = \mathcal{B}(H_i)$ and π_i is the identity representation. We call an element $\zeta \in \Gamma(H_1, \dots, H_n)$ thoroughly elementary if

$$\zeta = \eta_{1,2}^{\mathrm{d}} \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}$$

where all $\eta_{j,j+1}^{\mathrm{d}} = \eta_j^{\mathrm{d}} \otimes \eta_{j+1}^{\mathrm{d}}$ and $\xi_{j-1,j} = \xi_{j-1} \otimes \xi_j$ are elementary tensors. The linear span of the thoroughly elementary tensors is dense in the completion of $\Gamma(H_1, \dots, H_n)$ in $\|\cdot\|_{2,\wedge}$. Moreover, the linear span of the elementary tensors $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_n$ is dense in $\mathcal{B}(H_1) \otimes \cdots \otimes \mathcal{B}(H_n)$. By (7) and since $S_\varphi(\zeta)$ is linear in both φ and ζ, it suffices to show that $S_\varphi(\zeta)$ is compact when φ is an elementary tensor and ζ is a thoroughly elementary tensor. However, in this case $S_\varphi(\zeta)$ has rank at most 1, since for every $\xi_1 \in H_1$,

$$S_\varphi(\zeta)\xi_1 = \varphi_n \theta(\xi_{n-1,n}) \dots \varphi_2^{\mathrm{d}} \theta\big(\eta_{1,2}^{\mathrm{d}}\big)\varphi_1\xi_1 = \left(\prod_{j=1}^{n-1} (\varphi_j \xi_j, \eta_j)\right) \varphi_n \xi_n. \qquad \square$$

We now establish some notation. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. Assume that n is even and let $\pi_1, \dots, \pi_n$ be representations of $\mathcal{A}_1, \dots, \mathcal{A}_n$ on $H_1, \dots, H_n$, respectively. Set $\pi = \pi_1 \otimes \cdots \otimes \pi_n$. Using the natural identifications, we consider the map $S_{\pi(\varphi)} : \Gamma(H_1, \dots, H_n) \to H_1 \otimes H_n$ as a map (denoted in the same way)

$$S_{\pi(\varphi)} : \mathcal{C}_2\big(H_1^{\mathrm{d}}, H_2\big) \odot \cdots \odot \mathcal{C}_2\big(H_{n-1}^{\mathrm{d}}, H_n\big) \to \mathcal{C}_2\big(H_1^{\mathrm{d}}, H_n\big).$$

We let

$$\Phi_{\pi(\varphi)} : \mathcal{C}_2\big(H_{n-1}^{\mathrm{d}}, H_n\big) \odot \cdots \odot \mathcal{C}_2\big(H_1^{\mathrm{d}}, H_2\big) \to \mathcal{C}_2\big(H_1^{\mathrm{d}}, H_n\big)$$

be the map given on elementary tensors by

$$\Phi_{\pi(\varphi)}(T_{n-1} \otimes \cdots \otimes T_1) = S_{\pi(\varphi)}(T_1 \otimes \cdots \otimes T_{n-1}).$$

Note that if $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ then $\Phi_{\pi(\varphi)}$ is bounded when the domain is equipped with the Haagerup norm and the range with the operator norm. In this case, $\Phi_{\pi(\varphi)}$ has a unique extension (which will be denoted in the same way)

$$\Phi_{\pi(\varphi)} : \big(\mathcal{K}\big(H_{n-1}^{\mathrm{d}}, H_n\big) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}\big(H_1^{\mathrm{d}}, H_2\big), \|\cdot\|_{\mathrm{h}}\big) \to \big(\mathcal{K}\big(H_1^{\mathrm{d}}, H_n\big), \|\cdot\|_{\mathrm{op}}\big).$$

If n is odd then the map $\Phi_{\pi(\varphi)}$ is defined in a similar way. The map $\Phi_{\pi(\varphi)}$ will be used extensively hereafter.

The main result of this section is Theorem 4.3, where we explain how the complete boundedness of the mappings $\Phi_{\pi(\varphi)}$ relates to the property of φ being a multiplier. We will need the following lemma.

Lemma 4.2. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $i = 1, \dots, n$, and let $k \in \mathbb{N}$. Let $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ and write $\psi = (\mathrm{id}^{(k)} \otimes \cdots \otimes \mathrm{id}^{(k)})(\varphi)$. Suppose that n is even. If $T_i \in M_k(\mathcal{C}_2(H_i^{\mathrm{d}}, H_{i+1}))$ for odd i and $T_i \in M_k(\mathcal{C}_2(H_i, H_{i+1}^{\mathrm{d}}))$ for even i then*

$$\Phi_\varphi^{(k)}(T_{n-1} \odot \cdots \odot T_1) = \Phi_\psi(T_{n-1} \otimes \cdots \otimes T_1),$$

where we identify the operator spaces $M_k(\mathcal{C}_2(H_i^{\mathrm{d}}, H_{i+1}))$ and $\mathcal{C}_2((H_i^{\mathrm{d}})^{(k)}, H_{i+1}^{(k)})$ for odd i, and $M_k(\mathcal{C}_2(H_i, H_{i+1}^{\mathrm{d}}))$ and $\mathcal{C}_2(H_i^{(k)}, (H_{i+1}^{\mathrm{d}})^{(k)})$ for even i. A similar statement holds for odd n.

Proof. To simplify notation, we give the proof for $n = 2$; the proof of the general case is similar. If $\varphi = a_1 \otimes a_2$ is an elementary tensor then $\Phi_\varphi(T) = a_2 T a_1^{\mathrm{d}}$ for $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$ and it is easily checked that the statement holds. By linearity, it holds for each $\varphi \in \mathcal{A}_1 \odot \mathcal{A}_2$. Suppose now that $\varphi \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is arbitrary. Let $\{\varphi_m\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_2$ be a sequence converging in the operator norm to φ and $\psi_m = (\mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)})(\varphi_m)$. By (7), $\Phi_{\varphi_m}(T) \to \Phi_\varphi(T)$ in the operator norm, for all $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. This implies that if $S \in M_k(\mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$, then $\Phi_{\varphi_m}^{(k)}(S) \to \Phi_\varphi^{(k)}(S)$ in the operator norm of $M_k(\mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$. Since $\psi_m \to \psi$ in the operator norm, we conclude that $\Phi_{\psi_m}(S) \to \Phi_\psi(S)$ in the operator norm of $\mathcal{C}_2((H_1^{\mathrm{d}})^{(k)}, H_2^{(k)})$. It follows that $\Phi_\psi(S) = \Phi_\varphi^{(k)}(S)$. $\square$

Theorem 4.3. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. The following are equivalent:*

(i) $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$;
(ii) *if π_i is a representation of $\mathcal{A}_i$, $i = 1, \dots, n$, and $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is completely bounded*;
(iii) *there exist faithful representations π_i of $\mathcal{A}_i$, $i = 1, \dots, n$, such that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is completely bounded.*

Moreover, if the above conditions hold and π is as in (iii) *then $\|\varphi\|_{\mathrm{m}} = \|\Phi_{\pi(\varphi)}\|_{\mathrm{cb}}$.*

Proof. For technical simplicity we only consider the case $n = 3$.

(i) $\Rightarrow$ (ii) Let $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ be a representation, $i = 1, 2, 3$. Then $\pi(\varphi) \in M(\pi_1(\mathcal{A}_1), \pi(\mathcal{A}_2), \pi_3(\mathcal{A}_3))$; thus, it suffices to assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ are concrete C^*-algebras and that π_i is the identity representation, $i = 1, 2, 3$.

Fix $k \in \mathbb{N}$ and let $\psi = (\mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)})(\varphi)$. Since $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, the map

$$\Phi_\psi : \mathcal{K}\big(H_2^{\mathrm{d}(k)}, H_3^{(k)}\big) \odot \mathcal{K}\big(H_1^{(k)}, H_2^{\mathrm{d}(k)}\big) \to \mathcal{K}\big(H_1^{(k)}, H_3^{(k)}\big)$$

is bounded with norm not exceeding $\|\varphi\|_{\mathrm{m}}$. By Lemma 4.2, $\|\Phi_\varphi^{(k)}\| \leqslant \|\varphi\|_{\mathrm{m}}$. Since this inequality holds for every $k \in \mathbb{N}$, the map Φ_φ is completely bounded.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) ⇒ (i) We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ and that π_i is the identity representation, $i = 1, 2, 3$. Let λ be a cardinal number, $\rho_i = \mathrm{id}^{(\lambda)}$ be the ampliation of the identity representation of multiplicity λ, $\psi = (\rho_1 \otimes \rho_2 \otimes \rho_3)(\varphi)$, and $\tilde{H}_i = H_i^{\lambda}$, $i = 1, 2, 3$. Fix $\varepsilon > 0$ and $\zeta \in \Gamma(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$. Let

$$\tilde{T} = \tilde{T}_2 \odot \tilde{T}_1 \in \mathcal{C}_2\big(\tilde{H}_2^{\mathrm{d}}, \tilde{H}_3\big) \odot \mathcal{C}_2\big(\tilde{H}_1, \tilde{H}_2^{\mathrm{d}}\big)$$

be the element canonically corresponding to ζ. Then there exist $k \in \mathbb{N}$ and canonical projections P_i from $\tilde{H}_i$ onto the direct sum of k copies of H_i such that if $T_0 = (P_3 \tilde{T}_2 (P_2^{\mathrm{d}} \otimes I)) \odot ((P_2^{\mathrm{d}} \otimes I)\tilde{T}_1 P_1)$ and if ζ_0 is the element of $\Gamma(H_1^{(k)}, H_2^{(k)}, H_3^{(k)})$ corresponding to T_0 then $\|\zeta - \zeta_0\|_{2,\wedge} \leqslant \varepsilon$.

Set $\psi_0 = (\mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)} \otimes \mathrm{id}^{(k)})(\varphi)$. Arguing as in Lemma 4.2, we see that $\|\Phi_{\psi_0}(T_0)\|_{\mathrm{op}} = \|\Phi_{\psi}(T_0)\|_{\mathrm{op}}$. Using (7) and Lemma 4.2 we obtain

$$\begin{aligned}
\big\|S_\psi(\zeta)\big\|_{\mathrm{op}} &\leqslant \big\|S_\psi(\zeta - \zeta_0)\big\|_{\mathrm{op}} + \big\|S_\psi(\zeta_0)\big\|_{\mathrm{op}} = \big\|S_\psi(\zeta - \zeta_0)\big\|_{\mathrm{op}} + \big\|\Phi_\psi(T_0)\big\|_{\mathrm{op}} \\
&\leqslant \|\psi\|\|\zeta - \zeta_0\|_{2,\wedge} + \big\|\Phi_{\psi_0}(T_0)\big\|_{\mathrm{op}} \leqslant \varepsilon\|\varphi\| + \big\|\Phi_\varphi^{(k)}(T_0)\big\|_{\mathrm{op}} \\
&\leqslant \varepsilon\|\varphi\| + \|\Phi_\varphi\|_{\mathrm{cb}}\|T_0\|_{\mathrm{h}} \\
&\leqslant \varepsilon\|\varphi\| + \|\Phi_\varphi\|_{\mathrm{cb}}\big\|P_3\tilde{T}_2\big(P_2^{\mathrm{d}} \otimes I\big)\big\|_{\mathrm{op}}\big\|\big(P_2^{\mathrm{d}} \otimes I\big)\tilde{T}_1 P_1\big\|_{\mathrm{op}} \\
&\leqslant \varepsilon\|\varphi\| + \|\Phi_\varphi\|_{\mathrm{cb}}\|\tilde{T}_2\|_{\mathrm{op}}\|\tilde{T}_1\|_{\mathrm{op}}.
\end{aligned}$$

It follows that $\|\varphi\|_{\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)}} \leqslant \|\Phi_\varphi\|_{\mathrm{cb}}$.

Now let ρ_1, ρ_2, ρ_3 be arbitrary representations of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, respectively. Then there exists a cardinal number λ such that each of the representations ρ_i is approximately subordinate to the representation $\mathrm{id}^{(\lambda)}$ (see [26] and [10, Theorem 5.1]). By Theorem 5.1 of [12], $\|\varphi\|_{\rho_1,\rho_2,\rho_3} \leqslant \|\varphi\|_{\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)},\mathrm{id}^{(\lambda)}}$; now the previous paragraph implies that $\|\varphi\|_{\rho_1,\rho_2,\rho_3} \leqslant \|\Phi_\varphi\|_{\mathrm{cb}}$. It follows that $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $\|\varphi\|_{\mathrm{m}} \leqslant \|\Phi_\varphi\|_{\mathrm{cb}}$. As the reversed inequality was already established, we conclude that $\|\varphi\|_{\mathrm{m}} = \|\Phi_\varphi\|_{\mathrm{cb}}$. □

5. The symbol of a universal multiplier

Our aim in this section is to generalise the natural correspondence between a function $\varphi \in \ell^\infty \otimes_{\mathrm{eh}} \ell^\infty$ and the Schur multiplier S_φ on $\mathcal{B}(\ell^2(\mathbb{N}))$ given by $S_\varphi((a_{ij})) = (\varphi(i, j)a_{ij})$. To each universal operator multiplier we will associate an element of an extended Haagerup tensor product which we call its symbol. This will be used in the subsequent sections to identify certain classes of operator multipliers.

Recall that if $\mathcal{A}$ is a C^*-algebra, its opposite C^*-algebra $\mathcal{A}^o$ is defined to be the C^*-algebra whose underlying set, norm, involution and linear structure coincide with those of $\mathcal{A}$ and whose multiplication $\cdot$ is given by $a \cdot b = ba$. If $a \in \mathcal{A}$ we denote by a^o the element of $\mathcal{A}^o$ corresponding to a. If $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a representation of $\mathcal{A}$ then the map $\pi^{\mathrm{d}} : a^o \to \pi(a)^{\mathrm{d}}$ from $\mathcal{A}^o$ into $\mathcal{B}(H^{\mathrm{d}})$ is a representation of $\mathcal{A}^o$. Clearly, π is faithful if and only if π^{d} is faithful. If $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ are faithful representations, $i = 1, \ldots, n$ (n even), then by [5, Lemma 5.4] there exists a complete isometry $\pi_n \otimes_{\mathrm{eh}} \pi_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \pi_1^{\mathrm{d}}$ from $\mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_1^o$ into $\mathcal{B}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_1^{\mathrm{d}})$ which sends $a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o$ to $\pi_n(a_n) \otimes \pi_{n-1}(a_{n-1})^{\mathrm{d}} \otimes \cdots \otimes \pi_1(a_1)^{\mathrm{d}}$.

Henceforth, we will consistently write $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ and

$$\pi' = \begin{cases} \pi_n \otimes_{\mathrm{eh}} \pi_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \pi_2 \otimes_{\mathrm{eh}} \pi_1^{\mathrm{d}} & \text{if } n \text{ is even,} \\ \pi_n \otimes_{\mathrm{eh}} \pi_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \pi_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \pi_1 & \text{if } n \text{ is odd.} \end{cases}$$

Let $n \in \mathbb{N}$, $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C^*-algebras, π_i be a representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, and $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Assume that n is even. By Theorem 4.3, the map

$$\Phi_{\pi(\varphi)} : \mathcal{K}\big(H_{n-1}^{\mathrm{d}}, H_n\big) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}\big(H_1^{\mathrm{d}}, H_2\big) \to \mathcal{K}\big(H_1^{\mathrm{d}}, H_n\big)$$

is completely bounded. By Proposition 3.3, there exists a unique element $u_\varphi^\pi \in \mathcal{B}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_1^{\mathrm{d}})$ such that $\gamma_0(u_\varphi^\pi) = \Phi_{\pi(\varphi)}$. For example, if each $\mathcal{A}_i$ is a concrete C^*-algebra and $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$, then

$$u^{\mathrm{id}}_{a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n} = a_n \otimes a_{n-1}^{\mathrm{d}} \otimes \cdots \otimes a_2 \otimes a_1^{\mathrm{d}}.$$

If n is odd then we define u_φ^π similarly.

The main result of this section is the following.

Theorem 5.1. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C^*-algebras and $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. There exists a unique element*

$$u_\varphi \in \begin{cases} \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}_1^o & \textit{if } n \textit{ is even,} \\ \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^o \otimes_{\mathrm{eh}} \mathcal{A}_1 & \textit{if } n \textit{ is odd} \end{cases}$$

with the property that if π_i is a representation of $\mathcal{A}_i$ for $i = 1, \ldots, n$ then

$$u_\varphi^\pi = \pi'(u_\varphi). \tag{8}$$

The map $\varphi \mapsto u_\varphi$ is linear and if $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$, then

$$u_{a_1 \otimes \cdots \otimes a_n} = \begin{cases} a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_2 \otimes a_1^o & \textit{if } n \textit{ is even,} \\ a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_2^o \otimes a_1 & \textit{if } n \textit{ is odd.} \end{cases}$$

Moreover, $\|\varphi\|_{\mathrm{m}} = \|u_\varphi\|_{\mathrm{eh}}$.

Definition 5.2. The element u_φ defined in Theorem 5.1 will be called the *symbol* of the universal multiplier φ.

In order to prove Theorem 5.1 we have to establish a number of auxiliary results. If $\omega \in \mathcal{B}(H)^*$ we let $\tilde{\omega} \in \mathcal{B}(H^{\mathrm{d}})^*$ be the functional given by $\tilde{\omega}(a^{\mathrm{d}}) = \omega(a)$. Note that if $\omega = \omega_{\xi,\eta}$ is the vector functional $a \mapsto (a\xi, \eta)$ then $\tilde{\omega} = \omega_{\eta^{\mathrm{d}},\xi^{\mathrm{d}}}$.

Lemma 5.3. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $\xi_i, \eta_i \in H_i$ and $\omega_i = \omega_{\xi_i,\eta_i}$, $i = 1, \ldots, n$. Suppose that $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then*

$$\big(\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n\big) = \begin{cases} \langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle & \textit{if } n \textit{ is even,} \\ \langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \omega_1 \rangle & \textit{if } n \textit{ is odd.} \end{cases} \tag{9}$$

Proof. We only consider the case n is even; the proof for odd n is similar. Suppose that φ is an elementary tensor, say $\varphi = a_1 \otimes \cdots \otimes a_n$. Then $u_\varphi^{\mathrm{id}} = a_n \otimes a_{n-1}^{\mathrm{d}} \otimes \cdots \otimes a_1^{\mathrm{d}}$ and thus

$$\big(\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n\big) = \prod_{i=1}^{n} (a_i \xi_i, \eta_i) = \big\langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \big\rangle.$$

By linearity, (9) holds for each $\varphi \in \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$.

Now let φ be an arbitrary element of $M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. By Theorem 2.3, there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$ and representations $u_\varphi^{\mathrm{id}} = A_n \odot \cdots \odot A_1$ and $u_{\varphi_\nu}^{\mathrm{id}} = A_n^\nu \odot \cdots \odot A_1^\nu$, where A_i^ν are finite matrices with entries in $\mathcal{A}_i$ if i is even and in $\mathcal{A}_i^{\mathrm{d}}$ if i is odd, such that $\varphi_\nu \to \varphi$ semi-weakly, $A_i^\nu \to A_i$ strongly and all norms $\|A_i\|$, $\|A_i^\nu\|$ are bounded by a constant depending only on n. As in (2), we have

$$\big\langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \big\rangle = \langle A_n, \omega_n \rangle \langle A_{n-1}, \tilde{\omega}_{n-1} \rangle \ldots \langle A_1, \tilde{\omega}_1 \rangle. \tag{10}$$

Moreover, all norms $\|\langle A_i^\nu, \omega_i \rangle\|$ (for even i) and $\|\langle A_i^\nu, \tilde{\omega}_i \rangle\|$ (for odd i) are bounded by a constant depending only on n, and the strong convergence of A_i^ν to A_i implies that $\langle A_i^\nu, \omega_i \rangle$ converges strongly to $\langle A_i, \omega_i \rangle$. Indeed, it is easy to check that if $\xi, \eta \in H$, $A \in M_I(\mathcal{B}(H)) = \mathcal{B}(H \otimes \ell_2(I))$ and $\zeta \in \ell_2(I)$ for some index set I then

$$\big\| \langle A, \omega_{\xi,\eta} \rangle \zeta \big\|^2 = \big(A(\xi \otimes \zeta), \eta \otimes \langle A, \omega_{\xi,\eta} \rangle \zeta\big).$$

This implies that $\|\langle A_i - A_i^\nu, \omega_i \rangle \eta\| \leqslant C \|(A_i - A_i^\nu)(\xi_i \otimes \eta)\|$ for some constant $C > 0$, and the strong convergence follows.

Since operator multiplication is jointly strongly continuous on bounded sets, it now follows from (10) that

$$\big\langle u_{\varphi_\nu}^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \big\rangle \to \big\langle u_\varphi^{\mathrm{id}}, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \big\rangle.$$

On the other hand, since $\varphi_\nu \to \varphi$ semi-weakly,

$$\big(\varphi_\nu(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n\big) \to \big(\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n\big).$$

The proof is complete. $\square$

Lemma 5.4. *Let H_i be a Hilbert space and $\mathcal{E}_i \subseteq \mathcal{B}(H_i)$ be an operator space, $i = 1, \ldots, n$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are closed subspaces of $\mathcal{E}_1$ and $\mathcal{E}_n$, respectively and let $u, v \in \mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n$. If*

$$R_\omega(u) \in \mathcal{X} \quad \textit{and} \quad L_{\omega'}(v) \in \mathcal{Y}$$

whenever $\omega = \omega_2 \otimes \cdots \otimes \omega_n$ and $\omega' = \omega_1' \otimes \cdots \otimes \omega_{n-1}'$ where every $\omega_i, \omega_i' \in \mathcal{B}(H_i)_*$ is a vector functional, then

$$u \in \mathcal{X} \otimes_{\mathrm{eh}} \mathcal{E}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n \quad \textit{and} \quad v \in \mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_{n-1} \otimes_{\mathrm{eh}} \mathcal{Y}.$$

Proof. Let $\mathcal{F}_i$ be the span of the vector functionals on $\mathcal{B}(H_i)$. By linearity, $R_\omega(u) \in \mathcal{X}$ for each $\omega \in \mathcal{F}_2 \odot \cdots \odot \mathcal{F}_n$. Now suppose that

$$\omega \in \big(\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)\big)_* = \mathcal{C}_1(H_2) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{C}_1(H_n).$$

There exists a sequence $(\omega_m) \subseteq \mathcal{F}_2 \odot \cdots \odot \mathcal{F}_n$ such that $\omega_m \to \omega$ in norm. Hence

$$\big\| R_\omega(u) - R_{\omega_m}(u)\big\|_{\mathcal{B}(H_1)} \leqslant \|\omega - \omega_m\| \|u\|_{\mathrm{eh}} \to 0,$$

whence $R_\omega(u) = \lim_m R_{\omega_m}(u) \in \mathcal{X}$. Spronk's formula (5) now implies that $u \in \mathcal{X} \otimes_{\mathrm{eh}} \mathcal{E}_2 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n$. The assertion concerning v has a similar proof. □

We will use slice maps defined on the minimal tensor product of several C^*-algebras as follows. Assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ and $\omega_i \in \mathcal{B}(H_i)^*$, $i = 1, \ldots, n$, and let $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. If $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $\{\ell_1 < \ell_2 < \cdots < \ell_{n-k}\}$ is the complement of $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, n\}$, let

$$\Lambda_{\omega_{i_1}, \ldots, \omega_{i_k}} : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{A}_{\ell_1} \otimes \cdots \otimes \mathcal{A}_{\ell_{n-k}}$$

be the unique norm continuous linear mapping given on elementary tensors by

$$\Lambda_{\omega_{i_1}, \ldots, \omega_{i_k}}(a_1 \otimes \cdots \otimes a_n) = \omega_{i_1}(a_{i_1}) \ldots \omega_{i_k}(a_{i_k})\, a_{\ell_1} \otimes \cdots \otimes a_{\ell_{n-k}}.$$

Proposition 5.5. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, be C^*-algebras and let $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then*

$$u_\varphi^{\mathrm{id}} \in \begin{cases} \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}_1^{\mathrm{d}} & \textit{if } n \textit{ is even,} \\ \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \mathcal{A}_1 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. We only consider the case $n = 3$. Let $u = u_\varphi^{\mathrm{id}}$; by definition, $u \in \mathcal{B}(H_3) \otimes_{\mathrm{eh}} \mathcal{B}(H_2^{\mathrm{d}}) \otimes_{\mathrm{eh}} \mathcal{B}(H_1)$. Let $\xi_i, \eta_i \in H_i$ and $\omega_i = \omega_{\xi_i, \eta_i}$, $i = 1, 2, 3$. Then by (4) and Lemma 5.3,

$$\begin{aligned} \big(R_{\tilde{\omega}_2 \otimes \omega_1}(u)\xi_3, \eta_3\big) &= \big\langle R_{\tilde{\omega}_2 \otimes \omega_1}(u), \omega_3\big\rangle = \langle u, \omega_3 \otimes \tilde{\omega}_2 \otimes \omega_1\rangle \\ &= \big(\varphi(\xi_1 \otimes \xi_2 \otimes \xi_3), \eta_1 \otimes \eta_2 \otimes \eta_3\big) = \big(\Lambda_{\omega_1, \omega_2}(\varphi)\xi_3, \eta_3\big). \end{aligned}$$

Thus

$$R_{\tilde{\omega}_2 \otimes \omega_1}(u) = \Lambda_{\omega_1, \omega_2}(\varphi) \in \mathcal{A}_3.$$

Lemma 5.4 now implies that $u \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{B}(H_2^{\mathrm{d}}) \otimes_{\mathrm{eh}} \mathcal{B}(H_1)$.

Let $w = R_{\omega_1}(u)$. By the previous paragraph, $w \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{B}(H_2^{\mathrm{d}})$. By (4) and Lemma 5.3,

$$\begin{aligned} \big(L_{\omega_3}(w)\eta_2^{\mathrm{d}}, \xi_2^{\mathrm{d}}\big) &= \big\langle L_{\omega_3}(w), \tilde{\omega}_2\big\rangle = \big\langle R_{\omega_1}(u), \omega_3 \otimes \tilde{\omega}_2\big\rangle \\ &= \langle u, \omega_3 \otimes \tilde{\omega}_2 \otimes \omega_1\rangle = \big(\Lambda_{\omega_1, \omega_3}(\varphi)\xi_2, \eta_2\big) = \big(\Lambda_{\omega_1, \omega_3}(\varphi)^{\mathrm{d}}\eta_2^{\mathrm{d}}, \xi_2^{\mathrm{d}}\big). \end{aligned}$$

Hence $L_{\omega_3}(w) = \Lambda_{\omega_1,\omega_3}(\varphi)^{\mathrm{d}} \in \mathcal{A}_2^{\mathrm{d}}$ and, by Lemma 5.4, $w \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}}$. Applying this lemma again shows that $u \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \mathcal{B}(H_1)$. Continuing in this fashion we see that $u \in \mathcal{A}_3 \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}} \otimes_{\mathrm{eh}} \mathcal{A}_1$. $\square$

Lemma 5.6. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be C^*-algebras and let*

$$\rho_i : \mathcal{A}_i \to \mathcal{B}(K_i), \qquad \theta_i : \rho_i(\mathcal{A}_i) \to \mathcal{B}(H_i)$$

be representations, $i = 1, \dots, n$. Suppose that

(i) *for any cardinal number κ, the representations $\theta_i^{(\kappa)} : \rho_i(\mathcal{A}_i) \to \mathcal{B}(H_i^\kappa)$ are strongly continuous, and*
(ii) *whenever $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and $\{\varphi_\nu\}$ is a net in $\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$ such that $\rho(\varphi_\nu) \to \rho(\varphi)$ semi-weakly and $\sup_\nu \|\varphi_\nu\|_m < \infty$ then $\Phi_{\theta\circ\rho(\varphi_\nu)} \to \Phi_{\theta\circ\rho(\varphi)}$ pointwise weakly.*

Then $u_\varphi^{\theta\circ\rho} = \theta'(u_\varphi^\rho)$, for each $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$.

Proof. We suppose that n is even, the proof for odd n being similar. If $\varphi = a_1 \otimes \cdots \otimes a_n$ is an elementary tensor, then $u_\varphi^\rho = \rho'(a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o)$, so

$$u_\varphi^{\theta\circ\rho} = (\theta \circ \rho)'\bigl(a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o\bigr) = \theta'\bigl(u_\varphi^\rho\bigr).$$

By linearity, the claim also holds for $\varphi \in \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$.

If $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is arbitrary then $\rho(\varphi) \in M(\rho_1(\mathcal{A}_1), \dots, \rho_n(\mathcal{A}_n))$ and by Theorem 2.3 and Proposition 5.5, there exist a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$ such that $\rho(\varphi_\nu) \to \rho(\varphi)$ semi-weakly, a representation $u_\varphi^\rho = A_n \odot \cdots \odot A_1$, where $A_i \in M_\kappa(\rho_i(\mathcal{A}_i)) \subseteq \mathcal{B}(K_i^\kappa)$ if i is even and $A_i \in M_\kappa(\rho_i^{\mathrm{d}}(\mathcal{A}_i^o)) \subseteq \mathcal{B}(K_i^\kappa)^{\mathrm{d}}$ if i is odd (κ being a suitable index set), whose operator matrix entries belong to $\rho_i(\mathcal{A}_i)$ if i is even and to $\rho_i^{\mathrm{d}}(\mathcal{A}_i^o)$ if i is odd, and representations $u_{\varphi_\nu}^\rho = A_n^\nu \odot \cdots \odot A_1^\nu$ where the A_i^ν are finite matrices with operator entries in $\rho_i(\mathcal{A}_i)$ if i is even and $\rho_i^{\mathrm{d}}(\mathcal{A}_i^o)$ if i is odd such that $A_i^\nu \to A_i$ strongly and all norms $\|A_i^\nu\|$, $\|A_i\|$ are bounded.

Now $\theta'(u_\varphi^\rho) = \tilde{A}_n \odot \cdots \odot \tilde{A}_1$ and $\theta'(u_{\varphi_\nu}^\rho) = \tilde{A}_n^\nu \odot \cdots \odot \tilde{A}_1^\nu$ where $\tilde{A}_i$ and $\tilde{A}_i^\nu$ are the images of A_i and A_i^ν under $\theta_i^{(\kappa)}$ or $(\theta_i^{\mathrm{d}})^{(\kappa)}$ according to whether i is even or odd. By assumption (i),

$$\gamma_0\bigl(\theta'\bigl(u_{\varphi_\nu}^\rho\bigr)\bigr)(T_{n-1} \otimes \cdots \otimes T_1) \to \gamma_0\bigl(\theta'\bigl(u_\varphi^\rho\bigr)\bigr)(T_{n-1} \otimes \cdots \otimes T_1) \tag{11}$$

weakly for all $T_{n-1} \in \mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n)$,…,$T_1 \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. On the other hand, assumption (ii) and the first paragraph of the proof show that

$$\gamma_0\bigl(\theta'\bigl(u_{\varphi_\nu}^\rho\bigr)\bigr) = \gamma_0\bigl(u_{\varphi_\nu}^{\theta\circ\rho}\bigr) = \Phi_{\theta\circ\rho(\varphi_\nu)} \to \Phi_{\theta\circ\rho(\varphi)} = \gamma_0\bigl(u_\varphi^{\theta\circ\rho}\bigr)$$

pointwise weakly. Using (11) we conclude that $\gamma_0(u_\varphi^{\theta\circ\rho}) = \gamma_0(\theta'(u_\varphi^\rho))$; since γ_0 is injective we have that $u_\varphi^{\theta\circ\rho} = \theta'(u_\varphi^\rho)$. $\square$

Proof of Theorem 5.1. We will only consider the case n is even. Let $\rho_i : \mathcal{A}_i \to \mathcal{B}(K_i)$ be the universal representation of $\mathcal{A}_i$, $i = 1, \dots, n$. Set $\rho = \rho_1 \otimes \cdots \otimes \rho_n$ and $\rho' = \rho_n \otimes \rho_{n-1}^{\mathrm{d}} \otimes \cdots \otimes \rho_1^{\mathrm{d}}$. By Proposition 5.5, u_φ^ρ lies in the image of ρ'; we define $u_\varphi = (\rho')^{-1}(u_\varphi^\rho)$.

Let κ be a nonzero cardinal number and let $\sigma_i = \rho_i^{(\kappa)}$. If $\theta_i = \mathrm{id}^{(\kappa)}_{\rho_i(\mathcal{A}_i)} = \sigma_i \circ \rho_i^{-1}$ then it follows from the proof of Proposition 6.2 of [12] that the hypotheses of Lemma 5.6 are satisfied, so writing $\theta = \theta_1 \otimes \cdots \otimes \theta_n$, we have

$$u_\varphi^\sigma = u_\varphi^{\theta\circ\rho} = \theta'\big(u_\varphi^\rho\big) = (\theta' \circ \rho')(u_\varphi) = \sigma'(u_\varphi).$$

Now let π_i be an arbitrary representation of $\mathcal{A}_i$. It is well known (see e.g. [25]) that π_i is unitarily equivalent to a subrepresentation of $\sigma_i = \rho_i^{(\kappa)}$ for some κ. Hence there exist unitary operators v_i, $i = 1, \dots, n$ (acting between appropriate Hilbert spaces) and subspaces H_i of K_i^κ, such that if $\tau_i(x) = v_i x v_i^*|_{H_i}$ then $\pi_i = \tau_i \circ \sigma_i$. Examining the proof of Proposition 6.2 of [12], we see that $\tau = \tau_1 \otimes \cdots \otimes \tau_n$ satisfies the hypotheses of Lemma 5.6, so

$$u_\varphi^\pi = u_\varphi^{\tau\circ\sigma} = \tau'\big(u_\varphi^\sigma\big) = (\tau \circ \sigma)'(u_\varphi) = \pi'(u_\varphi).$$

The uniqueness of u_φ follows from the injectivity of γ_0. The linearity of the map $\varphi \mapsto u_\varphi$ and its values on elementary tensors are straightforward. The fact that $\|\varphi\|_{\mathrm{m}} = \|u_\varphi\|_{\mathrm{eh}}$ follows from Proposition 3.3 and Theorem 4.3. □

Remarks.

(i) Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \dots, n$ be concrete C^*-algebras of operators. Taking π_i to be the identity representation for $i = 1, \dots, n$ and writing $\mathrm{id} = \pi_1 \otimes \cdots \otimes \pi_n$ gives $u_\varphi = u_\varphi^{\mathrm{id}}$ if we identify $\mathcal{A}_i^o$ with $\mathcal{A}_i^{\mathrm{d}}$.

(ii) Theorem 5.1 implies that if $\mathcal{A}_i$, $i = 1, \dots, n$, are concrete C^*-algebras then the entries of the block operator matrices A_i appearing in the representation of φ in Theorem 2.3 can be chosen from $\mathcal{A}_i$, $i = 1, \dots, n$.

6. Completely compact multipliers

In this section we introduce the class of completely compact multipliers and characterise them within the class of all universal multipliers using the notion of the symbol introduced in Section 5. We will need the following lemma.

Lemma 6.1. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $i = 1, \dots, n$, $a \in \mathcal{A}_1$, $b \in \mathcal{A}_n$ and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Let $\psi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ be given by*

$$\psi = \begin{cases} (a \otimes I \otimes \cdots \otimes I \otimes b)\varphi & \text{if } n \text{ is even}, \\ (I \otimes \cdots \otimes I \otimes b)\varphi(a \otimes I \otimes \cdots \otimes I) & \text{if } n \text{ is odd}. \end{cases}$$

Then $\psi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and

$$\Phi_\psi(x) = \begin{cases} b\Phi_\varphi(x)a^{\mathrm{d}} & \text{if } n \text{ is even}, \\ b\Phi_\varphi(x)a & \text{if } n \text{ is odd}. \end{cases} \tag{12}$$

Proof. For technical simplicity, we will only consider the case $n = 2$. Let $a_i \in \mathcal{A}_i$, $i = 1, 2$, and $\varphi = a_1 \otimes a_2$. In this case $\psi = (aa_1) \otimes (ba_2)$ so

$$\Phi_\psi(T) = ba_2T(aa_1)^{\mathrm{d}} = ba_2Ta_1^{\mathrm{d}}a^{\mathrm{d}} = b\Phi_\varphi(T)a^{\mathrm{d}}.$$

By linearity, (12) holds whenever $\varphi \in \mathcal{A}_1 \odot \mathcal{A}_2$.

Assume that $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2)$ is arbitrary. Fix an operator $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. By Theorem 2.3, there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_2$ such that $\varphi_\nu \to \varphi$ semi-weakly, $\sup_\nu \|\varphi_\nu\|_{\mathrm{m}} < \infty$ and $\Phi_{\varphi_\nu}(T) \to \Phi_\varphi(T)$ weakly.

Let $\psi_\nu = (a \otimes b)\varphi_\nu$; then $\psi_\nu \to \psi$ semi-weakly. Clearly, $\psi_\nu \in \mathcal{A}_1 \odot \mathcal{A}_2$; in particular $\psi_\nu \in M(\mathcal{A}_1, \mathcal{A}_2)$. By the previous paragraph, $\Phi_{\psi_\nu}(\cdot) = b\Phi_{\varphi_\nu}(\cdot)a^{\mathrm{d}}$ and hence $\Phi_{\psi_\nu}(T) \to b\Phi_\varphi(T)a^{\mathrm{d}}$ weakly. If $\varphi_\nu = B_1^\nu \odot B_2^\nu$ then $\psi_\nu = (aB_1^\nu) \odot ((b \otimes I)B_2^\nu)$. It follows from Theorem 2.3 that $\psi \in M(\mathcal{A}_1, \mathcal{A}_2)$ and that $\Phi_{\psi_\nu}(T) \to \Phi_\psi(T)$ weakly. Thus $\Phi_\psi(T) = b\Phi_\varphi(T)a^{\mathrm{d}}$. $\Box$

Given faithful representations $\pi_1, \dots, \pi_n$ of the C^*-algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$, respectively, we define

$$M_{cc}^\pi(\mathcal{A}_1, \dots, \mathcal{A}_n) = \bigl\{\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)\colon \Phi_{\pi(\varphi)} \text{ is completely compact}\bigr\}$$
$$M_{ff}^\pi(\mathcal{A}_1, \dots, \mathcal{A}_n) = \bigl\{\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)\colon \text{the range of } \Phi_{\pi(\varphi)} \text{ is a finite dimensional space of finite-rank operators}\bigr\}.$$

Theorem 6.2. *Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ be a C^*-algebra, $i = 1, \dots, n$, and $\varphi \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$. The following are equivalent*:

(i) $\varphi \in M_{cc}^{\mathrm{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$;

(ii) $u_\varphi^{\mathrm{id}} \in \begin{cases} (\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} (\mathcal{K}(H_1^{\mathrm{d}}) \cap \mathcal{A}_1^{\mathrm{d}}) & \textit{if } n \textit{ is even,} \\ (\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2^{\mathrm{d}}) \otimes_{\mathrm{h}} (\mathcal{K}(H_1) \cap \mathcal{A}_1) & \textit{if } n \textit{ is odd}; \end{cases}$

(iii) *there exists a net $\{\varphi_\alpha\} \subseteq M_{ff}^{\mathrm{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ such that $\|\varphi_\alpha - \varphi\|_{\mathrm{m}} \to 0$.*

Proof. We will only consider the case n is even.

(i) $\Rightarrow$ (ii) Theorem 3.4 implies that

$$u_\varphi^{\mathrm{id}} \in \mathcal{K}(H_n) \otimes_{\mathrm{h}} \bigl(\mathcal{B}\bigl(H_{n-1}^{\mathrm{d}}\bigr) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2)\bigr) \otimes_{\mathrm{h}} \mathcal{K}\bigl(H_1^{\mathrm{d}}\bigr)$$

while, by Proposition 5.5,

$$u_\varphi^{\mathrm{id}} \in \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}_1^{\mathrm{d}}.$$

The conclusion now follows from Lemma 2.2.

(ii) $\Rightarrow$ (i) By Theorem 3.4, $\Phi_\varphi = \gamma_0(u_\varphi^{\mathrm{id}})$ is completely compact.

(ii) $\Rightarrow$ (iii) Let $p \in \mathcal{B}(H_1)$ (resp. $q \in \mathcal{B}(H_n)$) be the projection onto the span of all ranges of operators in $\mathcal{K}(H_1) \cap \mathcal{A}_1$ (resp. $\mathcal{K}(H_n) \cap \mathcal{A}_n$), and let $\{p_\alpha\} \subseteq \mathcal{K}(H_1) \cap \mathcal{A}_1$ (resp. $\{q_\alpha\} \subseteq \mathcal{K}(H_n) \cap \mathcal{A}_n$) be a net of finite rank projections which tends strongly to p (resp. q). It is easy to see that $\Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1) = q\Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1)p^{\mathrm{d}}$, for all $T_1 \in \mathcal{K}(H_1^{\mathrm{d}}, H_2), \dots, T_{n-1} \in \mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n)$. Let $\varphi_\alpha = (p_\alpha \otimes I \otimes \cdots \otimes I \otimes q_\alpha)\varphi$. By Lemma 6.1, $\varphi_\alpha \in M(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and $\Phi_{\varphi_\alpha}(\cdot) = q_\alpha \Phi_\varphi(\cdot) p_\alpha^{\mathrm{d}}$; hence $\varphi_\alpha \in M_{ff}^{\mathrm{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$. We have already seen that Φ_φ is completely compact, and it follows from the proof of Theorem 3.4 that $\Phi_{\varphi_\alpha} \to \Phi_\varphi$ in the cb norm. By Theorem 4.3, $\|\varphi - \varphi_\alpha\|_{\mathrm{m}} \to 0$.

(iii) $\Rightarrow$ (i) is immediate from Proposition 3.2 and Theorem 4.3 and the fact that finite rank maps are completely compact. $\Box$

Now consider the sets

$$M_{cc}(\mathcal{A}_1,\dots,\mathcal{A}_n)=\bigcup_{\pi} M^{\pi}_{cc}(\mathcal{A}_1,\dots,\mathcal{A}_n),$$
$$M_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)=\bigcup_{\pi} M^{\pi}_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)$$

where the unions are taken over all $\pi=\pi_1\otimes\cdots\otimes\pi_n$, each π_i being a faithful representation of $\mathcal{A}_i$. We refer to the first of these as the set of completely compact multipliers.

Lemma 6.3. *If ρ_i is the reduced atomic representation of $\mathcal{A}_i$, $i=1,\dots,n$, and $\rho=\rho_1\otimes\cdots\otimes\rho_n$ then $M_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)=M^{\rho}_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)$.*

Proof. Again, we give the proof for the even case only. We must show that $M^{\pi}_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)\subseteq M^{\rho}_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)$ whenever $\pi=\pi_1\otimes\cdots\otimes\pi_n$ where each π_i is a faithful representation of $\mathcal{A}_i$. Without loss of generality, we may assume that each π_i is the identity representation of $\mathcal{A}_i\subseteq\mathcal{B}(H_i)$. Let $\varphi\in M^{\pi}_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)$ so that the range of Φ_φ is finite dimensional and consists of finite rank operators. By Remark 3.5 (i) there exist finite rank projections p and q on H_1^{d} and H_n, respectively, such that u^{id}_φ lies in the intersection of

$$\big(q\mathcal{K}(H_n)\big)\otimes_{\mathrm{h}}\big(\mathcal{B}\big(H^{\mathrm{d}}_{n-1}\big)\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{B}(H_2)\big)\otimes_{\mathrm{h}}\big(\mathcal{K}\big(H^{\mathrm{d}}_1\big)p\big)$$

and $\mathcal{A}_n\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}^{\mathrm{d}}_1$. By Lemma 2.2, u^{id}_φ lies in

$$\big(q\mathcal{K}(H_n)\cap\mathcal{A}_n\big)\otimes_{\mathrm{h}}\big(\mathcal{B}\big(H^{\mathrm{d}}_{n-1}\big)\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{B}(H_2)\big)\otimes_{\mathrm{h}}\big(\mathcal{K}\big(H^{\mathrm{d}}_1\big)p\cap\mathcal{A}^{\mathrm{d}}_1\big).$$

Hence there exists a representation $u^{\mathrm{id}}_\varphi=A_n\odot\cdots\odot A_1$ of u^{id}_φ such that $A_n=qA_n$ and $A_1=A_1p$. Suppose that $A_n=[b_1,b_2,\dots]$, where $b_j\in\mathcal{A}_n$ for each j, and let q_j be the orthogonal projection onto the range of b_j. Setting $Q_m=\bigvee_{j=1}^m q_j$ we see that $\{Q_m\}$ is an increasing sequence of projections in $\mathcal{A}_n$ dominated by q. It follows that $\bigvee_{m=1}^\infty Q_m\in\mathcal{A}_n$. We may thus assume that $q\in\mathcal{A}_n$. Similarly, we may assume that $p\in\mathcal{A}^{\mathrm{d}}_1$. Now

$$\rho'(u_\varphi)=\big(\rho_n(q)\rho_n(A_n)\big)\odot\cdots\odot\big(\rho_1(A_1)\rho_1(p)\big).$$

By [29], $\rho_n(q)$ and $\rho_1(p)$ have finite rank. By Lemma 6.1, $\varphi\in M^{\rho}_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)$. □

We are now ready to prove the main result of this section.

Theorem 6.4. *Let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be C^*-algebras and $\varphi\in M(\mathcal{A}_1,\dots,\mathcal{A}_n)$. The following are equivalent:*

(i) $\varphi\in M_{cc}(\mathcal{A}_1,\dots,\mathcal{A}_n)$;

(ii) $u_\varphi\in\begin{cases}\mathcal{K}(\mathcal{A}_n)\otimes_{\mathrm{h}}(\mathcal{A}^o_{n-1}\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}_2)\otimes_{\mathrm{h}}\mathcal{K}(\mathcal{A}^o_1) & \textit{if } n \textit{ is even},\\ \mathcal{K}(\mathcal{A}_n)\otimes_{\mathrm{h}}(\mathcal{A}^o_{n-1}\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}^o_2)\otimes_{\mathrm{h}}\mathcal{K}(\mathcal{A}_1) & \textit{if } n \textit{ is odd};\end{cases}$

(iii) *there exists a net $\{\varphi_\alpha\}\subseteq M_{ff}(\mathcal{A}_1,\dots,\mathcal{A}_n)$ such that $\|\varphi_\alpha-\varphi\|_{\mathrm{m}}\to 0$.*

Proof. We will only consider the case n is even.

(i) $\Rightarrow$ (ii) Choose $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ such that $\varphi \in M^{\pi}_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$; after identifying $\mathcal{A}_i$ with its image under π_i, we may assume that each π_i is the identity representation of a concrete C^*-algebra $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$. By Theorem 6.2, $u^{\mathrm{id}}_{\varphi}$ lies in

$$\big(\mathcal{K}(H_n) \cap \mathcal{A}_n\big) \otimes_{\mathrm{h}} \big(\mathcal{A}^o_{n-1} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2\big) \otimes_{\mathrm{h}} \big(\mathcal{K}\big(H^{\mathrm{d}}_1\big) \cap \mathcal{A}^o_1\big).$$

The conclusion follows from the fact that $\mathcal{K}(H_i) \cap \mathcal{A}_i \subseteq \mathcal{K}(\mathcal{A}_i)$ for $i = 1, n$.

(ii) $\Rightarrow$ (i) Let ρ_i be the reduced atomic representation $\mathcal{A}_i \to \mathcal{B}(H_i)$ for $i = 1, \ldots, n$. Since ρ' is an isometry, $u^{\rho}_{\varphi} = \rho'(u_{\varphi})$ lies in

$$\rho_n\big(\mathcal{K}(\mathcal{A}_n)\big) \otimes_{\mathrm{h}} \big(\rho^{\mathrm{d}}_{n-1}\big(\mathcal{A}^o_{n-1}\big) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \rho_2(\mathcal{A}_2)\big) \otimes_{\mathrm{h}} \rho^{\mathrm{d}}_1\big(\mathcal{K}\big(\mathcal{A}^o_1\big)\big).$$

By Theorem 7.5 of [28], $\mathcal{K}(H_i) \cap \rho_i(\mathcal{A}_i) = \rho_i(\mathcal{K}(\mathcal{A}_i))$. By Theorem 6.2, $\varphi \in M^{\rho}_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

(i) $\Rightarrow$ (iii) is immediate from Theorem 6.2.

(iii) $\Rightarrow$ (i) Suppose that $\{\varphi_\alpha\} \subseteq M_{\mathit{ff}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a net such that $\|\varphi_\alpha - \varphi\|_{\mathrm{m}} \to 0$. By Lemma 6.3, $\{\varphi_\alpha\} \subseteq M^{\rho}_{\mathit{ff}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, where ρ is the tensor product of the reduced atomic representations of $\mathcal{A}_1, \ldots, \mathcal{A}_n$. By Theorem 6.2, $\varphi \in M^{\rho}_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. □

In the next theorem we show that in the case $n = 2$ one more equivalent condition can be added to those of Theorem 6.4.

Theorem 6.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\varphi \in M(\mathcal{A}, \mathcal{B})$. The following are equivalent*:

(i) $\varphi \in M_{cc}(\mathcal{A}, \mathcal{B})$;
(ii) *there exists a sequence $\{\varphi_k\}_{k=1}^{\infty} \subseteq \mathcal{K}(\mathcal{A}) \odot \mathcal{K}(\mathcal{B})$ such that $\|\varphi_k - \varphi\|_{\mathrm{m}} \to 0$ as $k \to \infty$.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 6.4, $u_{\varphi} \in \mathcal{K}(\mathcal{B}) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}^o)$; thus $u_{\varphi} = \sum_{i=1}^{\infty} b_i \otimes a^o_i$ where $a^o_i \in \mathcal{K}(\mathcal{A}^o)$, $b_i \in \mathcal{K}(\mathcal{B})$, $i \in \mathbb{N}$, and the series $\sum_{i=1}^{\infty} b_i b_i^*$ and $\sum_{i=1}^{\infty} a^{o*}_i a^o_i$ converge in norm. Let $\varphi_k = \sum_{i=1}^{k} a_i \otimes b_i \in \mathcal{A} \odot \mathcal{B}$. By Theorem 5.1, $u_{\varphi_k} = \sum_{i=1}^{k} b_i \otimes a^o_i$ and $\|\varphi - \varphi_k\|_{\mathrm{m}} = \|u_{\varphi} - u_{\varphi_k}\|_{\mathrm{eh}} \to 0$ as $k \to \infty$.

(ii) $\Rightarrow$ (i) Assume that $\mathcal{A}$ and $\mathcal{B}$ are represented concretely. It is clear that $\varphi_k \in M_{cc}(\mathcal{A}, \mathcal{B})$. By Theorem 4.3, $\|\Phi_{\mathrm{id}(\varphi)} - \Phi_{\mathrm{id}(\varphi_k)}\|_{\mathrm{cb}} = \|\varphi - \varphi_k\|_{\mathrm{m}}$. Proposition 3.2 now implies that $\Phi_{\mathrm{id}(\varphi)}$ is completely compact, in other words, $\varphi \in M_{cc}(\mathcal{A}, \mathcal{B})$. □

7. Compact multipliers

In this section we compare the set of completely compact multipliers with that of compact multipliers. We exhibit sufficient conditions for these two sets of multipliers to coincide, and show that in general they are distinct. Finally, we address the question of when any universal multiplier in the minimal tensor product of two C^*-algebras is automatically compact. We show that this happens precisely when one of the C^*-algebras is finite dimensional while the other coincides with the set of its compact elements.

7.1. Automatic complete compactness

We will need the following result complementing Theorem 3.4. Notation is as in Section 3.

Proposition 7.1. *If $\Phi : \mathcal{K}_{\mathrm{h}} \to \mathcal{K}(H_n, H_1)$ is a compact completely bounded map then $\gamma_0^{-1}(\Phi) \in \mathcal{K}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}) \otimes_{\mathrm{eh}} \mathcal{K}(H_n)$.*

Proof. Fix $\varepsilon > 0$. By compactness, there exist $y_1, \ldots, y_\ell \in \mathcal{K}(H_n, H_1)$ such that $\min_{1\leqslant i\leqslant \ell} \|\Phi(x) - y_i\| < \varepsilon$ for each $x \in \mathcal{K}_{\mathrm{h}}$ with $\|x\| \leqslant 1$.

Let $\{p_\alpha\}$ (resp. $\{q_\alpha\}$) be a net of finite rank projections in $\mathcal{K}(H_1)$ (resp. $\mathcal{K}(H_n)$) such that $p_\alpha \to I$ (resp. $q_\alpha \to I$) strongly and let $\Phi_\alpha : \mathcal{K}_{\mathrm{h}} \to \mathcal{K}(H_n, H_1)$ be the map given by $\Phi_\alpha(x) = p_\alpha \Phi(x) q_\alpha$. Let $u = \gamma_0^{-1}(\Phi)$ and $u_\alpha = \gamma_0^{-1}(\Phi_\alpha)$. Since each y_i is compact there exists α_0 such that $\|p_\alpha y_i q_\alpha - y_i\| < \varepsilon$ for $i = 1, \ldots, \ell$ and $\alpha \geqslant \alpha_0$. Moreover, for any $x \in \mathcal{K}_{\mathrm{h}}$, $\|x\| \leqslant 1$ and $\alpha \geqslant \alpha_0$, we have

$$\begin{aligned}\big\|\Phi_\alpha(x) - \Phi(x)\big\| &\leqslant \min_{1\leqslant i\leqslant \ell}\big\{\big\|\Phi_\alpha(x) - p_\alpha y_i q_\alpha\big\| + \|p_\alpha y_i q_\alpha - y_i\| + \big\|y_i - \Phi(x)\big\|\big\} \\ &\leqslant \min_{1\leqslant i\leqslant \ell}\big\{2\big\|\Phi(x) - y_i\big\| + \|p_\alpha y_i q_\alpha - y_i\|\big\} \leqslant 3\varepsilon,\end{aligned}$$

so $\|\Phi_\alpha - \Phi\| \to 0$. Remark 3.5 (i) shows that $u_\alpha \in \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$; it follows that for every $\omega \in (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1}) \otimes_{\mathrm{eh}} \mathcal{B}(H_n))_*$ we have $R_\omega(u_\alpha) \in \mathcal{K}(H_1)$.

Suppose that $\xi_i, \eta_i \in H_i$ and let $\omega_i = \omega_{\xi_i,\eta_i}$ be the corresponding vector functional. Lemma 5.3 and a straightforward verification shows that if $v \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ has a representation of the form $v = A_1 \odot \cdots \odot A_n$ and $\omega = \omega_2 \otimes \cdots \otimes \omega_n$ then

$$\big(R_\omega(v)\xi_1, \eta_1\big) = \langle v, \omega_1 \otimes \cdots \otimes \omega_n\rangle = \big(\gamma_0(v)(\zeta)\xi_n, \eta_1\big), \tag{13}$$

where

$$\zeta = \big(\big(\eta_2^* \otimes \xi_1\big) \otimes \big(\eta_3^* \otimes \xi_2\big) \otimes \cdots \otimes \big(\eta_{n-1}^* \otimes \xi_{n-2}\big) \otimes \big(\eta_n^* \otimes \xi_{n-1}\big)\big) \in \mathcal{K}_{\mathrm{h}}$$

is an elementary tensor whose components are rank one operators.

Since $\gamma_0(u_\alpha) \to \gamma_0(u)$ in norm, (13) implies that $R_\omega(u_\alpha) \to R_\omega(u)$ in the operator norm of $\mathcal{K}(H_1)$. Since $R_\omega(u_\alpha) \in \mathcal{K}(H_1)$, we obtain $R_\omega(u) \in \mathcal{K}(H_1)$. By Lemma 5.4, $u \in \mathcal{K}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$. Similarly we see that $u \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{K}(H_n)$; the conclusion now follows. □

Remark. The converse of Proposition 7.1 does not hold, even for $n = 2$. Indeed, let $\{p_i\}_{i=1}^\infty$ be a family of pairwise orthogonal rank one projections on a Hilbert space H and let $u = \sum_{i=1}^\infty p_i \otimes p_i$. Then $u \in \mathcal{K}(H) \otimes_{\mathrm{eh}} \mathcal{K}(H)$ and the range of $\gamma_0(u)$ consists of compact operators, but $\gamma_0(u)(p_i) = p_i$ for each i, so $\gamma_0(u)$ is not compact.

Given C^*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$, we let $M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be the collection of all $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ for which there exist faithful representations $\pi_1, \ldots, \pi_n$ of $\mathcal{A}_1, \ldots, \mathcal{A}_n$, respectively, such that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is compact. We call the elements of $M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *compact multipliers*.

As a consequence of the previous result we obtain the following fact.

Proposition 7.2. *Let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be C^*-algebras and let $\varphi\in M_c(\mathcal{A}_1,\dots,\mathcal{A}_n)$. Then*

$$u_\varphi\in\begin{cases}\mathcal{K}(\mathcal{A}_n)\otimes_{\mathrm{eh}}\mathcal{A}_{n-1}^o\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}_2\otimes_{\mathrm{eh}}\mathcal{K}(\mathcal{A}_1^o) & \text{if } n \text{ is even},\\ \mathcal{K}(\mathcal{A}_n)\otimes_{\mathrm{eh}}\mathcal{A}_{n-1}^o\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}_2^o\otimes_{\mathrm{eh}}\mathcal{K}(\mathcal{A}_1) & \text{if } n \text{ is odd}.\end{cases}$$

Proof. We only consider the case n is even. We may assume that $\mathcal{A}_i\subseteq\mathcal{B}(H_i)$ is a concrete non-degenerate C^*-algebra, $i=1,\dots,n$, and that Φ_φ is compact. By Propositions 5.5 and 7.1, u_φ^{id} belongs to

$$\big(\mathcal{K}(H_n)\otimes_{\mathrm{eh}}\mathcal{B}\big(H_{n-1}^{\mathrm{d}}\big)\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{B}(H_2)\otimes_{\mathrm{eh}}\mathcal{K}\big(H_1^{\mathrm{d}}\big)\big)\cap\big(\mathcal{A}_n\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}_1^{\mathrm{d}}\big).$$

Since $\mathcal{K}(H_n)\cap\mathcal{A}_n\subseteq\mathcal{K}(\mathcal{A}_n)$ and $\mathcal{K}(H_1^{\mathrm{d}})\cap\mathcal{A}_1^{\mathrm{d}}\subseteq\mathcal{K}(\mathcal{A}_1^{\mathrm{d}})$, an application of (5) shows that $u_\varphi^{\mathrm{id}}\in\mathcal{K}(\mathcal{A}_n)\otimes_{\mathrm{eh}}\mathcal{A}_{n-1}^{\mathrm{d}}\otimes_{\mathrm{eh}}\cdots\otimes_{\mathrm{eh}}\mathcal{A}_2\otimes_{\mathrm{eh}}\mathcal{K}(\mathcal{A}_1^{\mathrm{d}})$. □

If $\{\mathcal{A}_j\}_{j\in J}$ is a family of C^*-algebras, we will denote by $\bigoplus_{j\in J}^{c_0}\mathcal{A}_j$ and $\bigoplus_{j\in J}^{\ell_\infty}\mathcal{A}_j$ their c_0- and ℓ_∞-direct sums, respectively.

Theorem 7.3. *Let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be C^*-algebras, and suppose that $\mathcal{K}(\mathcal{A}_1)$ is isomorphic to $\bigoplus_{j\in J}^{c_0}M_{m_j}$ and $\mathcal{K}(\mathcal{A}_n)$ is isomorphic to $\bigoplus_{j\in J}^{c_0}M_{n_j}$ where J is some index set and $\sup_{j\in J}m_j$ and $\sup_{j\in J}n_j$ are finite. Then*

$$M_c(\mathcal{A}_1,\dots,\mathcal{A}_n)=M_{cc}(\mathcal{A}_1,\dots,\mathcal{A}_n).$$

Proof. We give the proof for $n=3$; the case of a general n is similar. Let $m=\sup\{m_j,n_j\colon j\in J\}$. By hypothesis, $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_3)$ may both be embedded in the C^*-algebra $\mathcal{C}\stackrel{\mathrm{def}}{=}\bigoplus_{j\in J}^{c_0}M_m$ for some $m\in\mathbb{N}$; without loss of generality, we may assume that this embedding is an inclusion and that $\mathcal{A}_i$ is represented faithfully on some Hilbert space H_i such that H_1 and H_3 both contain the Hilbert space $H=\bigoplus_{j\in J}\mathbb{C}^m$. Given $\varphi\in M_c(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$, Proposition 7.2 implies that the symbol u_φ of φ can be written in the form $u_\varphi=A_3\odot A_2\odot A_1$, where the entries of A_3 and A_1 belong to $\mathcal{C}$. Let us write $\{e_{ij}: i,j=1,\dots,m\}$ for the canonical matrix unit system of M_m and let $P_k=\bigoplus_{j\in J}e_{kk}\in\bigoplus_{j\in J}^{\ell^\infty}M_m$, $k=1,\dots,m$. For $k,\ell,s,t=1,\dots,m$, we set $A_3^{k,\ell}=P_kA_3(P_\ell\otimes I)$ and $A_1^{s,t}=(P_s\otimes I)A_1P_t$ and define

$$u_{k,\ell,s,t}=A_3^{k,\ell}\odot A_2\odot A_1^{s,t}\quad\text{and}\quad\Phi_{k,\ell,s,t}=\gamma_0(u_{k,\ell,s,t}).$$

Then $\gamma_0(u_\varphi)=\Phi=\sum_{k,\ell,s,t}\Phi_{k,\ell,s,t}$ so it suffices to show that each of the maps $\Phi_{k,\ell,s,t}$ is completely compact. Now

$$\Phi_{k,\ell,s,t}(T_2\otimes T_1)=P_k\Phi(P_\ell T_2\otimes T_1P_s)P_t=A_3^{k,\ell}\big((P_\ell T_2)\otimes I\big)A_2\big((T_1P_s)\otimes I\big)A_1^{s,t}.$$

Thus, $\Phi_{k,\ell,s,t}$ can be considered as a completely bounded multilinear map from $\mathcal{K}(H_2^{\mathrm{d}},P_\ell H)\times\mathcal{K}(P_sH,H_2^{\mathrm{d}})$ into $\mathcal{K}(P_tH,P_kH)$. Since Φ is compact, it follows that $\Phi_{k,\ell,s,t}$ is compact.

Take a basis $\{e_i^j\colon i=1,\dots,m,\ j\in J\}$ of $H=\bigoplus_{j\in J}\mathbb{C}^m$, where for each $j\in J$, the standard basis of the jth copy of $\mathbb{C}^m$ is $\{e_i^j\colon i=1,\dots,m\}$. Let $U_k:P_kH\to P_1H$ be the unitary operator defined by $U_k e_k^j = e_1^j$. Consider the mapping $\Psi : \mathcal{K}(H_2^{\mathrm{d}}, P_1 H) \times \mathcal{K}(P_1 H, H_2^{\mathrm{d}}) \to \mathcal{K}(P_1 H, P_1 H)$ given by

$$\Psi(T_2 \otimes T_1) = U_k \Phi_{k,\ell,s,t}(U_\ell T_2 \otimes T_1 U_s) U_t.$$

To show that $\Phi_{k,\ell,s,t}$ is completely compact it suffices to show that Ψ is. Let $\mathcal{C}_0 = P_1 \mathcal{C} P_1$; then $\mathcal{C}_0$ is isomorphic to c_0 and its commutant $\mathcal{C}_0'$ has a cyclic vector. Moreover, Ψ is a $\mathcal{C}_0'$-modular multilinear map. Let $\{p_\alpha\}$ be a net of finite rank projections belonging to $\mathcal{C}_0$, such that s-lim $p_\alpha = I_{P_1 H}$. Consider the completely bounded multilinear maps $\Psi_\alpha(x) = p_\alpha \Psi(x) p_\alpha$. Since the range of each p_α is finite dimensional, Ψ_α has finite rank, so is completely compact. Since Ψ is compact, we may argue as in the proof of Proposition 7.1 to show that $\|\Psi_\alpha - \Psi\| \to 0$. Now the maps Ψ and Ψ_α are $\mathcal{C}_0'$-modular and $\mathcal{C}_0'$ has a cyclic vector, so by the generalisation [12, Lemma 3.3] of a result of Smith [23, Theorem 2.1],

$$\|\Psi_\alpha - \Psi\|_{\mathrm{cb}} = \|\Psi_\alpha - \Psi\| \to 0.$$

Proposition 3.2 now implies that Ψ is completely compact. □

The following corollary extends Proposition 5 of [11] to the case of multidimensional Schur multipliers. Let $n \geqslant 2$ be an integer. We recall from [12] that with every $\varphi \in \ell_\infty(\mathbb{N}^n)$ we associate a mapping $S_\varphi : \ell_2(\mathbb{N}^2) \odot \cdots \odot \ell_2(\mathbb{N}^2) \to \ell_2(\mathbb{N}^2)$ which extends the usual Schur multiplication in the case $n = 2$. We equip the domain of S_φ with the Haagerup norm where each of the terms is given its operator space structure arising from its embedding into the corresponding space of Hilbert–Schmidt operators endowed with the operator norm.

Corollary 7.4. *Let $n > 2$ and $\varphi \in \ell_\infty(\mathbb{N}^n)$. The following are equivalent*:

(i) *S_φ is compact*;
(ii) $\varphi \in c_0 \otimes_{\mathrm{h}} (\underbrace{\ell_\infty \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \ell_\infty}_{n-2}) \otimes_{\mathrm{h}} c_0$.

Proof. Assume first that S_φ is compact. It follows from [12, Section 3] that the map S_φ induces a completely bounded compact map

$$\hat{S}_\varphi : \mathcal{C}_2 \times \cdots \times \mathcal{C}_2 \to \mathcal{C}_2$$

defined by $\hat{S}_\varphi(T_{f_1}, \ldots, T_{f_n}) = T_{S_\varphi(f_1,\ldots,f_n)}$, where T_f is the Hilbert–Schmidt operator with kernel f. By Proposition 7.1, $\varphi = \gamma_0^{-1}(\hat{S}_\varphi) \in \mathcal{K}(\ell_2) \otimes_{\mathrm{eh}} \mathcal{B}(\ell_2) \otimes_{\mathrm{eh}} \ldots \otimes_{\mathrm{eh}} \mathcal{B}(\ell_2) \otimes_{\mathrm{eh}} \mathcal{K}(\ell_2)$. Since S_φ is bounded, φ is a Schur multiplier and by [12, Theorem 3.4], $\varphi \in \ell_\infty \otimes_{\mathrm{eh}} \ldots \otimes_{\mathrm{eh}} \ell_\infty$. Hence $\varphi \in c_0 \otimes_{\mathrm{eh}} \ell_\infty \otimes_{\mathrm{eh}} \ldots \otimes_{\mathrm{eh}} \ell_\infty \otimes_{\mathrm{eh}} c_0$. We may now argue as in the last paragraph of the preceding proof to show that $\varphi \in c_0 \otimes_{\mathrm{h}} (\ell_\infty \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \ell_\infty) \otimes_{\mathrm{h}} c_0$. □

Our next aim is to show that if both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes then the completely compact multipliers form a proper subset of the compact multipliers. Saar [21] has provided an example of a compact completely bounded map on $\mathcal{K}(H)$ (where H is a separable Hilbert space) which is not completely compact. It turns out that Saar's example also shows that the sets of compact and completely compact multipliers are distinct, in the case under consideration.

We will need some preliminary results. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Recall that a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called symmetric (or hermitian) if $\Phi = \Phi^*$ where $\Phi^* : \mathcal{A} \to \mathcal{B}$ is the map given by $\Phi^*(a) = (\Phi(a^*))^*$. By $S_{\mathcal{A}}$ we denote the unit ball of $\mathcal{A}$ and set $S^h_{\mathcal{A}} = \{a \in S_{\mathcal{A}} : a = a^*\}$. The following lemma is a special case of Satz 6 of [21]. We include a direct proof for the convenience of the reader.

Lemma 7.5. *Let H be a Hilbert space. If $\Phi : \mathcal{A} \to \mathcal{K}(H)$ is a symmetric, completely compact linear map with $\|\Phi\|_{\mathrm{cb}} \leqslant 1$, then there exists a positive operator $c \in \mathcal{K}(H)$ such that $\Phi^{(n)}(a) \leqslant c \otimes 1_n$ for all $a \in S^h_{M_n(\mathcal{A})}$ and all $n \in \mathbb{N}$. Moreover, c can be chosen to have norm arbitrarily close to one.*

Proof. We first show that for a given $\varepsilon > 0$ there exists a finite rank projection p on H such that

$$\left\|\Phi^{(n)}(a) - (p \otimes 1_n)\Phi^{(n)}(a)(p \otimes 1_n)\right\| \leqslant \varepsilon \quad \text{for any } a \in S_{M_n(\mathcal{A})}. \tag{14}$$

Since Φ is completely compact, there exists a finite dimensional subspace $F \subset \mathcal{K}(H)$ such that $\operatorname{dist}(\Phi^{(n)}(a), M_n(F)) \leqslant \varepsilon/3$ for any $a \in M_n(\mathcal{A})$, $\|a\| \leqslant 1$ and any $n \in \mathbb{N}$. Let $S_{F,1+\varepsilon} = \{x \in F\colon \|x\| \leqslant 1 + \varepsilon\}$ and let $k = \dim F$. Choose a finite rank projection $p \in \mathcal{K}(H)$ such that

$$\|x - pxp\| < \frac{\varepsilon}{k(3+\varepsilon)} \quad \text{for all } x \in S_{F,1+\varepsilon}$$

and let $\Psi : F \to \mathcal{K}(H)$ be defined by $\Psi(x) = x - pxp$. By [6, Corollary 2.2.4], Ψ is completely bounded and $\|\Psi\|_{\mathrm{cb}} \leqslant k\|\Psi\|$. This implies that

$$\left\|\Psi^{(n)}(y)\right\| \leqslant k\|\Psi\|\|y\| \leqslant \frac{\varepsilon}{3+\varepsilon}\|y\| \leqslant \frac{\varepsilon}{3}$$

for all $y \in M_n(F)$ with $\|y\| \leqslant 1 + \varepsilon/3$.

Now for $a \in S^h_{M_n(\mathcal{A})}$ let $y \in M_n(F)$ be such that $\|\Phi^{(n)}(a) - y\| \leqslant \varepsilon/3$. Then $\|y\| \leqslant \|\Phi^{(n)}(a)\| + \varepsilon/3 \leqslant 1 + \varepsilon/3$. Hence

$$\begin{aligned}
&\left\|\Phi^{(n)}(a) - (p \otimes 1_n)\Phi^{(n)}(a)(p \otimes 1_n)\right\| \\
&\quad\leqslant \left\|\Phi^{(n)}(a) - y\right\| + \left\|\Psi^{(n)}(y)\right\| + \left\|(p \otimes 1_n)\big(y - \Phi^{(n)}(a)\big)(p \otimes 1_n)\right\| \\
&\quad\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\end{aligned}$$

proving (14). Next we fix $\varepsilon > 0$ and choose a finite rank projection q_1 on H such that

$$\left\|\Phi^{(n)}(a) - (q_1 \otimes 1_n)\Phi^{(n)}(a)(q_1 \otimes 1_n)\right\| \leqslant \frac{\varepsilon}{2}, \quad a \in M_n(\mathcal{A}),\ \|a\| \leqslant 1,\ n \in \mathbb{N}.$$

Let $r_1 : \mathcal{A} \to \mathcal{K}(H)$ be the mapping given by $r_1(a) = \Phi(a) - q_1\Phi(a)q_1$, $a \in \mathcal{A}$. Then $r_1 = \Psi \circ \Phi$, where $\Psi : \mathcal{K}(H) \to \mathcal{K}(H)$ is the completely bounded map given by $\Psi(x) = x - q_1xq_1$. By Proposition 3.2, r_1 is completely compact. Moreover, $\|r_1\|_{\mathrm{cb}} \leqslant \varepsilon/2$ and $\Phi(a) =$

$q_1\Phi(a)q_1 + r_1(a)$, $a \in \mathcal{A}$. Proceeding by induction, we can find sequences of finite rank projections q_i and completely compact symmetric mappings r_i such that $\|r_i\|_{\mathrm{cb}} \leqslant \varepsilon/2^i$ and

$$\Phi(a) = q_1\Phi(a)q_1 + \sum_{i=1}^{\infty} q_{i+1}r_i(a)q_{i+1}, \quad a \in \mathcal{A}.$$

Let $c = q_1 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} q_{i+1}$. We have that $\Phi^{(n)}$ and $r_i^{(n)}$ are symmetric and

$$\Phi^{(n)}(a) = (q_1 \otimes 1_n)\Phi^{(n)}(a)(q_1 \otimes 1_n) + \sum_{i=1}^{\infty}(q_{i+1} \otimes 1_n)r_i^{(n)}(a)(q_{i+1} \otimes 1_n),$$

for each $a \in \mathcal{A}$. Now

$$\Phi^{(n)}(a) \leqslant (q_1 \otimes 1_n)\|\Phi\|_{\mathrm{cb}} + \sum_{i=1}^{\infty}(q_{i+1} \otimes 1_n)\|r_i\|_{\mathrm{cb}} \leqslant \left(q_1 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} q_{i+1}\right) \otimes 1_n = c \otimes 1_n$$

for all $a \in S^h_{M_n(\mathcal{A})}$. By construction, c is compact and $\|c\| \leqslant 1 + \varepsilon$. $\square$

Let H be an infinite dimensional separable Hilbert space and $\{q_k\}_{k\in\mathbb{N}}$ be a family of pairwise orthogonal projections in $\mathcal{B}(H)$ with rank $q_k = k$ and $\sum_{k=1}^{\infty} q_k = I$. Set $p_n = \sum_{k=1}^{n} q_k$, $n \in \mathbb{N}$. Let $\Phi_k : \mathcal{B}(q_kH) \to \mathcal{B}(q_kH)$, $k \in \mathbb{N}$, be symmetric linear maps such that

$$\|\Phi_k\|_{\mathrm{cb}} = 1, \qquad \|\Phi_k\| \to 0 \quad \text{as } k \to \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \|\Phi_k\|_2^2 < \infty, \tag{15}$$

where $\|\Phi_k\|_2$ denotes the norm of the mapping Φ_k when $\mathcal{B}(q_kH) \simeq \mathcal{C}_2(q_kH)$ is equipped with the Hilbert–Schmidt norm. Identifying $\mathcal{B}(q_kH)$ with $q_k\mathcal{B}(H)q_k$, let $\Phi : \mathcal{K}(H) \to \mathcal{B}(H)$ be the map given by the norm-convergent sum

$$\Phi(x) = \sum_{k=1}^{\infty}{}^{\oplus} \Phi_k(q_kxq_k), \quad x \in \mathcal{K}(H). \tag{16}$$

An example of such a map is obtained by taking $\Phi_k = k^{-1}\tau_k$ where τ_k is the transposition map $\mathcal{B}(q_kH) \simeq M_k \to M_k \simeq \mathcal{B}(q_kH)$, which is symmetric and an isometry for both the operator and the Hilbert–Schmidt norm. It is well known [20, p. 419] that $\|\tau_k\|_{\mathrm{cb}} = k$ and hence conditions (15) are satisfied.

The next lemma is a straightforward extension of [21, pp. 32–34].

Lemma 7.6. *If Φ is a map satisfying* (15) *and* (16) *then the range of Φ consists of compact operators. Moreover, Φ is completely contractive and compact but not completely compact.*

Proof. Fix $x \in \mathcal{K}(H)$. Since $\|\Phi_k\| \to 0$ as $k \to \infty$, we have $p_n\Phi(x)p_n \to \Phi(x)$ in norm, so $\Phi(x) \in \mathcal{K}(H)$. Each of the maps $x \mapsto \Phi_k(q_kxq_k)$ is completely contractive, so Φ is completely contractive.

Next, note that Φ maps the unit ball of $\mathcal{K}(H)$ into $U \stackrel{\text{def}}{=} U_1 \oplus U_2 \oplus \cdots$, where U_k is the ball of radius $\|\Phi_k\|$ in $q_k\mathcal{B}(H)q_k$. Since U is compact, the map Φ is compact.

If Φ were completely compact then by Lemma 7.5, there would exist a positive compact operator c on H such that

$$\Phi^{(k)}(x) \leqslant c \otimes 1_k \quad \text{for all } x \in S^h_{M_k(\mathcal{K}(H))} \text{ and all } k \in \mathbb{N}.$$

Hence for every $k \in \mathbb{N}$ and $x \in S^h_{M_k(\mathcal{K}(H))}$,

$$\Phi_k^{(k)}\big((q_k \otimes 1_k)x(q_k \otimes 1_k)\big) = (q_k \otimes 1_k)\Phi^{(k)}(x)(q_k \otimes 1_k) \leqslant q_k c q_k \otimes 1_k.$$

However, $\|\Phi_k^{(k)}\| = \|\Phi_k\|_{\mathrm{cb}} = 1$ by [22], so

$$\|q_k c q_k\| = \|q_k c q_k \otimes 1_k\| \geqslant \sup\big\{\big\|\Phi_k^{(k)}(x)\big\|\colon x \in S^h_{M_k(q_k\mathcal{K}(H)q_k)}\big\} \geqslant \frac{1}{2},$$

which is impossible since c is compact. $\square$

Lemma 7.7. *Given a map Φ be as above, let $\mathcal{C} = \bigoplus_{k\in\mathbb{N}}^{c_0} \mathcal{B}(q_k H) \subseteq \mathcal{K}(H)$. There exists a universal multiplier $\varphi \in M(\mathcal{C}^{\mathrm{d}}, \mathcal{C})$ with $\Phi = \Phi_{\mathrm{id}(\varphi)}$.*

Proof. Let $\varphi_k \in \mathcal{B}(q_k H)^{\mathrm{d}} \otimes \mathcal{B}(q_k H)$ be such that $\Phi_{\mathrm{id}(\varphi_k)} = \Phi_k$, $k \in \mathbb{N}$, where the family $\{\Phi_k\}_{k=1}^{\infty}$ satisfies (15). Then $\|\varphi_k\|_{\min} = \|\Phi_k\|_2$. Let $\psi_n = \sum_{k=1}^n \varphi_k$. If $n < m$ then $\|\psi_m - \psi_n\|_{\min} = \|\sum_{k=n+1}^m \Phi_k\|_2$ so

$$\|\psi_m - \psi_n\|_{\min} \leqslant \left(\sum_{k=n+1}^{m} \|\Phi_k\|_2^2\right)^{1/2}.$$

By (15), the sequence $\{\psi_n\}$ converges to an element $\varphi \in \mathcal{C}^{\mathrm{d}} \otimes \mathcal{C}$. Moreover, for every $x \in \mathcal{C}_2(H)$ we have

$$\Phi_{\mathrm{id}(\varphi)}(x) = \lim_{n\to\infty} p_n \Phi_{\mathrm{id}(\varphi)}(x) p_n = \lim_{n\to\infty} \Phi_{\mathrm{id}(\psi_n)}(x) = \Phi(x),$$

where the limits are in the operator norm. So $\Phi_{\mathrm{id}(\varphi)} = \Phi$ which is completely contractive by Lemma 7.6, so $\varphi \in M(\mathcal{C}^{\mathrm{d}}, \mathcal{C})$ by Theorem 4.3. $\square$

Given C^*-algebras $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, and $\psi = c_2 \otimes \cdots \otimes c_{n-1} \in \mathcal{A}_2 \odot \cdots \odot \mathcal{A}_{n-1}$, we may define a bounded linear map $\mathcal{A}_1 \otimes \mathcal{A}_n \to \mathcal{B}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$, where $\mathcal{B}_1 = \mathcal{A}_1$ if n is even and $\mathcal{B}_1 = \mathcal{A}_1^{\mathrm{d}}$ if n is odd, by

$$a \otimes b \mapsto \begin{cases} a \otimes \psi \otimes b & \text{if } n \text{ is even},\\ a^{\mathrm{d}} \otimes \psi \otimes b & \text{if } n \text{ is odd}. \end{cases}$$

We write ι_ψ for the restriction of this map to $M(\mathcal{A}_1, \mathcal{A}_n)$.

Lemma 7.8.

(i) *The range of ι_ψ is contained in $M(\mathcal{B}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$.*
(ii) $\iota_\psi(M_c^{\mathrm{id}}(\mathcal{A}_1, \mathcal{A}_n)) \subseteq M_c^{\mathrm{id}}(\mathcal{B}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$.
(iii) *Suppose that n is even and $\omega \in (\mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2))_*$. Writing*

$$M_\omega : \mathcal{B}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}\big(H_{n-1}^{\mathrm{d}}\big) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2) \otimes_{\mathrm{eh}} \mathcal{B}\big(H_1^{\mathrm{d}}\big) \to \mathcal{B}(H_n) \otimes_{\mathrm{eh}} \mathcal{B}\big(H_1^{\mathrm{d}}\big)$$

for the "middle slice map" $M_\omega = R_\omega \otimes_{\mathrm{eh}} \mathrm{id}_{\mathcal{B}(H_1^{\mathrm{d}})}$, we have

$$M_\omega(u_{\iota_\psi(\varphi)}) = \omega(\tilde{\psi})u_\varphi$$

where $\tilde{\psi} = c_{n-1}^{\mathrm{d}} \otimes \cdots \otimes c_2$. The same is true, mutatis mutandis, if n is odd.

Proof. Let $\varphi \in M(\mathcal{A}_1, \mathcal{A}_n)$. By Theorem 2.3, there exist a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \mathcal{A}_n$ and representations $u_{\varphi_\nu}^{\mathrm{id}} = A_2^\nu \odot A_1^\nu$ and $u_\varphi^{\mathrm{id}} = A_2 \odot A_1$, where A_i^ν are finite matrices with entries in $\mathcal{A}_1^{\mathrm{d}}$ if $i = 1$ and in $\mathcal{A}_n$ if $i = 2$, such that $\varphi_\nu \to \varphi$ semi-weakly, $A_i^\nu \to A_i$ strongly and $\sup_{i,\nu} \|A_i^\nu\| < \infty$.

(i) It is easy to see that $\iota_\psi(\varphi_\nu)$ satisfies the boundedness conditions of Theorem 2.3 and converges semi-weakly to $\iota_\psi(\varphi)$, which is therefore a universal multiplier.

(ii) Suppose that n is even and let $\iota = \iota_\psi$. It is immediate to check that if $\varphi \in \mathcal{A}_1 \odot \mathcal{A}_n$ and $T_1 \in \mathcal{K}(H_1^{\mathrm{d}}, H_2), \ldots, T_{n-1} \in \mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n)$ then

$$\Phi_{\iota(\varphi)}(T_{n-1} \otimes \cdots \otimes T_1) = \Phi_\varphi\big(T_{n-1}c_{n-1}^{\mathrm{d}} \ldots c_2 T_1\big).$$

Note that this equation holds for any $\varphi \in M(\mathcal{A}_1, \mathcal{A}_n)$ since $\Phi_{\varphi_\nu}(T) \to \Phi_\varphi(T)$ and $\Phi_{\iota(\varphi_\nu)}(T_{n-1} \otimes \cdots \otimes T_1) \to \Phi_{\iota(\varphi)}(T_{n-1} \otimes \cdots \otimes T_1)$ weakly for any $T, T_1, \ldots, T_{n-1}$. Since $\Phi_{\iota(\varphi)}$ is the composition of the bounded mapping $X_{n-1} \otimes \cdots \otimes X_1 \mapsto X_{n-1}c_{n-1}^{\mathrm{d}} \ldots c_2 X_1$ with Φ_φ, it follows that if φ is a compact operator multiplier then so is $\iota(\varphi)$.

(iii) We have that

$$\begin{aligned}\Phi_{\iota(\varphi_\nu)}(T_{n-1} \otimes \cdots \otimes T_1) &= A_2^\nu(T_{n-1} \otimes I)\big(c_{n-1}^{\mathrm{d}} \otimes I\big) \ldots (c_2 \otimes I)(T_1 \otimes I)A_1^\nu \\ &\to A_2(T_{n-1} \otimes I)\big(c_{n-1}^{\mathrm{d}} \otimes I\big) \ldots (c_2 \otimes I)(T_1 \otimes I)A_1\end{aligned}$$

weakly. On the other hand, $\Phi_{\iota(\varphi_\nu)}(T_{n-1} \otimes \cdots \otimes T_1) \to \Phi_{\iota(\varphi)}(T_{n-1} \otimes \cdots \otimes T_1)$ which implies that $u_{\iota(\varphi)} = A_2 \odot (c_{n-1}^{\mathrm{d}} \otimes I) \odot \cdots \odot (c_2 \otimes I) \odot A_1$. It follows that $M_\omega(u_{\iota(\varphi)}) = \omega(\tilde{\psi})u_\varphi$. $\square$

Theorem 7.9. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C^*-algebras with the property that both $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes. Then the inclusion $M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_c(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is proper.*

Proof. We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$ for some Hilbert spaces $H_1, \ldots, H_n$. First suppose that $n = 2$. By hypothesis, we may assume that there is an infinite dimensional separable Hilbert space H with $H^{\mathrm{d}} \subseteq H_1$ and $H \subseteq H_2$, and a C^*-algebra $\mathcal{C} = \bigoplus_{k\in\mathbb{N}}^{c_0} M_k \subseteq \mathcal{K}(H)$ as in Lemma 7.7 with $\mathcal{C}^{\mathrm{d}} \subseteq \mathcal{A}_1$ and $\mathcal{C} \subseteq \mathcal{A}_2$. By the injectivity of the minimal tensor product of C^*-algebras, $\mathcal{C}^{\mathrm{d}} \otimes \mathcal{C} \subseteq \mathcal{A}_1 \otimes \mathcal{A}_2$.

Let $\varphi \in \mathcal{C}^{\mathrm{d}} \otimes \mathcal{C}$ be given by Lemma 7.7. It follows from Lemma 7.6 that $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_2) \setminus M_{cc}^{\mathrm{id}}(\mathcal{A}_1, \mathcal{A}_2)$. Since faithful representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ restrict to representations of $\mathcal{C}$ containing the identity subrepresentation up to unitary equivalence, we have that $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_2) \setminus M_{cc}(\mathcal{A}_1, \mathcal{A}_2)$.

Suppose now that n is even. Let $\varphi \in M_c(\mathcal{A}_1, \mathcal{A}_n) \setminus M_{cc}(\mathcal{A}_1, \mathcal{A}_n)$, fix any non-zero $\psi = c_2 \otimes \cdots \otimes c_{n-1} \in \mathcal{A}_2 \odot \cdots \odot \mathcal{A}_{n-1}$ and let us write $\iota = \iota_\psi$. Suppose that $\iota(\varphi)$ is a completely compact multiplier. By Theorem 6.4, $u_{\iota(\varphi)} \in \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1^o)$.

Let $\tilde{\psi} = c_{n-1}^{\mathrm{d}} \otimes \cdots \otimes c_2 \in \mathcal{A}_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2$ and fix $\omega \in (\mathcal{B}(H_{n-1}^{\mathrm{d}}) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_2))_*$ such that $\omega(\tilde{\psi}) \neq 0$. By Lemma 7.8 (iii), $M_\omega(u_{\iota(\varphi)}) = \omega(\tilde{\psi}) u_\varphi$ and hence $u_\varphi \in \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1^o)$ which by Theorem 6.4 contradicts the assumption that φ is not a completely compact multiplier.

If n is odd then the same proof works with minor modifications. $\square$

Remark 7.10. We do not know whether the sets $M_{cc}(\mathcal{A}, \mathcal{B})$ and $M_c(\mathcal{A}, \mathcal{B})$ are distinct if $\mathcal{K}(\mathcal{A})$ contains matrix algebras of arbitrarily large sizes, while $\mathcal{K}(\mathcal{B})$ does not (and vice versa). Let $\mathcal{C}$ be the C^*-algebra defined in Lemma 7.7. To show that the inclusion $M_{cc}(\mathcal{C}, c_0) \subseteq M_c(\mathcal{C}, c_0)$ is proper it would suffice to exhibit mappings $\Phi_k : M_k \to M_k$ which satisfy (15) and are left D_k-modular (where D_k is the subalgebra of all diagonal matrices of M_k). This modularity condition would enable us to find $\varphi_k \in M_k^{\mathrm{d}} \otimes D_k$ such that $\Phi_k = \Phi_{\mathrm{id}(\varphi_k)}$ using the method of Lemma 7.7 and we could then conclude from Lemma 7.6 that $M_{cc}(\mathcal{C}, c_0) \subsetneq M_c(\mathcal{C}, c_0)$. However, we do not know if such mappings Φ_k exist.

This prompts the following question: if $\mathcal{D}$ is a masa in $\mathcal{B}(H)$, does there exist a constant C such that whenever $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ is a bounded and left $\mathcal{D}$-modular map then $\|\Phi\|_{\mathrm{cb}} \leqslant C\|\Phi\|$? If such a version of Smith's automatic complete boundedness result holds then it would follow that $M_{cc}(\mathcal{C}, c_0) = M_c(\mathcal{C}, c_0)$.

7.2. *Automatic compactness*

We now turn to the question of when every universal multiplier is automatically compact. We will restrict to the case $n = 2$ for the rest of the paper. We will first establish an auxiliary result in a different but related setting. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras and assume that $\mathcal{A} = C_0(X)$ and $\mathcal{B} = C_0(Y)$ for some locally compact Hausdorff spaces X and Y. The C^*-algebra $C_0(X) \otimes C_0(Y)$ will be identified with $C_0(X \times Y)$ and $M(\mathcal{A}, \mathcal{B})$ with a subset of $C_0(X \times Y)$. Elements of the Haagerup tensor product $C_0(X) \otimes_{\mathrm{h}} C_0(Y)$, as well as of the projective tensor product $C_0(X) \hat{\otimes} C_0(Y)$, will be identified with functions in $C_0(X \times Y)$ in the natural way. Note that, by Grothendieck's inequality, $C_0(X) \otimes_{\mathrm{h}} C_0(Y)$ and $C_0(X) \hat{\otimes} C_0(Y)$ coincide as sets of functions.

Proposition 7.11. *Let X and Y be infinite, locally compact Hausdorff spaces. Then $C_0(X) \otimes_{\mathrm{h}} C_0(Y) \subseteq M(C_0(X), C_0(Y))$ and this inclusion is proper.*

Proof. The inclusion $C_0(X) \otimes_{\mathrm{h}} C_0(Y) \subseteq M(C_0(X), C_0(Y))$ follows from Corollary 6.7 of [14]. To show that this inclusion is proper, suppose first that X and Y are compact. By Theorem 11.9.1 of [8], there exists a sequence $(f_i)_{i=1}^\infty \subseteq C(X) \otimes_{\mathrm{h}} C(Y)$ such that $\sup_{i \in \mathbb{N}} \|f_i\|_{\mathrm{h}} < \infty$, converging uniformly to a function $f \in C(X \times Y) \setminus C(X) \otimes_{\mathrm{h}} C(Y)$. By Corollary 6.7 of [14], $f \in M(C(X), C(Y))$. The conclusion now follows.

Now assume that both X and Y are locally compact but not compact (the case where one of the spaces is compact while the other is not is similar). Let $\tilde{X} = X \cup \{\infty\}$ and $\tilde{Y} = Y \cup \{\infty\}$ be the one point compactifications of X and Y. Then $C(\tilde{X}) = C_0(X) + \mathbb{C}1$ and $C(\tilde{Y}) = C_0(Y) + \mathbb{C}1$, where 1 denotes the constant function taking the value one. Moreover, it is easy to see that

$$C(\tilde{X}) \otimes C(\tilde{Y}) = C_0(X \times Y) + C_0(X) + C_0(Y) + \mathbb{C}1$$

and

$$C(\tilde{X}) \hat{\otimes} C(\tilde{Y}) = C_0(X) \hat{\otimes} C_0(Y) + C_0(X) + C_0(Y) + \mathbb{C}1. \tag{17}$$

By the first part of the proof, there exists $\varphi \in M(C(\tilde{X}), C(\tilde{Y})) \setminus C(\tilde{X}) \otimes_{\mathrm{h}} C(\tilde{Y})$. Write $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ where $\varphi_1 \in C_0(X \times Y)$, $\varphi_2 \in C_0(X)$, $\varphi_3 \in C_0(Y)$ and $\varphi_4 \in \mathbb{C}1$. Suppose that $\varphi_1 \in C_0(X) \otimes_{\mathrm{h}} C_0(Y)$. By (17), $\varphi \in C(\tilde{X}) \hat{\otimes} C(\tilde{Y})$, a contradiction. $\square$

Theorem 7.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The following are equivalent*:

(i) *either $\mathcal{A}$ is finite dimensional and $\mathcal{K}(\mathcal{B}) = \mathcal{B}$, or $\mathcal{B}$ is finite dimensional and $\mathcal{K}(\mathcal{A}) = \mathcal{A}$*;
(ii) $M_c(\mathcal{A}, \mathcal{B}) = M(\mathcal{A}, \mathcal{B})$;
(iii) $M_{cc}(\mathcal{A}, \mathcal{B}) = M(\mathcal{A}, \mathcal{B})$.

Proof. (i) $\Rightarrow$ (iii) Suppose that $\mathcal{A}$ is finite dimensional and $\mathcal{K}(\mathcal{B}) = \mathcal{B}$, and that $\mathcal{A} \subseteq \mathcal{B}(H_1)$ and $\mathcal{B} \subseteq \mathcal{B}(H_2)$ for some Hilbert spaces H_1 and H_2 where H_1 is finite dimensional. Fix $\varphi \in M(\mathcal{A}, \mathcal{B})$. Then φ is the sum of finitely many elements of the form $a \otimes b$ where a has finite rank and $b \in \mathcal{K}(H_2)$; such elements are completely compact multipliers by Theorem 6.4.

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i) Assume that both $\mathcal{A}$ and $\mathcal{B}$ are infinite dimensional and are identified with their image under the reduced atomic representation. If either $\mathcal{K}(\mathcal{A})$ or $\mathcal{K}(\mathcal{B})$ is finite dimensional then there exists an elementary tensor $a \otimes b \in (\mathcal{A} \odot \mathcal{B}) \setminus (\mathcal{K}(\mathcal{A}) \odot \mathcal{K}(\mathcal{B}))$. By Proposition 7.2, $a \otimes b \notin M_c(\mathcal{A}, \mathcal{B})$. We can therefore assume that both $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$ are infinite dimensional. Then, up to a $*$-isomorphism, c_0 is contained in both $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$. By Proposition 7.11, there exists $\varphi \in M(c_0, c_0) \setminus (c_0 \otimes_{\mathrm{h}} c_0)$. Then $\varphi \in M(\mathcal{A}, \mathcal{B})$ and $\Phi_{\mathrm{id}(\varphi)}$ is not compact by Hladnik's characterisation [11]. Since the restrictions to c_0 of any faithful representations of $\mathcal{A}$, $\mathcal{B}$ contain representations unitarily equivalent to the identity representations, we see that φ is not a compact multiplier.

Thus at least one of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is finite dimensional; assume without loss of generality that this is $\mathcal{A}$. Suppose that $\mathcal{B} \neq \mathcal{K}(\mathcal{B})$ and fix an element $b \in \mathcal{B} \setminus \mathcal{K}(\mathcal{B})$. Let $a \in \mathcal{A}$ be a non-zero element. By Proposition 7.2, the elementary tensor $a \otimes b$ is not a compact multiplier. $\square$

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