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Complex Chebyshev Polynomials and Generalizations with an Application to the Optimal Choice of Interpolating Knots in Complex Planar Splines*

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This paper contains a brief account on complex planar splines which are complex valued functions defined piecewise on a grid. For noncontinuous (so called nonconforming) splines the problem of the placement of knots at which these splines are required to be continuous is investigated. It is shown that this problem reduces to finding complex Chebyshev polynomials under the additional requirement that the zeros of the polynomials are on the boundary of the corresponding domains. It is proved that the zeros of a generalized Chebyshev polynomial are in the convex hull of the domain on which the Chebyshev polynomials are defined. Some open problems are stated. A numerical and graphical display for the optimal location of three and six points on certain triangles is provided.

1. INTRODUCTION

To motivate this study, a brief account of the so-called complex planar splines is given. These splines were introduced by Opfer and Puri [15] and were further studied by Opfer and Schober [16]. The problem of the optimal

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choice of interpolating knots is closely related to the concept of nonconforming complex planar splines.

Let $f: D \subset \mathbb{C} \to \mathbb{C}$ be a continuous mapping, where D is the closure of some bounded region in $\mathbb{C}$. If f is complicated, then it is desirable to find an approximation for f which is easier to compute than f itself. Since f is intrinsically connected to its domain of definition D, D in general needs to be approximated by a simpler set, too.

One approach to this problem is to replace D by a union of suitable simple configuration like triangles or rectangles and to approximate f piecewise by simple *elements*.

Since triangles play a special role when dividing up a given two-dimensional set like D, the study here will be restricted to triangular grids.

It is explained in Opfer and Puri [15] that the continuity requirement which is imposed upon the complex planar splines has the consequence that holomorphic elements like polynomials cannot be used reasonably.

However, if one requires continuity of the approximation of f only in the interior points and in certain selected points (knots) of the boundary of the triangles, one obtains approximations which (in analogy to the real case) should be called *nonconforming* complex planar splines.

The advantage of relaxing the continuity requirement is that one can use holomorphic functions like complex polynomials as elements.

In this connection the question of placing the knots in a certain optimal fashion when using polynomials on triangles is of much interest and is the subject of study in this paper. In a forthcoming paper by Opfer and Werner some further properties of nonconforming elements will be exhibited.

To illustrate the problem suppose we take a triangular grid and define a quadratic complex polynomial in each triangle of that grid. Suppose further that we require that the resulting function is either continuous in all grid-points (i.e., vertices of the triangles) or continuous in all the midpoints of the edges. The question is which of the two requirements is favorable.

To illustrate further we interpolate $f(z) = \exp(z)$ at the vertices (case 1) or at the midpoints (case 2) of the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : |y| \leqslant x \leqslant \sqrt{0, 5}\} \tag{1.1}$$

by a complex polynomial of degree two and suppose we denote the resulting polynomials by p_1 and p_2 respectively. We now compare the uniform errors on T and we obtain after some numerical calculations that

$$\|f - p_1\|_\infty = 0.0984, \qquad \|f - p_2\|_\infty = 0.1106 \tag{1.2}$$

which indicates that case 1 is preferable to case 2.

2. Complex Chebyshev Polynomials and Generalizations

Let Π_n denote the set of all polynomials with complex coefficients and degree at most $n \in \mathbb{N}^0 = \{0, 1, ...\}$. Let $\Pi_n^{\mathbb{R}}$ denote the subset of Π_n consisting of all the polynomials with real coefficients.

If $p \in \Pi_n$ interpolates a given function $f: D \to \mathbb{C}$ at points $z_0, z_1, ..., z_n \in D$, where D is a compact set in $\mathbb{C}$, then the error takes the form

$$r(z) = f(z) - p(z) = f[z_0, z_1, ..., z_n, z] \prod_{j=0}^{n} (z - z_j) \tag{2.1}$$

(Davis [7, p. 64]) regardless of the assumptions on f since expression (2.1) can be derived by purely algebraic means. Here the expression $f[z_0, z_1, ..., z_n, z]$ is defined in terms of divided differences. Implicity we always assume that D contains sufficiently many points; in this context at least $n + 2$.

However, the assumptions imposed on f play an important role in deriving equivalent expressions for $f[z_0, z_1, ..., z_n, z]$ or estimates for $|f[z_0, z_1, ..., z_n, z]|$.

For a selection of these expressions and estimates see Davis [7, pp. 67–69] or Abramowitz and Stegun [1, Chap. 25].

For any $z_0, z_1, ..., z_n, z \in \mathbb{C}$ we set

$$\zeta = (z_0, z_1, ..., z_n) \tag{2.2}$$

and

$$\omega(z, \zeta) = \prod_{j=0}^{n} (z - z_j).$$

A uniformly best choice of $z_0, z_1, ..., z_n$ in (2.1) which is independent of f is apparently made if we require that

$$\|\omega(\cdot, \zeta)\|_\infty = \max_{z \in D} |\omega(z, \zeta)| \tag{2.3}$$

is as small as possible by appropriate choice of ζ. Since

$$\omega(z, \zeta) = z^{n+1} + p(z), \qquad p \in \Pi_n, \quad n \in \mathbb{N}^0, \tag{2.4}$$

the problem of minimizing $\|\omega(\cdot, \zeta)\|_\infty$ may be regarded as a linear complex approximation problem. Π_n is a Haar space of finite dimension $n + 1$ which implies that the stated approximation problem is uniquely soluble (cf. Meinardus [14, p. 16]). It also implies that the zeros of the best $\omega(z, \zeta)$ are unique. For each fixed $n \in \mathbb{N}^0$, the polynomial

$$t_{n+1}(z) = z^{n+1} + \hat{p}(z), \qquad z \in D, \quad \hat{p} \in \Pi_n \tag{2.5}$$

defined by

$$\|t_{n+1}\|_\infty \leqslant \|z^{n+1}+p\|_\infty \qquad \text{for all } p \in \Pi_n \tag{2.6}$$

wil be called the *Chebyshev polynomial* (or for short T-polynomial) of degree $n+1$ with respect to D, and the corresponding zeros the *Chebyshev points* of D. In addition, we define

$$t_0(z) = 1 \qquad \text{for all } z \in D. \tag{2.7}$$

The ordinary T-polynomials usually called T_n (cf. Rivlin [17]) are obtained by letting $D = [-1, 1]$. However note that usually the normalization $T_n = t_n/\|t_n\|_\infty$ implying $\|T_n\|_\infty = 1$ is used.

Interestingly the ordinary T-polynomials for $D = [-1, 1]$ are at the same time the T-polynomials for all confocal ellipses with foci -1 and $+1$. This was already observed by Faber [8].

Since the T-polynomials are essentially uneffected by the linear transformation of D,

$$w(z) = az + b, \qquad z \in D, \quad a \neq 0, \quad a, b \in \mathbb{C}, \tag{2.8}$$

one can compute the nth degree T-polynomial of $w(D)$ if the nth degree T-polynomial of D is known. We note that the inverse of w in (2.8) is given by $z(w) = (w-b)/a$.

To be precise let

$$t_n(z) = z^n + \sum_{j=0}^{n-1} \alpha_j z^j \tag{2.9}$$

be the nth T-polynomial of any compact nonempty set $D \subset \mathbb{C}$ and $\tilde{D} = w(D)$. Then

$$\begin{aligned} \tilde{t}_n(w) &= a^n t_n\left(\frac{w-b}{a}\right) \\ &= (w-b)^n + \sum_{j=0}^{n-1} \alpha_j a^{n-j}(w-b)^j \end{aligned} \tag{2.10}$$

is the T-polynomial of degree n with respect to $\tilde{D}$. We also note that T-polynomials with respect to D are in $\Pi_n^{\mathbb{R}}$, $n \in \mathbb{N}$, provided D is symmetric with respect to the real axis (cf. Meinardus [14, p. 28]).

EXAMPLE 2.1. Let E be any ellipse in the plane with foci $f_1, f_2 \in \mathbb{C}$, $f_1 \neq f_2$. Then

$$w(z) = [(f_2 - f_1)\, z + (f_2 + f_1)]/2$$

maps $D=[-1, 1]$ onto the segment $[f_1, f_2]$ in $\mathbb{C}$. Let

$$t_n(z)=z^n+\alpha_{n-2}z^{n-2}+\alpha_{n-4}z^{n-4}+\cdots+\alpha_{n-2[n/2]}z^{n-2[n/2]} \quad (2.11)$$

be the (ordinary) T-polynomial for $D=[-1, 1]$ where the coefficients α_{n-2j}, $j=1, 2, \ldots, [n/2]$ are given up to a factor 2^{n-1} in Abramowitz and Stegun [1, see 22.3.6].

Explicit numbers for all $n \leqslant 20$ are given by Luke [13, p. 458]. Then the T-polynomials of E are given explicitly by

$$t_n(z)=\left(z-\frac{f_2+f_1}{2}\right)^n+\sum_{j=1}^{[n/2]}\alpha_{n-2j}\left(\frac{f_2-f_1}{2}\right)^{2j}\left(z-\frac{f_2+f_1}{2}\right)^{n-2j}. \quad (2.12)$$

It should be noticed that for ellipses E the polynomials given in (2.12) are at the same time also the so called Faber polynomials of E (cf. Gaier [10, p. 47]).

EXAMPLE 2.2. Define

$$D=\{z\in\mathbb{C}: |z-2|\leqslant 1\}\cup\{z\in\mathbb{C}: |z+2|\leqslant 1\}. \quad (2.13)$$

In this case the first degree T-polynomial with respect to D is

$$t_1(z)=z, \qquad z\in D, \quad (2.14)$$

and the only zero of t_1 is at $z_0=0\notin D$. However

$$t_2(z)=z^2-5, \qquad z\in D \quad (2.15)$$

has its zeros $z_0=-\sqrt{5}$, $z_1=+\sqrt{5}$ in D. It may be remarked that applications do occur where the approximation problems are treated on unconnected sets (cf. de Boor and Rice [6] and Fuchs [9]). A slightly more general problem than stated so far is to find a vector $\zeta=(z_0, z_1, \ldots, z_n)\in\mathbb{C}^{n+1}$ of knots such that for a given vector $\nu=(\nu_0, \nu_1, \ldots, \nu_n)\in\mathbb{N}^{n+1}$ of natural numbers the uniform norm of a polynomial ω_ν defined by

$$\omega_\nu(z, \zeta)=\prod_{j=0}^{n}(z-z_j)^{\nu_j}, \qquad z\in D, \quad (2.16)$$

is minimal. This allows for multiple knots in the mentioned interpolation problem.

The problem of minimizing the norm of ω_ν by appropriate choice of the knots ζ was recently treated for the real case by Bojanov [4] under the additional requirement that the knots are pairwise distinct. Related problems for the real case were investigated by Boyanov [5], Barrar and Loeb [2], and Barrar, Loeb and Werner [3].

A polynomial ω_ν with the described minimal property will be called *generalized Chebyshev polynomial* or generalized T-polynomial.

By the following example it is shown that in the complex case we cannot require the knots to be pairwise distinct, or in other words the requirement of distinct knots would imply that a generalized T-polynomial will not necessarily exist.

EXAMPLE 2.3. The uniform norm of the polynomial ω defined by

$$\omega(z, \zeta) = (z - z_0)(z - z_1), \qquad z \in D = \{z : |z| \leqslant 1\}$$

is minimal if and only if $z_0 = z_1 = 0$. The proof is elementary and omitted.

THEOREM 2.1. *Let $\nu = (\nu_0, \nu_1, \dots, \nu_n) \in \mathbb{N}^{n+1}$ be given. Then there exists a vector $\hat{\zeta} = (\hat{z}_0, \hat{z}_1, \dots, \hat{z}_n) \in \mathbb{C}^{n+1}$ such that*

$$\|\omega_\nu(\cdot, \hat{\zeta})\|_\infty \leqslant \|\omega_\nu(\cdot, \zeta)\|_\infty \qquad \textit{for all } \zeta \in \mathbb{C}^{n+1}.$$

If H is the convex hull of D in $\mathbb{C}$, then $\hat{z}_j \in H$ for all $j = 0, 1, \dots, n$.

Proof. Let $\Theta = (0, 0, \dots, 0) \in \mathbb{C}^{n+1}$. The set

$$K = \{\zeta \in \mathbb{C}^{n+1} : \|\omega_\nu(\cdot, \zeta)\|_\infty \leqslant \|\omega_\nu(\cdot, \Theta)\|_\infty\}$$

is nonempty and compact in $\mathbb{C}^{n+1}$ and $\|\omega_\nu(\cdot, \zeta)\|_\infty$ as a function of $\zeta \in \mathbb{C}^{n+1}$ is continuous on K. Therefore a vector $\hat{\zeta}$ which minimizes $\|\omega_\nu(\cdot, \zeta)\|_\infty$ exists. Thus the existence of generalized T-polynomials is established. To show that all $\hat{z}_j \in H$, $j = 0, 1, \dots, n$, we assume the contrary, namely, that one of the $\hat{z}_j$, say $\hat{z}_0$ is outside of H. Since H is compact and convex there is a uniquely determined point $u \in H$ with

$$0 < |\hat{z}_0 - u| \leqslant |\hat{z}_0 - z| \qquad \text{for all } z \in H.$$

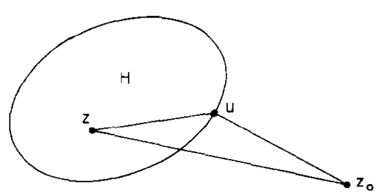

FIG. 1. Zero z_0 outside of H.

In addition, since H can be separated strictly from $\{\hat{z}_0\}$ (see Fig. 1) we have

$$|z-u| < |z-\hat{z}_0| \qquad \text{for all } z \in H. \tag{2.17}$$

Let

$$\omega_\nu(z, \hat{\zeta}) = \prod_{j=0}^{n} (z-\hat{z}_j)^{\nu_j},$$

then there is a $\hat{\hat{z}} \in D$ with

$$\begin{aligned} |\hat{\hat{z}}-u|^{\nu_0} \prod_{j=1}^{n} |\hat{\hat{z}}-z_j|^{\nu_j} &= \max_{z\in D} \left\{ |z-u|^{\nu_0} \prod_{j=1}^{n} |z-\hat{z}_j|^{\nu_j} \right\} \\ &\geqslant \max_{z\in D} \prod_{j=0}^{n} |z-\hat{z}_j|^{\nu_j} \geqslant \prod_{j=0}^{n} |z-\hat{z}_j|^{\nu_j} \qquad \text{for all } z \in D. \end{aligned} \tag{2.18}$$

By letting $z = \hat{\hat{z}}$ in (2.18) we obtain

$$|\hat{\hat{z}}-u|^{\nu_0} \prod_{j=1}^{n} |\hat{\hat{z}}-\hat{z}_j|^{\nu_j} \geqslant |\hat{\hat{z}}-\hat{z}_0|^{\nu_0} \prod_{j=1}^{n} |\hat{\hat{z}}-\hat{z}_j|^{\nu_j}$$

which contradicts (2.17) in case $\prod_{j=1}^{n} |\hat{\hat{z}}-\hat{z}_j|^{\nu_j} > 0$. If, however, $\prod_{j=1}^{n} |\hat{\hat{z}}-\hat{z}|^{\nu_j} = 0$, then ω_ν vanishes identically, which implies $D \subset \{z_0, z_1, \dots, z_n\}$, a case which we have excluded.

If all ν_j, $j = 0, 1, \dots, n$, are the same, then the problem of finding the corresponding generalized T-polynomial may be regarded as finding a best approximation of z^{n+1} by polynomials of degree n or less. This problem has a unique solution. However, in general the unicity problem is open here.

It is easy to see that $t_n(z) = z^n$ is the nth T-polynomial with respect to all disks centered at zero. An even sharper result, given by Geiger and Opfer [7], says that $t_n(z) = z^n$ is the nth T-polynomial with respect to any sector.

$$S^\alpha = \{z \in \mathbb{C} : |z| \leqslant 1, |\arg z| \leqslant \alpha\}, \tag{2.19}$$

provided $\alpha \geqslant n\pi/(n+1)$. In example 2.2 we have also observed that $t_1(z) = z$ is the T-polynomial for the unconnected set D defined in (2.13). Also the ordinary T-polynomials are T-polynomials on a whole family of sets (see Example 2.1).

On the other hand, Rivlin [18] established that any monic polynomial (i.e., the coefficient of the highest power is one) is *the* T-polynomial for a certain set D.

It seems therefore natural to pose the following:

Problem 2.1. Let $t_n(z) = z^n + p(z)$, $p \in \Pi_{n-1}$, $n \in \mathbb{N}$ be any given polynomial.

(A) Characterize (or determine explicitly) those compact sets $D \subset \mathbb{C}$ for which t_n is the nth T-polynomial with respect to D.

(B) Problem A with additional restriction that D is convex.

(C) Consider problems A and B for the special case when $p \in \Pi_{n-1}^{\mathbb{R}}$.

As mentioned earlier a compact set D which is symmetric with respect to the real axis yields real T-polynomials. To see that the converse is not true, assume that $t_n(z) = z^n$ is the T-polynomial with respect to a compact set D which is not a disk centered at zero. Examples of such a set D are the sectors S^α introduced in (2.19) for $\alpha \in [\pi n/(n+1), \pi]$ (cf. Geiger and Opfer [11]).
An application of formula (2.10) shows that $t_n(z) = z^n$ is also the T-polynomial for $e^{i\phi}D$ for all angles $\phi \in [0, 2\pi]$. Since D is not a disk, the sets $e^{i\phi}D$ cannot be symmetric with respect to the real axis for all ϕ.

DEFINITION 2.1. Let $t_n(z) = z^n + p(z)$, $p \in \Pi_{n-1}$, $n \in \mathbb{N}$, be a monic polynomial. All compact sets $D \subset \mathbb{C}$ which have the property that t_n is the T-polynomial with respect to D will be called *Chebyshev sets* with respect to t_n. The family of all Chebyshev sets with respect to t_n will be called *Chebyshev cluster* with respect to t_n.

It seems even difficult to determine the Chebyshev cluster $\mathscr{C}$ with respect to $t_1(z) = z$. Clearly all compact sets D which are symmetric with respect to the real and imaginary axes belong to $\mathscr{C}$. Also all sectors $S^\alpha \in \mathscr{C}$ for $\alpha \in (\pi/2, \pi)$. It may be noted that S^α is not convex for $\alpha \in (\pi/2, \pi)$. If $D \in \mathscr{C}$, then $a \cdot D \in \mathscr{C}$ for all $a \in \mathbb{C}$, $a \neq 0$.

Let $f\colon D \subset \mathbb{C} \to \mathbb{C}$ be any continuous function. Then

$$E_f = \{z \in D : |f(z)| = \|f\|_\infty\} \tag{2.20}$$

will be called the *set of extremal points of f*. The number and distribution of extremal points is studied by Grothkopf and Opfer [12] for T-polynomials on sector S^α.

3. Optimal Knots on Triangles

There are two aspects of choosing knots. One is the purely interpolating aspect to reduce the error in (2.1) by an appropriate choice of the zeros of $\omega(z, \zeta)$ as defined in (2.2).

The other aspect is relevant if we want to use the knots for constructing splines. In that case we have to impose the additional side condition that the knots are on the boundary of the corresponding meshes.

If, for example, the underlying set D is a triangle, then Theorem 2.1 implies that all zeros of the corresponding T-polynomial with respect to D

are in D. However, the zeros will usually not be on the boundary ∂D of D as can be seen in the following:

EXAMPLE 3.1. Let D be any triangle. The first degree T-polynomial $t_1(z) = z - a$ is obtained either by letting a be the center of the circumscribed circle about D (in case D is acute, i.e., all angles are less or equal to $\pi/2$), or by letting a be the midpoint of the longest side of D (in case D is obtuse). It may also be noted that in the acute case t_1 has three extremal points, namely, the vertices of D, whereas in the obtuse case there are only two extremal points, namely, the endpoints of the longest side of D.

If we want to construct splines on triangular grids as explained earlier, then the knots must be on the edges of the corresponding triangles which imposes additional side conditions. Usually the desired number of knots is a multiple of three since we want to place the same number of knots on each edge of the triangles.

Let us now study the problem for the triangles

$$D = \text{convex hull}\{0, e^{i\alpha}, e^{-i\alpha}\}, \qquad \alpha \in [0, \pi/2]. \tag{3.1}$$

Let $t_1(z) = z - a$ be the first T-polynomial with respect to D^α. Then

$$\begin{aligned} a &= 1/(2\cos\alpha) && \text{for} \quad 0° \leqslant \alpha \leqslant 45° \\ &= \cos\alpha && \text{for} \quad 45° \leqslant \alpha \leqslant 90°, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|T_1\|_\infty &= 1/(2\cos\alpha) && \text{for} \quad 0° \leqslant \alpha \leqslant 45° \\ &= \sin\alpha && \text{for} \quad 45° \leqslant \alpha \leqslant 90°. \end{aligned} \tag{3.3}$$

It may be noted that the coefficient a as a function of α is not differentiable at $\alpha = 45°$, whereas $\|T_1\|_\infty$ is differentiable in $(0, \pi/2)$ with respect to α.

If we call t_n^α the nth T-polynomial with respect to D^α for $\alpha \in [0, \pi/2]$, then t_n^α may be regarded as a homotopy from t_n^0 to $t_n^{\pi/2}$. The T-polynomial t_n^0 is the usual T-polynomial (up to normalization) on $[0, 1]$ and $t_n^{\pi/2}$ is the T-polynomial with respect to the segment $[-i, i]$ which can easily be determined from the ordinary T-polynomials on $[-1, 1]$ just by changing all coefficients into their absolute values. This follows from (2.10).

Thus the number of extremal points of t_n will start with $n+1$ for $\alpha = 0$ and will end with the same number for $\alpha = \pi/2$. Even for a small n explicit expressions for the nth T-polynomials t_n with respect to α are difficult to obtain. The additional requirement that the zeros of t_n should be on ∂D^α makes the problem even more complicated.

Thus we have made an attempt to find the optimal knots for the interesting cases $n = 3$ and $n = 6$ by numerical means. The T-polynomials

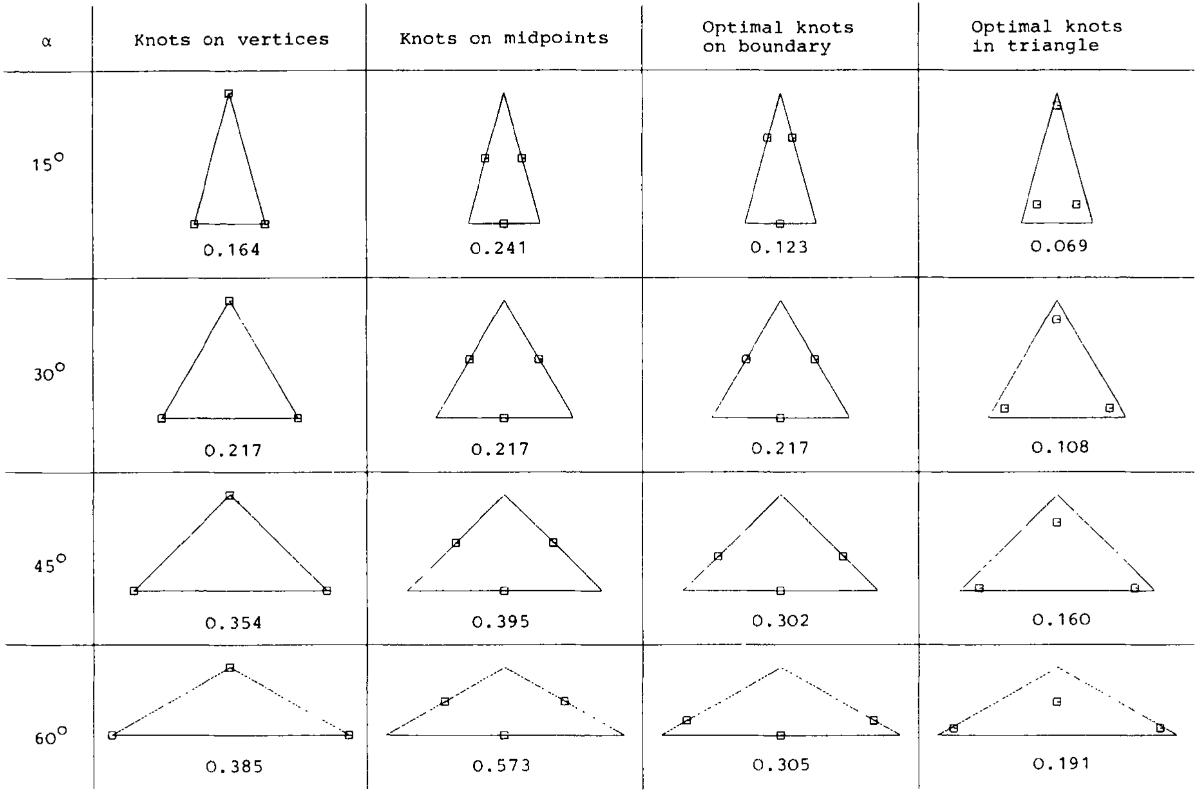

FIG. 2. Location of knots z_1, z_2, z_3 in $T_3(z) = (z - z_1)(z - z_2)(z - z_3)$ and $\|T_3\|_\alpha$ for different cases.

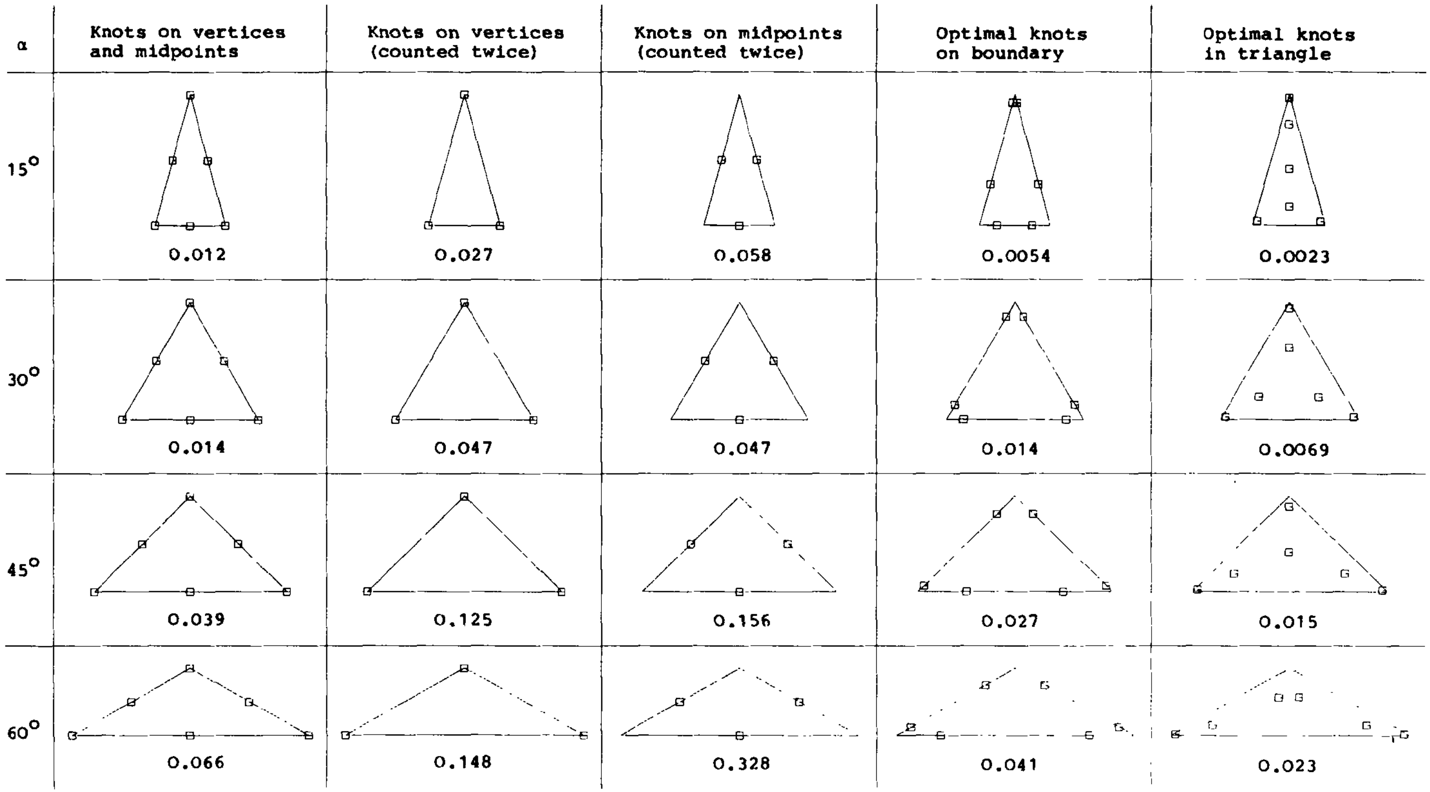

FIG. 3. Location of knots $z_1, z_2, \ldots, z_6$ in $T_6(z) = \prod_{j=1}^{6} (z - z_j)$ and $\|T_6\|_\infty$ for different cases.

were computed by a method which is described in Grothkopf and Opfer [12]. Optimal knots on the boundary ∂D (which are not necessarily unique) were computed by a direct search method.

We selected the values $\alpha = 15°, 30°, 45°, 60°$, including the equilateral and the rectangular isosceles triangle. The results are displayed in Figs. 2 and 3.

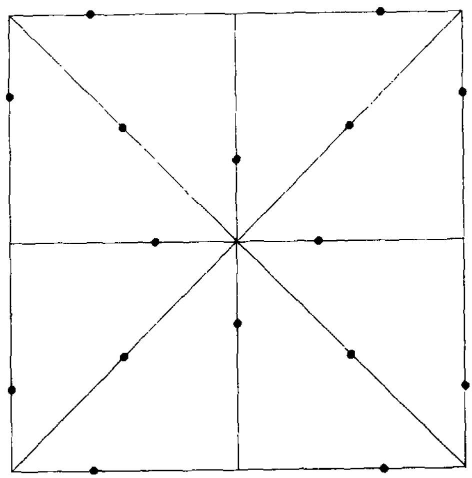

FIG. 4. "Union Jack" triangulation, optimal knots coincide on common edges of neighboring triangles

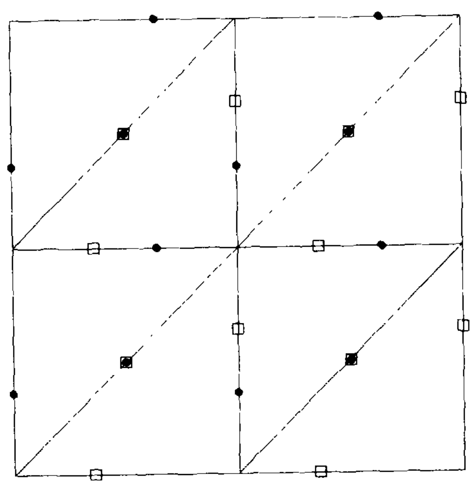

FIG. 5. Triangulation with noncoinciding optimal knots.

If one wants to use the optimal knots in the grids, then the so called "Union Jack" triangulation of a rectangular grid of congruent rectangles is well suited (see Fig. 4).

On the other hand, an unsymmetric triangulation of the same rectangular grid cannot be used since, on a common edge of two triangles, the optimal knots will, in general, not coincide (see Fig. 5).

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