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Solvability of Volterra-Stieltjes Operator-Integral Equations and Their Applications

J. BANAŚ
Department of Mathematics, Rzeszów University of Technology
35-959 Rzeszów, W. Pola 2, Poland

K. SADARANGANI
Department of Mathematics, University of Las Palmas de Gran Canaria
Campus Universitario de Tafira, 35017 Las Palmas de Gran Canaria, Spain

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Abstract—We investigate a class of operator-integral equations of Volterra-Stieltjes type and we study the solvability of those equations in the space of continuous functions. Equations in question create a generalization of numerous integral equations considered in nonlinear analysis. The main tool used in our considerations is the technique associated with measures of noncompactness. We show the applicability of our existence result in the study of a few integral equations of Volterra type.

Keywords—Stieltjes integral, Functions of bounded variation, Integral equation, Measure of noncompactness, Darbo condition.

1. INTRODUCTION

Integral equations create a very important and significant part of mathematical analysis and its applications to real world problems (cf. [1–6], among others). The theory of integral equations is now very developed with help of various tools of functional analysis, topology, and fixed-point theory, for example.

In this paper, we are going to investigate a class of operator-integral equations of Volterra-Stieltjes type which create a generalization of numerous integral equations appearing in mathematical literature. For example, this class covers linear Volterra integral equations, several types of nonlinear Volterra integral equations and a class of the so-called quadratic integral equations of Volterra-Stieltjes type, among others. The mentioned integral equations are frequently used in the description of many real world problems in engineering, physics, biology, and economics, among others (cf. [3–5,7,8]).

We will show that the operator-integral equations investigated here are solvable in the space of continuous functions on some closed bounded interval. This class is sufficient in our study because in applications one seeks mainly continuous solutions of considered equations. The main

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tool used in our study is associated with the technique of measures of noncompactness. This technique is recently successfully applied in several branches of nonlinear analysis (see [5,9–11] and references therein). We will also show how our existence result may be applied to concrete types of nonlinear Volterra integral equations.

The results of this paper generalize a lot of ones obtained up to now. Especially, we extend the results from the paper [12], where the so-called quadratic integral equations of Urysohn-Stieltjes type were treated.

2. AUXILIARY FACTS AND RESULTS

In this section, we give a collection of results which will be needed further on. At the beginning, we start with some facts concerning functions of bounded variation and Stieltjes integral.

If x is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b x$ indicates the variation of x on $[a,b]$. We say that x is of bounded variation whenever $\bigvee_a^b x$ is finite. If $u(t,s) = u : [a,b] \times [c,d] \to \mathbb{R}$, then we denote by $\bigvee_{t=p}^q u(t,s)$ the variation of the function $t \to u(t,s)$ on the interval $[p,q] \subset [a,b]$, where s is arbitrarily fixed in $[c,d]$. In the similar way, we define $\bigvee_{s=p}^q u(t,s)$. We refer to [13,14] for the properties of functions of bounded variation.

If x and φ are two real bounded functions defined on the interval $[a,b]$, then under some additional conditions [14], we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b x(t)\, d\varphi(t)$$

of the function x with respect to φ. In this case, we say that x is Stieltjes integrable on $[a,b]$ with respect to φ. There are known several conditions guaranteeing the Stieltjes integrability [14,15]. One of the most frequently used requires that x is continuous and φ is of bounded variation on the interval $[a,b]$.

The properties of the Stieltjes integral used in the sequel are contained in below-given lemmas (cf. [16]).

LEMMA 2.1. *If x is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then*

$$\left| \int_a^b x(t)\, d\varphi(t) \right| \le \left(\sup_{a \le t \le b} |x(t)| \right) \bigvee_a^b \varphi.$$

Moreover, the following inequality holds:

$$\left| \int_a^b x(t)\, d\varphi(t) \right| \le \int_a^b |x(t)|\, d\left(\bigvee_a^t \varphi \right).$$

LEMMA 2.2. *Let x_1, x_2 be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ and such that $x_1(t) \le x_2(t)$ for $t \in [a,b]$. Then*

$$\int_a^b x_1(t)\, d\varphi(t) \le \int_a^b x_2(t)\, d\varphi(t).$$

In what follows, we will also consider the Stieltjes integral of the form

$$\int_a^b x(t)\, d_s g(t,s),$$

where $g : [a, b] \times [a, b] \to \mathbb{R}$ and the symbol d_s indicates the integration with respect to s. The details concerning the integral of this type will be described later.

In nonlinear analysis there is frequently used the so-called superposition operator F, defined by the formula $(Fx)(t) = f(t, x(t))$, where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function and x is an arbitrary real function defined on $[a, b]$. For the properties of this operator, we refer to [17].

Now assume that x is a real function defined on $[a, b]$. Then by $\omega(x, \varepsilon)$, we define the modulus of continuity of the function x, i.e.,

$$\omega(x, \varepsilon) = \sup[|x(t) - x(s)| : t,\ s \in [a, b],\ |t - s| \le \varepsilon].$$

If $p(t, s) = p : [a, b] \times [c, d] \to \mathbb{R}$, then the formula

$$\omega(p, \varepsilon) = \sup[|p(t, s) - p(u, v)| : t, u \in [a, b],\ s, v \in [c, d],\ |t - u| \le \varepsilon,\ |s - v| \le \varepsilon]$$

defines the modulus of continuity of the function $p(t, s)$ with respect to both variables t and s. We can also use the modulus of continuity of $p(t, s)$ with respect to one variable. For example,

$$\omega(p(t, \cdot), \varepsilon) = \sup[|p(t, s) - p(t, u)| : s, v \in [c, d],\ |s - v| \le \varepsilon],$$

where t is a fixed number in the interval $[a, b]$.

Now we give a few facts concerning measures of noncompactness.

Assume that $(E, \|\cdot\|)$ is a given Banach space. Denote by $B(x, r)$ the closed ball centered at x and with radius r. For a given nonempty bounded subset X of E, we denote by $\chi(X)$ the so-called Hausdorff measure of noncompactness of X. This quantity is defined by the formula

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon- \text{ net in } E\}.$$

Let us mention that the concept of a measure of noncompactness may be defined in other way than that given above (cf. [9,10]). Nevertheless, the Hausdorff measure χ seems to be the most useful and important in application. It is caused mainly by the fact that in some Banach spaces this measure can be expressed by useful and handy formulas. For example, let $C[a, b]$ be the space consisting of all real functions defined and continuous on the interval $[a, b]$ with the standard maximum norm. Then, for a nonempty and bounded subset X of $C[a, b]$, we have [10]

$$\chi(X) = \frac{1}{2}\omega_0(X),$$

where $\omega_0(X) = \lim_{\varepsilon \to 0}\{\sup[\omega(x, \varepsilon) : x \in X]\}$.

Now, let us suppose that M is a nonempty subset of a Banach space E and $T : M \to E$ is a continuous operator that transforms bounded sets onto bounded ones. We say that T satisfies the Darbo condition (with a constant $k \ge 0$) if for any bounded subset X of M the following inequality holds:

$$\chi(TX) \le k\chi(X).$$

If T satisfies the Darbo condition with $k < 1$, then it is said to be a contraction with respect to χ.

THEOREM 2.1. *(See [18].) Let Ω be a nonempty bounded closed convex subset of E and let $T : \Omega \to \Omega$ be a contraction with respect to χ. Then T has at least one fixed point in the set Ω.*

3. MAIN RESULT

In this section, we will study the solvability of the following operator-integral equation of Volterra-Stieltjes type

$$x(t) = h(t) + (Tx)(t) \int_0^t u(t, s, x(s))\, d_s g(t, s), \tag{3.1}$$

where $t \in [0,1] = I$ and T is an operator acting from the space $C(I)$ into itself.

In our investigations, we assume that the functions involved in equation (3.1) satisfy the following conditions:

(i) $h \in C(I)$;

(ii) the operator $T : C(I) \to C(I)$ is continuous and satisfies the Darbo condition with a constant Q;

(iii) there exist nonnegative constants a and b such that

$$|(Tx)(t)| \leq a + b\|x\|,$$

for each $t \in I$ and $x \in C(I)$;

(iv) $g : I \times I \to \mathbb{R}$ and the function $s \to g(t, s)$ is of bounded variation on I for each $t \in I$;

(v) for every $\varepsilon > 0$, there is $\delta > 0$ such that for $t_1, t_2 \in I,\ t_1 < t_2$, and $t_2 - t_1 \leq \delta$ the following inequality holds:

$$\bigvee_{s=0}^{1} [g(t_2, s) - g(t_1, s)] \leq \varepsilon;$$

(vi) the function $s \to g(t, s)$ is continuous on I for any $t \in I$;

(vii) $u : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|u(t, s, x)| \leq f(|x|),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$.

Now let us observe that, from Assumptions (iv) and (v), it follows [19] that the function

$$t \to \bigvee_{s=0}^{1} g(t, s)$$

is continuous on the interval I. This implies that there exists a finite constant K such that

$$K = \sup \left\{ \bigvee_{s=0}^{1} g(t, s) : t \in I \right\}.$$

Further, let us assume additionally that

(viii) there exists a positive solution $r = r_0$ of the inequality

$$\|h\| + K(a + br) f(r) \leq r,$$

such that $KQf(r_0) < 1$.

Now we can formulate our main existence result.

THEOREM 3.1. *Under Assumptions (i)–(viii), there exists at least one solution $x = x(t)$ of equation (3.1).*

Before we proceed to the proof, we notice the following remark which will be used in it.

REMARK 3.1. Let us note that, under Assumption (iv) Hypothesis (vi), is equivalent to the condition claiming that the function

$$p \to \bigvee_{s=0}^{p} g(t,s)$$

is continuous on the interval I for any $t \in I$ (cf. [14,15], for instance).

PROOF. At the beginning, let us denote

$$M(\varepsilon) = \sup\left\{ \bigvee_{s=0}^{1} [g(t_2,s) - g(t_1,s)] : t_1, t_2 \in I,\ t_1 < t_2,\ t_2 - t_1 \le \varepsilon \right\}.$$

In view of Hypothesis (v), we deduce that $M(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Further, for any $x \in C(I)$ and $t \in I$, let us denote

$$(Ux)(t) = \int_0^t u(t,s,x(s))\, d_s g(t,s),$$

$$(Fx)(t) = h(t) + (Tx)(t) \cdot (Ux)(t).$$

Next, fix arbitrarily $\varepsilon > 0$ and take $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \le \varepsilon$. Then, in view of our assumptions, for an arbitrary function $x \in C(I)$, we obtain

$$\begin{aligned}
&|(Ux)(t_2) - (Ux)(t_1)| \\
&\le \left| \int_0^{t_2} u(t_2,s,x(s))\, d_s g(t_2,s) - \int_0^{t_1} u(t_2,s,x(s))\, d_s g(t_2,s) \right| \\
&\quad + \left| \int_0^{t_1} u(t_2,s,x(s))\, d_s g(t_2,s) - \int_0^{t_1} u(t_1,s,x(s))\, d_s g(t_2,s) \right| \\
&\quad + \left| \int_0^{t_1} u(t_1,s,x(s))\, d_s g(t_2,s) - \int_0^{t_1} u(t_1,s,x(s))\, d_s g(t_1,s) \right| \\
&\le \int_{t_1}^{t_2} |u(t_2,s,x(s))|\, d_s \left(\bigvee_{p=0}^{s} g(t_2,p) \right) \\
&\quad + \int_0^{t_1} |u(t_2,s,x(s)) - u(t_1,s,x(s))|\, d_s \left(\bigvee_{p=0}^{s} g(t_2,p) \right) \\
&\quad + \int_0^{t_1} |u(t_1,s,x(s))|\, d_s \left(\bigvee_{p=0}^{s} [g(t_2,p) - g(t_1,p)] \right) \\
&\le f(\|x\|) \int_{t_1}^{t_2} d_s \left(\bigvee_{p=0}^{s} g(t_2,p) \right) + \int_0^{t_1} |u(t_2,s,x(s)) - u(t_1,s,x(s))|\, d_s \left(\bigvee_{p=0}^{s} g(t_2,p) \right) \\
&\quad + f(\|x\|) \int_0^{t_1} d_s \left(\bigvee_{p=0}^{s} [g(t_2,p) - g(t_1,p)] \right).
\end{aligned}$$

Now, let us denote

$$\omega(\varepsilon) = \sup\{|u(t_2,s,y) - u(t_1,s,y)| : t_1,t_2,s \in I,\ |t_2 - t_1| \le \varepsilon,\ y \in [-\|x\|, \|x\|]\}.$$

Then we get

$$\begin{aligned} |(Ux)(t_2) - (Ux)(t_1)| &\le f(\|x\|)\left[\bigvee_{s=0}^{t_2} g(t_2,s) - \bigvee_{s=0}^{t_1} g(t_2,s)\right] \\ &\quad + \omega(\varepsilon)\bigvee_{s=0}^{t_1} g(t_2,s) + f(\|x\|)\bigvee_{s=0}^{t_1}[g(t_2,s) - g(t_1,s)] \\ &\le f(\|x\|)\left[\bigvee_{s=0}^{t_2} g(t_2,s) - \bigvee_{s=0}^{t_1} g(t_2,s)\right] + \omega(\varepsilon)\bigvee_{s=0}^{1} g(t_2,s) + f(\|x\|)M(\varepsilon). \end{aligned} \tag{3.2}$$

Observe that, in virtue of the uniform continuity of the function u on the set $I \times I \times [-\|x\|, \|x\|]$, we have that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Linking this fact with Remark 3.1, we can easily deduce from estimate (3.2) that the function Ux is continuous on the interval I.

Hence, keeping in mind Assumptions (i) and (ii), we conclude that $Fx \in C(I)$, i.e., the operator F maps the space $C(I)$ into itself.

Next, let us notice that in order to show that the operator F is continuous on the space $C(I)$ it is sufficient to prove (in view of (ii)) the continuity of U on $C(I)$). To do this, fix $\varepsilon > 0$ and take arbitrary $x, y \in C(I)$ with $\|x - y\| \le \varepsilon$. Put $P = \|x\| + \varepsilon$.

Then, using Lemmata 2.1 and 2.2, we have

$$\begin{aligned} |(Ux)(t) - (Uy)(t)| &\le \int_0^t |u(t,s,x(s)) - u(t,s,y(s))|\, d_s\left(\bigvee_{p=0}^{s} g(t,p)\right) \\ &\le \int_0^1 |u(t,s,x(s)) - u(t,s,y(s))|\, d_s\left(\bigvee_{p=0}^{s} g(t,p)\right). \end{aligned}$$

Hence, if we denote

$$\omega_P(u(t,s,\cdot),\varepsilon) = \sup\{|u(t,s,v) - u(t,s,q)| : v,q \in [-P,P],\ |v - q| \le \varepsilon\},$$

from the previously written estimate, we obtain

$$\begin{aligned} |(Ux)(t) - (Uy)(t)| &\le \int_0^1 \omega_P(u(t,s,\cdot),\varepsilon)\, d_s\left(\bigvee_{z=0}^{s} g(t,z)\right) \\ &\le \sup\{\omega_P(u(t,s,\cdot),\varepsilon) : t,s \in I\}\bigvee_{s=0}^{1} g(t,s) \\ &\le K \sup\{\omega_P(u(t,s,\cdot),\varepsilon) : t,s \in I\}. \end{aligned}$$

Thus, taking into account the uniform continuity of the function $u(t,s,x)$ on the set $I \times I \times [-P,P]$, we derive that U is continuous on the space $C(I)$.

Now, let us fix an arbitrary $x \in C(I)$. Then, in view of our assumptions, Lemmata 2.1 and 2.2, we get

$$\begin{aligned}|(Fx)(t)| &\le |h(t)| + |(Tx)(t)| \int_0^t |u(t,s,x(s))| \, d_s \left(\bigvee_{p=0}^{s} g(t,p) \right) \\ &\le \|h\| + (a + b\|x\|) \int_0^1 f(\|x\|) \, d_s \left(\bigvee_{p=0}^{s} g(t,p) \right) \\ &\le \|h\| + (a + b\|x\|) f(\|x\|) \bigvee_{s=0}^{1} g(t,s) \\ &\le \|h\| + (a + b\|x\|) f(\|x\|) \cdot K.\end{aligned}$$

This implies

$$\|Fx\| \le \|h\| + K(a + b\|x\|) f(\|x\|).$$

Hence, in view of Assumption (viii), we deduce that $F : B(\Theta, r_0) \to B(\Theta, r_0)$, where r_0 is a positive solution of the inequality $\|h\| + K(a + br) f(r) \le r$ such that $KQf(r_0) < 1$.

In what follows, let us take a nonempty subset X of the ball $B(\Theta, r_0)$ and $x \in X$. Then, for a fixed $\varepsilon > 0$ and $t_1, t_2 \in I$, $t_1 < t_2$ such that $t_2 - t_1 \le \varepsilon$, in view of estimate (3.2), we obtain

$$\begin{aligned}&|(Fx)(t_2) - (Fx)(t_1)| \le |h(t_2) - h(t_1)| \\ &\quad + |(Tx)(t_2)(Ux)(t_2) - (Tx)(t_2)(Ux)(t_1)| + |(Tx)(t_2)(Ux)(t_1) - (Tx)(t_1)(Ux)(t_1)| \\ &\le \omega(h,\varepsilon) + |(Tx)(t_2)| \cdot |(Ux)(t_2) - (Ux)(t_1)| + |(Ux)(t_1)| \cdot |(Tx)(t_2) - (Tx)(t_1)| \\ &\le \omega(h,\varepsilon) + (a + b\|x\|) \left[f(\|x\|) \left\{ \bigvee_{s=0}^{t_2} g(t_2,s) - \bigvee_{s=0}^{t_1} g(t_2,s) \right\} \right. \\ &\quad \left. + \omega(\varepsilon) \bigvee_{s=0}^{1} g(t_2,s) + f(\|x\|) M(\varepsilon) \right] + \|Ux\| \omega(Tx,\varepsilon) \le \omega(h,\varepsilon) \\ &\quad + (a + br_0) \left[f(r_0) \left\{ \bigvee_{s=0}^{t_2} g(t_2,s) - \bigvee_{s=0}^{t_1} g(t_2,s) \right\} + K\omega(\varepsilon) + f(r_0) M(\varepsilon) \right] \\ &\quad + Kf(r_0)\omega(Tx,\varepsilon).\end{aligned}$$

Hence, keeping in mind the properties of the terms appearing in the above estimate which were established previously, we arrive at the following evaluation:

$$\omega_0(FX) \le Kf(r_0)\omega_o(TX) \le KQf(r_0)\omega_o(X).$$

Now, taking into account the choice of the number r_0 and Theorem 2.1, we complete the proof.

4. APPLICATIONS AND FINAL REMARKS

We are going to discuss here some special cases of the operator-integral equation of Volterra-Stieltjes type investigated in the previous section. Namely, we show that our existence result contained in Theorem 3.1 can be applied to some special integral equations.

At the beginning, we will discuss the same equation as before, i.e., equation (3.1) assuming that the function f appearing in Assumption (vii) has the form

$$f(x) = c + dx,$$

where $x \ge 0$ and c and d are nonnegative constants. In such a case, equation (3.1) is called quadratic integral equation of Volterra-Stieltjes type (cf. [8,12,20,21]). Observe that in the considered situation Assumption (viii) has the form:

(viii′) there exists a positive solution $r = r_0$ of the inequality

$$\|h\| + K(a + br)(c + dr) \le r$$

such that $KQ(c + dr_0) < 1$.

Of course, in this case, we have to impose some assumptions on the constants K, Q, a, b, c, d, and $\|h\|$ involved in the above inequalities. For example, if we restrict ourselves to the case $h = 0$ and $\alpha = a = b$, $\gamma = c = d$, then the assumption to be required has the form:

(viii″) $\alpha\gamma K \leq 1/4$ and moreover, either
 (1) $Q < 2\alpha$, or
 (2) $Q \geq 2\alpha$ and $Q - \alpha \geq Q^2 K\gamma$.

Now, let us consider the special case of equation (3.1) having the form

$$x(t) = h(t) + (Tx)(t) \int_0^t k(t,s)\varphi(s)x(s)\,ds. \tag{4.1}$$

This equation is a counterpart of the equation

$$x(t) = h(t) + (Tx)(t) \int_0^1 k(t,s)\varphi(s)x(s)\,ds$$

considered in [21].

Following the mentioned paper, we will assume the hypotheses:

(1) $h, \varphi \in C(I)$;
(2) T satisfies Assumptions (ii) and (iii) of Theorem 3.1;
(3) $k : I \times I \setminus \{(0,0)\} \to \mathbb{R}$ is continuous and for each $t \in I$ there exists the integral

$$\int_0^1 |k(t,s)|\,ds;$$

(4) there exists a bounded function $w : I \to \mathbb{R}$ with the property $w(0) = \lim_{t\to 0} w(t) = 0$ and such that

$$\int_0^1 |k(t_2,s) - k(t_1,s)|\,ds \leq w(|t_2 - t_1|),$$

for $t_1, t_2 \in I$.

Now, let us put

$$p = \sup\left\{\int_0^1 |k(t,s)|\,ds : t \in I\right\},$$
$$q = \sup\left\{\int_0^1 |k(t,s)\varphi(s)|\,ds : t \in I\right\}.$$

In view of the above assumptions, we have that $p, q < \infty$.

Further, we assume that:

(5) there exists a positive solution r of the inequalities

$$\|h\| + (a + br)qr \leq r \qquad \text{and} \qquad qQr < 1.$$

Then we can prove the following result.

THEOREM 4.1. *Under Assumptions (1)–(5), equation (4.1) is solvable in the space $C(I)$.*

PROOF. We check that the assumptions of Theorem 3.1 are satisfied.

Thus, let us put

$$g(t,s) = \int_0^s k(t,z)\,dz.$$

It may be shown that

$$\bigvee_{s=0}^{1} g(t,s) \le \int_0^1 |k(t,z)|\,dz \le p,$$

which means that Assumption (iv) is satisfied.

Further, taking $t_1, t_2 \in I$, $t_1 < t_2$, and an arbitrary partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of the interval I, it is easily seen that

$$\sum_{i=1}^{n} |[g(t_2,s_i) - g(t_1,s_i)] - [g(t_2,s_{i-1}) - g(t_1,s_{i-1})]| \le w(|t_2 - t_1|).$$

Linking the above estimate with Assumption (4), we derive that Assumption (v) of Theorem 3.1 is also satisfied.

Moreover, in view of the equi-integrability of integral [15] it is easy to deduce that Assumption (vi) is fulfilled.

Finally, let us observe that, for $u(t,s,x) = \varphi(s)x$, we have

$$|u(t,s,x)| \le \|\varphi\| \cdot |x|,$$

which implies that the function φ appearing in Assumption (vii) has the form

$$f(r) = \|\varphi\| r.$$

We omit other details.

As a special case of equation (4.1), we can consider the following Volterra counterpart of the famous Chandrasekhar quadratic integral equation:

$$x(t) = 1 + x(t) \int_0^t \frac{t}{t+s} \varphi(s) x(s)\,ds.$$

Indeed, here we can put $Tx = x$, $u(t,s,x) = \varphi(s) \cdot x$, $h = 1$, and

$$g(t,s) = \begin{cases} t \ln \dfrac{t+s}{t}, & \text{for } t \in (0,1], \\ 0, & \text{for } t = 0. \end{cases}$$

Then it is easily seen that the assumptions of Theorem 4.1 are satisfied. In fact, it is sufficient to take $Q = 1$, $q \le \|\varphi\| \ln 2$, $\|h\| = 1$, $K = \ln 2$, $a = c = 0$, $b = 1$, $d = \|\varphi\|$, and to choose φ in such a way that there exists a positive solution of the inequalities $1 + \|\varphi\| r^2 \ln 2 \le 2$, $\|\varphi\| r \ln 2 < 1$. Moreover, as the function w appearing in Assumption (4), we can take the modulus of continuity of the function $t \to g(t,1)$.

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