De Branges–Rovnyak spaces and Dirichlet spaces

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Abstract

Sarason has shown that the local Dirichlet spaces $D_\lambda$ may be considered as manifestations of de Branges–Rovnyak spaces $H(b)$, and has used this identification to give a new proof that the spaces $D_\lambda$ are star-shaped. We investigate which other Dirichlet spaces $D(\mu)$ arise as de Branges–Rovnyak spaces, and which other de Branges–Rovnyak spaces $H(b)$ are star-shaped. We also prove a transfer principle which represents $H(b)$-spaces inside $D_\lambda$.

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1. Introduction

The spaces now called de Branges–Rovnyak spaces were introduced by de Branges and Rovnyak in the appendix of [1] and further studied in [2]. Subsequently, thanks in large part to the work of Sarason [6–10], it was realized that these spaces have numerous connections with other topics in complex analysis and operator theory.

De Branges–Rovnyak spaces on the unit disk $\mathbb{D}$ are a family of subspaces $H(b)$ of the Hardy space $H^2$, parametrized by $b$ in the closed unit ball of $H^\infty$. We shall give the precise definition...
of $H(b)$ in Section 2. In general $H(b)$ is not closed in $H^2$, but it carries its own norm $\| \cdot \|_b$ making it a Hilbert space.

The general theory of $H(b)$-spaces subdivides into two cases, according to whether or not $b$ is an extreme point of the unit ball of $H^\infty$. Perhaps the most important examples of extreme $b$ are inner functions. If $b$ is inner, then it turns out that $H(b) = (bH^2)^\perp$, sometimes called the model space associated to $b$. These spaces have been studied extensively in the literature.

In this article we shall concentrate on the case where $b$ is not extreme. An interesting example is obtained by taking $b_\lambda(z) := (1 - \tau)z/(1 - \tau z)$, where $\lambda \in \Bbb T$, the unit circle, and $\tau = (3 - \sqrt{5})/2$. With this choice, it turns out that $H(b_\lambda) = D_\lambda$, the so-called local Dirichlet space at $\lambda$. The space $D_\lambda$ was studied in detail by Richter and Sundberg in [5], and the identification $H(b_\lambda) = D_\lambda$ is due to Sarason [9]. The underlying theme of the present paper is to investigate to what extent this example may be considered typical.

Both the local Dirichlet spaces $D_\lambda$ and the classical Dirichlet space $D$ are instances of a more general family of Dirichlet spaces $D(\mu)$, indexed by finite positive measures $\mu$ on the unit circle $\Bbb T$. Indeed, $D_\lambda = D(\delta_\lambda)$, where $\delta_\lambda$ is the unit mass at $\lambda$, and $D = D(m)$, where $m$ is normalized Lebesgue measure on $\Bbb T$. The spaces $D(\mu)$ first arose in [4], in connection with the problem of classifying the shift-invariant subspaces of $D$. For which measures $\mu$ does $D(\mu)$ arise as a de Branges–Rovnyak space $H(b)$? In Section 3 we shall show that only such measures are multiples of $\delta_\lambda$ ($\lambda \in \Bbb T$), at the same time recovering Sarason’s identification of the corresponding functions $b$.

The proof of this result is based on a formula for the inner products of monomials in $H(b)$. This is a special case of a formula, established in Section 4, for the $H(b)$-norm of functions holomorphic on a neighborhood of $\overline{D}$. To extend this further and treat general holomorphic functions in $\mathbb{D}$, we are led to consider the problem of approximation of a function $f$ by its expansions $f_r(z) := f(rz)$ ($r < 1$). The spaces $D_\lambda$ (and more generally $D(\mu)$) enjoy the property of being star-shaped, in the sense that $f_r$ always converges to $f$. Is the same true of de Branges–Rovnyak spaces $H(b)$? In [9], it is mentioned that a counterexample can be constructed, but, as far as we know, it has never been published. In Section 5 we shall provide two different families of counterexamples, as well as a sufficient condition for $H(b)$ to be star-shaped which covers the case $H(b_\lambda) = D_\lambda$.

In Section 6 we prove a transfer principle which represents $H(b)$ inside $D_\lambda$. Thus, despite our results to the effect that $H(b)$ is almost never a local Dirichlet space, it can always be represented inside such a space.

The paper concludes with some open problems.

2. Background

2.1. De Branges–Rovnyak spaces

For $\chi \in L^\infty(\Bbb T)$, the Toeplitz operator $T_\chi : H^2 \to H^2$ is defined by $T_\chi f := P_+ (\chi f)$, where $P_+ : L^2(\Bbb T) \to H^2$ is the canonical projection. Given $b$ in the unit ball of $H^\infty$, the de Branges–Rovnyak space $H(b)$ is the image of $H^2$ under the operator $(I - T_b T_b^*)^{1/2}$. We define an inner product on $H(b)$ so as to make $(I - T_b T_b^*)^{1/2}$ an isometry from $H^2$ onto $H(b)$, namely

$$\langle (I - T_b T_b^*)^{1/2} f, (I - T_b T_b^*)^{1/2} g \rangle_b := \langle f, g \rangle_2 \quad (f, g \in (\ker(I - T_b T_b^*)^{1/2})^\perp).$$
The norm of $f$ in $\mathcal{H}(b)$ is denoted by $\|f\|_b$. The space $\mathcal{H}(b)$ is a reproducing kernel Hilbert space, with reproducing kernel

$$k^b_w(z) := \frac{1 - \overline{b(w)}b(z)}{1 - w\overline{z}} \quad (z, w \in \mathbb{D}).$$

For example, if $b \equiv 0$, then $\mathcal{H}(b) = H^2$, and if $b$ is inner, then $\mathcal{H}(b) = (bH^2)^\perp$, the model subspace of $H^2$. The book [7] contains a wealth of information about the spaces $\mathcal{H}(b)$.

As mentioned in the introduction, the study of de Branges–Rovnyak spaces is governed by a fundamental dichotomy, namely whether or not $b$ is an extreme point of the unit ball of $H^\infty$ (see [7, Chapters IV and V]). For instance, $\mathcal{H}(b)$ contains all functions holomorphic in a neighborhood of $\mathbb{D}$ if and only if $b$ is non-extreme [7, Theorem V-1].

In what follows, we are only interested in the non-extreme case. According to a well-known theorem [3, p. 138], the function $b$ is non-extreme if and only if $\log(1 - |b|^2) \in L^1(\mathbb{T})$. In this case, there is an outer function $a \in H^\infty$ for which $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Multiplying $a$ by a constant, we may further suppose that $a(0) > 0$, and $a$ is then uniquely determined. Following Sarason [8], we call $(b,a)$ a pair.

Using the pair $(b,a)$, we can express the norm in $\mathcal{H}(b)$ in terms of two $H^2$-norms.

**Theorem 2.1.** (See [6, Lemma 2, p. 77].) Let $(b,a)$ be a pair. A function $f \in H^2$ belongs to $\mathcal{H}(b)$ if and only if $T_b f$ belongs to $T_\overline{a} H^2$. In this case there exists a unique function $f^+ \in H^2$ such that $T_b f = T_\overline{a} f^+$, and

$$\|f\|_b^2 = \|f\|_2^2 + \|f^+\|_2^2.$$  

Many properties of $\mathcal{H}(b)$ can be expressed in terms of the pair $(b,a)$ and more particularly, in terms of the quotient $\phi := b/a$. Notice that $\phi \in N^+$, the Smirnov space. Conversely, for every function $\phi \in N^+$, there exists a unique pair $(b,a)$ such that $\phi = b/a$ [10, Proposition 3.1].

We next consider Toeplitz operators with unbounded symbols. Given $\phi$ holomorphic on $\mathbb{D}$, we define $T_\phi$ to be the operator of multiplication by $\phi$ on the domain $D(T_\phi) := \{f \in H^2 : \phi f \in H^2\}$. The bounded analytic Toeplitz operators are those with a symbol in $H^\infty$, and the norm of $T_\phi$ is then equal to $\|\phi\|_\infty$. For a general $\phi$, it can be shown that $T_\phi$ is densely defined on $H^2$ if and only if $\phi \in N^+$ [10, Lemma 5.2]. In this case, $T_\phi$ has a unique adjoint $T^*_{\overline{\phi}}$, and we henceforth define $T_{\overline{\phi}} := T^*_{\overline{\phi}}$. The next theorem shows that de Branges–Rovnyak spaces occur naturally as the domain of such adjoint operators.

**Theorem 2.2.** (See [10, Proposition 5.4].) Let $(b,a)$ be a pair and let $\phi := b/a$. Then the domain of $T_{\overline{\phi}}$ is $\mathcal{H}(b)$, and $T_{\overline{\phi}} f = f^+ \quad (f \in \mathcal{H}(b))$. Consequently,

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\overline{\phi}} f\|_2^2 \quad (f \in \mathcal{H}(b)).$$  \hspace{1cm} (1)

In what follows, we shall sometimes need to exchange the order of Toeplitz operators. According to a classical lemma, if $\phi, \psi \in L^\infty(\mathbb{T})$ and if at least one of them belongs to $H^\infty$, then $T_{\overline{\phi}} T_{\overline{\psi}} = T_{\overline{\phi \psi}}$ (see, e.g., [7, p. 9]). As an obvious consequence, if both $\phi$ and $\psi$ are in $H^\infty$, then $T_{\overline{\phi}}$ and $T_{\overline{\psi}}$ commute. The next result extends this to the case when one of the symbols belongs to $N^+$. 


Theorem 2.3. (See [10, Proposition 6.5].) Let \( \phi \in N^+ \) and \( \psi \in H^\infty \). Then

\[
T_\phi T_\psi f = T_{\phi \psi} f = T_{\overline{\psi} \phi} f \quad (f \in D(T_\phi)).
\]

2.2. Dirichlet spaces

For \( \lambda \in \mathbb{T} \) and \( f \in H^2 \), the local Dirichlet integral of \( f \) at \( \lambda \) is defined by

\[
D_\lambda(f) := \frac{1}{2\pi} \int_\mathbb{T} \left| \frac{f(e^{it}) - f(\lambda)}{e^{it} - \lambda} \right|^2 dt.
\]

Here \( f(\lambda) \) denotes the value of the radial limit of \( f \) at \( \lambda \), assuming that it exists. If \( f \) does not have a radial limit at \( \lambda \), then we set \( D_\lambda(f) := \infty \). The local Dirichlet space at \( \lambda \) is the Hilbert space

\[
D_\lambda := \{ f \in H^2 : \| f \|_2^2 := \| f \|_T^2 + D_\lambda(f) < \infty \}.
\]

Given a finite positive Borel measure \( \mu \) on \( \mathbb{T} \), we define

\[
D_\mu(f) := \int_\mathbb{T} D_\lambda(f) \, d\mu(\lambda) \quad (f \in H^2),
\]

and we associate to \( \mu \) the Hilbert space

\[
D(\mu) := \{ f \in H^2 : \| f \|_\mu^2 := \| f \|_T^2 + D_\mu(f) < \infty \}.
\]

Note that \( D_\lambda \) is just \( D(\delta_\lambda) \), where \( \delta_\lambda \) is the Dirac measure at \( \lambda \).

The Dirichlet integral \( D_\mu(f) \) can also be expressed as an area integral on the disk. Writing \( P_\mu \) for the Poisson integral of \( \mu \), and \( dA \) for area Lebesgue measure, we have

\[
D_\mu(f) = \frac{1}{\pi} \int_\mathbb{D} |f'(z)|^2 P_\mu(z) \, dA(z) \quad (f \in H^2).
\]

For a proof of this, see, e.g., [5, Proposition 2.2]. Thus, in particular, if \( \mu \) is normalized Lebesgue measure on \( \mathbb{T} \), then \( D_\mu(f) \) is just the usual Dirichlet integral of \( f \) and \( D(\mu) \) is the classical Dirichlet space.

For further information on the local Dirichlet integral, we refer to [5].

3. Coincidence of de Branges–Rovnyak spaces and Dirichlet spaces

Our goal in this section is to identify the functions \( b \) and the measures \( \mu \) for which \( \mathcal{H}(b) = D(\mu) \).
Theorem 3.1. Let $b$ be an element of the unit ball of $H^\infty$, and let $\mu$ be a finite positive Borel measure on $\mathbb{T}$. Then $\mathcal{H}(b) = D(\mu)$, with equality of norms, if and only if

$$\mu = c\delta_\lambda \quad \text{and} \quad b(z) = \frac{\sqrt{\tau \alpha \lambda z}}{1 - \tau \lambda z},$$

where $\lambda \in \mathbb{T}$, $c \geq 0$, $\alpha \in \mathbb{C}$ with $|\alpha|^2 = c$, and $\tau \in (0, 1]$ with $\tau + 1/\tau = 2 + c$.

The proof is based on a comparison of inner products of monomials in the two spaces $\mathcal{H}(b)$ and $D(\mu)$. We begin by computing these inner products in $\mathcal{H}(b)$. The first part of the following lemma was already proved in [6, p. 81].

Lemma 3.2. Let $(b, a)$ be a pair and let $\phi := b/a$, say $\phi(z) = \sum_{j \geq 0} c_j z^j$. Then

$$\|z^n\|_b^2 = 1 + \sum_{j=0}^n |c_j|^2 \quad (n \geq 0),$$

$$\langle z^{n+k}, z^n \rangle_b = \sum_{j=0}^n \bar{c}_{j+k} c_j \quad (n \geq 0, \ k \geq 1).$$

Proof. By (1) and the polarization identity, we have

$$\langle f, g \rangle_b = \langle f, g \rangle_2 + \langle T_{\bar{\phi}} f, T_{\bar{\phi}} g \rangle_2 \quad (f, g \in \mathcal{H}(b)). \quad (3)$$

It therefore suffices to compute $\langle T_{\bar{\phi}} (z^{n+k}), T_{\bar{\phi}} (z^n) \rangle_2$. For each $n \geq 0$, we can write $\phi(z) = \sum_{k=0}^n c_k z^k + z^{n+1} \psi_n(z)$, where $\psi_n \in N^+$. Thus

$$T_{\bar{\phi}} (z^n) = \sum_{k=0}^n \bar{c}_k T_{\bar{\psi}} (z^n) + T_{z^{n+1} \psi_n} (z^n).$$

Now $T_{z^k} (z^n) = z^{n-k} \ (0 \leq k \leq n)$. Also, by Theorem 2.3, we have

$$T_{z^{n+1} \psi_n} (z^n) = T_{\bar{\psi}_n} T_{z^{n+1}} (z^n) = T_{\bar{\psi}_n} (0) = 0.$$

It follows that $T_{\bar{\phi}} (z^n) = \sum_{m=0}^n \bar{c}_{n-m} z^m$. Hence

$$\langle T_{\bar{\phi}} (z^{n+k}), T_{\bar{\phi}} (z^n) \rangle_2 = \sum_{m=0}^n \bar{c}_{n+k-m} c_{n-m} = \sum_{j=0}^n \bar{c}_{j+k} c_j.$$

Together with (3) this gives the result. \qed

The next lemma is the corresponding result for $D(\mu)$. We denote by $\langle \cdot, \cdot \rangle_\mu$ the inner product in $D(\mu)$. Also we write $\hat{\mu}(k) := \int_{\mathbb{T}} e^{-ikt} \, d\mu(e^{it}) \ (k \in \mathbb{Z})$. 
Lemma 3.3. Let \( \mu \) be a finite positive measure on \( T \). Then

\[
\| z^n \|^2_\mu = 1 + n \mu(T) \quad (n \geq 0),
\]

\[
\langle z^{n+k}, z^n \rangle_\mu = n \hat{\mu}(-k) \quad (n \geq 0, \ k \geq 1).
\]

Proof. By (2) and the polarization identity, we have

\[
\langle f, g \rangle_\mu = \langle f, g \rangle_2 + \frac{1}{\pi} \int_D f'(z) \overline{g'(z)} P_\mu(z) dA(z) \quad (f, g \in D(\mu)).
\]

It thus suffices to compute the last integral with \( f(z) = z^{n+k} \) and \( g(z) = z^n \). With this choice of \( f, g \), we get

\[
\frac{1}{\pi} \int_D (n + k)z^{n+k-1}n\bar{z}^{n-1} P_\mu(z) dA(z) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} (n + k)n r^{2n+k-1} e^{ikt} P_\mu(re^{it}) dt dr
\]

\[
= \int_0^{2\pi} 2(n + k)n r^{2n+2k-1} \hat{\mu}(-k) dr
\]

\[
= n \hat{\mu}(-k).
\]

The result follows. \( \square \)

Proof of Theorem 3.1. Suppose that \( H(b) = D(\mu) \), with equality of norms. Notice first that every function holomorphic on a neighborhood of \( \mathbb{D} \) belongs to \( D(\mu) \), and therefore also to \( H(b) \). By [7, p. 37], this implies that \( b \) is not an extreme point in the unit ball of \( H^\infty \). Thus there exists an outer function \( a \) such that \( (b, a) \) forms a pair, and we may consider \( \phi(z) := b(z)/a(z) = \sum_{j \geq 0} c_j z^j \), say.

The next step is to determine the coefficients \( c_j \). Since \( \| z^n \|_b = \| z^n \|_\mu \) for all \( n \), Lemmas 3.2 and 3.3 give

\[
1 + \sum_{j=0}^{n} |c_j|^2 = 1 + n \mu(T) \quad (n \geq 0).
\]

Hence \( c_0 = 0 \) and \( |c_j|^2 = \mu(T) \) for all \( j \geq 1 \). Also, since \( \langle z^{n+1}, z^n \rangle_b = \langle z^{n+1}, z^n \rangle_\mu \) for all \( n \), the same lemmas imply that

\[
\sum_{j=0}^{n} \bar{c}_{j+1} c_j = n \hat{\mu}(-1) \quad (n \geq 0).
\]

Hence \( \bar{c}_{j+1} c_j = \hat{\mu}(-1) \) for all \( j \geq 1 \). Putting these facts together, it follows that \( c_j = \alpha \bar{\lambda}^j \) for all \( j \geq 1 \), where \( \lambda \in \mathbb{T} \) and \( \alpha \in \mathbb{C} \) with \( |\alpha|^2 = \mu(T) \).
Next, we determine $\mu$. Using Lemmas 3.2 and 3.3 once again, we have

$$\hat{\mu}(-k) = \langle z^{k+1}, z \rangle_{\mu} = \langle z^{k+1}, z \rangle_{b} = \sum_{j=0}^{1} c_{j+k} c_{j} = |\alpha|^2 \lambda^k = \mu(\mathbb{T}) \lambda^k \quad (k \geq 1).$$

Since $\mu$ is a real measure, the same relation holds for all $k \leq -1$, and clearly it is also true for $k = 0$. Thus $\mu$ has the same Fourier coefficients as the measure $c \delta_{\lambda}$, where $c = \mu(\mathbb{T})$, and we conclude that $\mu = c \delta_{\lambda}$.

It remains to determine $b$. To do this, we follow the method in [9]. Note first that

$$\phi(z) = \sum_{j \geq 0} c_{j} z^{j} = \sum_{j \geq 1} \alpha^{j} \lambda^{j} z^{j} = \frac{\alpha \lambda z}{1 - \lambda z} \quad (z \in \mathbb{D}).$$

Since $\phi = b/a$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, it follows that $|a|^2 = 1/(1 + |\phi|^2)$ a.e. on $\mathbb{T}$. Thus

$$|a(z)|^2 = \frac{|1 - \lambda z|^2}{|1 - \lambda z|^2 + |a|^2} \quad \text{a.e. on } \mathbb{T}.$$

A simple calculation shows that $|1 - \lambda z|^2 + |a|^2 = \tau^{-1} |1 - \tau \lambda z|^2$ for $z \in \mathbb{T}$, where $\tau \in (0, 1]$ is chosen so that $\tau + 1/\tau = 2 + |a|^2 = 2 + c$. As $a$ is an outer function, it follows that

$$a(z) = \sqrt{\tau} \frac{1 - \lambda z}{1 - \tau \lambda z} \quad (z \in \mathbb{D}).$$

Hence, finally,

$$b(z) = a(z) \phi(z) = \frac{\sqrt{\tau} \alpha \lambda z}{1 - \tau \lambda z} \quad (z \in \mathbb{D}).$$

This completes the proof of the “only if”.

For the “if”, note that with the given choice of $b, \mu$, working back through the calculations above we get $(z^{n+k}, z^{n})_{b} = (z^{n+k}, z^{n})_{\mu}$ for all $n, k \geq 0$. Since polynomials are dense both in $\mathcal{H}(b)$ [7, IV-3, p. 25] and in $\mathcal{D}(\mu)$ [4, Corollary 3.8], we deduce that $\mathcal{H}(b) = \mathcal{D}(\mu)$, with equality of norms.

What if $\mathcal{H}(b) = \mathcal{D}(\mu)$ without equality of norms? Since both $\mathcal{H}(b)$ and $\mathcal{D}(\mu)$ embed boundedly into $H^2$, using the closed graph theorem it is easy to see that the norms $\| \cdot \|_{b}$ and $\| \cdot \|_{\mu}$ must be equivalent. Do there exist measures $\mu$, other than point masses, for which $\mathcal{D}(\mu) = \mathcal{H}(b)$ with equivalence of norms?

4. Formulas for the norm in de Branges–Rovnyak spaces

Lemma 3.2 provides a formula for the inner product of monomials in $\mathcal{H}(b)$, expressed in terms of the coefficients $c_{j}$ of the function $\phi$. Since polynomials are dense in $\mathcal{H}(b)$, we might expect there to be an analogous formula for the norms of more general functions. The following theorem, which is implicit in [10], is a first step in this direction.
Theorem 4.1. Let \((b,a)\) be a pair, and let \(\phi := b/a\), say \(\phi(z) = \sum_{j \geq 0} c_j z^j\). Let \(f\) be holomorphic in a neighborhood of \(\mathbb{D}\), say \(f(z) = \sum_{j \geq 0} \hat{f}(j) z^j\). Then the series \(\sum_{j \geq 0} \hat{f}(j + k) \hat{\epsilon}_j\) converges absolutely for each \(k\), and

\[
\|f\|_b^2 = \sum_{k \geq 0} |\hat{f}(k)|^2 + \sum_{k \geq 0} \left| \sum_{j \geq 0} \hat{f}(j + k) \hat{\epsilon}_j \right|^2.
\]

(4)

Proof. Suppose first that \(f\) is a polynomial, of degree \(n\) say. In this case, we argue as in the proof of Lemma 3.2. Writing \(\phi(z) = \sum_{n \geq 0} c_j z^j + z^{n+1} \psi_n(z)\), where \(\psi_n \in \mathbb{N}^+\), we have

\[
T_{z^{n+1} \psi_n}(f) = T_{\psi_n} T_{z^{n+1}}(f) = T_{\psi_n}(0) = 0,
\]

and so

\[
T_{\hat{\phi}}(f) = \sum_{j=0}^n \hat{\epsilon}_j T_{z^j}(f) = \sum_{j=0}^n \hat{\epsilon}_j \sum_{k=0}^{n-j} \hat{f}(j + k) z^k = \sum_{k=0}^n \sum_{j=0}^{n-k} \hat{f}(j + k) \hat{\epsilon}_j z^k.
\]

Using Theorem 2.2, we obtain

\[
\|f\|_b^2 = \|f\|_2^2 + \|T_{\hat{\phi}} f\|_2^2 = \sum_{k=0}^n |\hat{f}(k)|^2 + \sum_{k=0}^n \sum_{j=0}^{n-k} |\hat{f}(j + k) \hat{\epsilon}_j|^2,
\]

which proves the theorem in this case.

For the general case, let us write \(f_n(z) := \sum_{j=0}^n \hat{f}(j) z^j\). By what we have already proved, we have

\[
\|f_n\|_b^2 = \sum_{k=0}^n |\hat{f}(k)|^2 + \sum_{k=0}^n \sum_{j=0}^{n-k} |\hat{f}(j + k) \hat{\epsilon}_j|^2.
\]

(5)

Fix \(R > 1\) such that \(f\) is holomorphic in a neighborhood of \(\mathbb{D}(0, R)\). Then \(\hat{f}(j) = O(R^{-j})\) as \(j \to \infty\). Since \(c_j = O(R^j)\) for each \(R' \in (1, R)\), it follows that the series \(\sum_{j \geq 0} \hat{f}(j + k) \hat{\epsilon}_j\) converges absolutely for each \(k\). Thus, as \(n \to \infty\), the right-hand side of (5) converges to the right-hand side of (4). Also, using Lemma 3.2, we have, for each \(R' \in (1, R)\),

\[
\|f_n z^k\|_b = |\hat{f}(k)| \left( 1 + \sum_{j=0}^k |c_j|^2 \right)^{1/2} = O((R'/R)^k) \quad \text{as} \quad k \to \infty.
\]

Thus the Taylor series of \(f\) converges in the norm of \(\mathcal{H}(b)\). The norm limit agrees with \(f\) on the unit disk, because norm convergence implies pointwise convergence. Therefore the left-hand side of (5) converges to the left-hand side of (4) as \(n \to \infty\). This completes the proof. □
It is instructive to look at what formula (4) gives when \( \mathcal{H}(b) = D_\lambda \). Recall that, in this case, 
\( c_0 = 0 \) and 
\( c_j = \tilde{\lambda}^j \) for all \( j \geq 1 \). Thus formula (4) becomes

\[
D_\lambda(f) = \sum_{k \geq 0} \left| \sum_{j \geq 1} \hat{f}(j + k) \lambda^j \right|^2.
\]

Writing \( S \) for the shift operator on \( H^2 \), and \( S^* \) for its adjoint, namely \( S^* f(z) := (f(z) - f(0))/z \), we obtain

\[
D_\lambda(f) = \sum_{k \geq 1} |(S^k f)(\lambda)|^2.
\]

This formula is already known. It is implicit in [5], and explicit in [11].

Although we have proved (6) only for functions holomorphic on a neighborhood of \( \bar{D} \), when suitably interpreted it is actually valid for all functions holomorphic in \( D \), thereby providing a test for membership of \( D_\lambda \). For the formula to make sense, we interpret \( S^k f(\lambda) \) as the radial limit of \( S^k f \) at \( \lambda \) if this limit exists, and we set \( |S^k f(\lambda)| := \infty \) otherwise. This version of the formula can be deduced from the more restricted version by considering the functions \( f_r(z) := f(rz) \) and using the fact that \( D_\lambda(f_r) \to D_\lambda(f) \) as \( r \to 1 \) (see [5, p. 377]).

This naturally raises the question of whether a similar approximation procedure is possible in general \( \mathcal{H}(b) \)-spaces. This is the subject of the next section.

5. Star-shapedness of de Branges–Rovnyak spaces

Throughout this section we assume that \( b \) is a non-extreme point of the unit ball of \( H^\infty \), that \( (b, a) \) is a pair, and that \( \phi = b/a \) is the associated function in \( N^+ \).

Given \( f \in \mathcal{H}(b) \) and \( r \in (0, 1) \), we write \( f_r(z) := f(rz) \). As \( f_r \) is holomorphic on a neighborhood \( \bar{D} \), we certainly have \( f_r \in \mathcal{H}(b) \). By the closed graph theorem, \( C_r : \mathcal{H}(b) \to \mathcal{H}(b) \), defined by \( C_r f := f_r \), is bounded linear map.

We seek to determine whether \( \lim_{r \to 1} \| f_r - f \|_b = 0 \) for all \( f \in \mathcal{H}(b) \). A space \( \mathcal{H}(b) \) for which this holds is called star-shaped. The following proposition provides some criteria for \( \mathcal{H}(b) \) to be star-shaped.

**Proposition 5.1.** The following are equivalent:

(i) \( \lim_{r \to 1} \| f_r - f \|_b = 0 \) for all \( f \in \mathcal{H}(b) \);
(ii) \( \sup_{r < 1} \| f_r \|_b < \infty \) for all \( f \in \mathcal{H}(b) \);
(iii) \( \sup_{r < 1} \| C_r \|_{\mathcal{H}(b) \to \mathcal{H}(b)} < \infty \).

**Proof.** Obviously (i) implies (ii), and the Banach–Steinhaus theorem shows that (ii) implies (iii). Finally (iii) implies (i), because \( \lim_{r \to 0} \| f_r - f \|_b = 0 \) when \( f \) is a polynomial, and polynomials are dense in \( \mathcal{H}(b) \) (see [7, p. 25]). □

The following weak version of (ii) always holds.
Theorem 5.2. If \( f \in \mathcal{H}(b) \), then
\[
\log^+ \| f_r \|_b = o\left( \frac{1}{1-r} \right) \quad \text{as} \ r \to 1.
\] (7)

Proof. Let \( g \in aH^2 \). Then
\[
\| \langle T_{\phi} f_r, g \rangle \|_2 = \| \langle f, \phi_r g \rangle \|_2 \leq \| f \|_2 \| \phi_r \|_\infty \| g \|_2.
\]
As \( a \) is outer, \( aH^2 \) is dense in \( H^2 \), and therefore \( \| T_{\phi} f_r \|_2 \leq \| \phi_r \|_\infty \| f \|_2 \). From (1) we get \( \| f_r \|_b \leq \max\{\| \phi_r \|_\infty, 1\} \| f \|_2 \). Thus, to prove the theorem, it suffices to show that \( \log^+ \| \phi_r \|_\infty = o(1/(1-r)) \) as \( r \to 1^- \).

Let us write \( \phi^* \) for the radial limit function of \( \phi \) on \( \mathbb{T} \). Then, for all \( z \in \mathbb{D} \), all \( r \in (0,1) \) and all \( K > 1 \), we have
\[
\log^+ |\phi(rz)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|rz|^2}{|e^{it} - rz|^2} \log^+ |\phi^*(e^{it})| \, dt
\]
\[
\leq \log K + \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|rz|^2}{|e^{it} - rz|^2} \log^+ |\phi^*(e^{it})| \, dt
\]
\[
\leq \log K + \frac{2}{1-r} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\phi^*(e^{it})| \, dt.
\]
Therefore
\[
\log^+ \| \phi_r \|_\infty \leq \log K + \frac{2}{1-r} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\phi^*(e^{it})| \, dt.
\]
As \( K \) is arbitrary, we get \( \log^+ \| \phi_r \|_\infty = o(1/(1-r)) \) as \( r \to 1 \), as required. \( \square \)

We shall see shortly that (7) cannot be improved, in general. However, the first part of the preceding argument can be adapted to provide a simple condition on \( \phi \) which guarantees that \( \mathcal{H}(b) \) is star-shaped.

Theorem 5.3. If \( \phi_r/\phi \) is bounded on \( \mathbb{D} \), then
\[
\| C_r \|_{\mathcal{H}(b) \to \mathcal{H}(b)} \leq \max\{ \| \phi_r/\phi \|_\infty, 1\}.
\]
Consequently, if \( \sup_{r<1} \| \phi_r/\phi \|_\infty < \infty \), then \( \mathcal{H}(b) \) is star-shaped.
Proof. Let \( f \in \mathcal{H}(b) \) and let \( g \in aH^2 \). Then
\[
\left| \langle T_{\phi} fr, g \rangle \right|_2 = \left| \langle f, \phi g_r \rangle \right|_2 = \left| \langle T_{\phi} f, (\phi r/\phi)g_r \rangle \right|_2 \leq \|T_{\phi} f\|_2 \|\phi r/\phi\| \|g\|_2.
\]
As \( a \) is outer, \( aH^2 \) is dense in \( H^2 \), and therefore \( \|T_{\phi} fr\|_2 \leq \|T_{\phi} f\|_2 \|\phi r/\phi\|_\infty \). From (1) we get
\[
\|fr\|_b \leq \max\{\|\phi r/\phi\|_\infty, 1\} \|f\|_b,
\]
whence the result.

As a special case, we recover a result that we cited in Section 4. This is essentially Sarason’s proof in [9].

Corollary 5.4. The space \( \mathcal{D}_\lambda \) is star-shaped and \( \|Cr\| \rightarrow \mathcal{D}_\lambda \leq 1 \) for all \( r \in (0, 1) \).

Proof. We have \( \mathcal{D}_\lambda = \mathcal{H}(b) \) with \( \phi(z) = \frac{\lambda z}{1 - \lambda z} \). Therefore, for \( r \in (0, 1) \), we obtain
\[
\|Cr\| \rightarrow \mathcal{D}_\lambda \leq \max\left\{\frac{2r}{1 + r}, 1\right\} = 1.
\]

If \( \sup_{r \leq 1} \|\phi r/\phi\|_\infty < \infty \), then necessarily \( \phi(z) = z^k \phi_o(z) \), where \( \phi_o \) is outer and \( 1/\phi_o \) is bounded (or, equivalently, \( b(z) = z^k b_o(z) \), where \( b_o \) is outer and \( 1/b_o \) is bounded). The example in Corollary 5.4 is thus rather typical. Based on this, one might guess that \( \mathcal{H}(b) \) is star-shaped whenever \( 1/b \) is bounded. We shall now prove that this is not the case.

Theorem 5.5. Let \( \rho : (0, 1) \rightarrow \mathbb{R}^+ \) be a function such that \( \rho(r) = o(1/(1 - r)) \) as \( r \rightarrow 1 \). Then there exist \( b \) (non-extreme in the ball of \( H^\infty \)) and \( f \in \mathcal{H}(b) \) such that
\[
\limsup_{r \rightarrow 1} \frac{\log \|fr\|_b}{\rho(r)} = \infty.
\]
The function \( b \) may be chosen to be outer with \( 1/b \) bounded.

Remarks.

(i) Taking \( \rho \equiv 1 \) in the theorem, we obtain the promised example showing that \( \mathcal{H}(b) \) need not be star-shaped, even if \( 1/b \) is bounded.

(ii) The theorem also shows that the estimate (7) cannot be improved.

The proof of the theorem is based on two lemmas.

Lemma 5.6. Given \( b, \phi \) as above,
\[
\|Cr\|_{\mathcal{H}(b) \rightarrow \mathcal{H}(b)}^2 \geq \sup_{w \in \mathbb{D}} \frac{1 + |\phi(rw)|^2}{1 + |\phi(w)|^2} \frac{1 - |w|^2}{1 - r^2 |w|^2}.
\]

Proof. Let \( k_w(z) := 1/(1 - wz) \) be the reproducing kernel for \( H^2 \). Then \( T_{\phi} k_w = \overline{\phi(w)} k_w \). Hence
\[
\|k_w\|^2_b = \|k_w\|^2 + \|T_{\phi} k_w\|^2 = \|k_w\|^2 + |\phi(w)|^2 \|k_w\|^2 = \frac{1 + |\phi(w)|^2}{1 - |w|^2}.
\]
Likewise
\[ \|C_r k_w\|_b^2 = \|k_{rw}\|_b^2 = \frac{1 + |\phi(rw)|^2}{1 - r^2|w|^2}. \]

The result follows. \( \Box \)

**Lemma 5.7.** Let \( \rho : (0, 1) \to \mathbb{R}^+ \) be a function such that \( \rho(r) = o(1/(1 - r)) \) as \( r \to 1 \). Then there exists an outer function \( \phi \) on \( \mathbb{D} \) such that \( |\phi| \geq 1 \) and

\[ \limsup_{r \to 1} \frac{\log(|\phi(r^2)|/|\phi(r)|)}{\rho(r)} = \infty. \]

**Proof.** Fix a positive sequence \( (\epsilon_n) \) such that the series \( \sum k \epsilon_k \) converges and satisfies \( \sum_{k>n} \epsilon_k = o(\epsilon_n) \) as \( n \to \infty \). For example, \( \epsilon_n := e^{-n^2} \) will do. Since \( \lim_{r \to 1} \rho(r)(1-r) = 0 \), there exists an increasing sequence \( r_n \to 1 \) such that \( \rho(r_n)(1-r_n)/\epsilon_n \to 0 \) as \( n \to \infty \). Define

\[ s_n := \frac{1 - r_n}{1 + r_n} \quad \text{and} \quad t_n := \frac{1 - t_n^2}{1 + t_n^2}. \]

Let \( \psi \) be the outer function on the upper half-plane whose non-tangential limit \( \psi^* \) on \( \mathbb{R} \) satisfies

\[ \log |\psi^*| = \sum_{k \geq 1} \left( \epsilon_k / t_k \right) 1_{[t_k, 2t_k]} \quad \text{a.e. on } \mathbb{R}. \]

Note that

\[ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |\psi^*(x)|}{1 + x^2} \, dx \leq \sum_{k \geq 1} \epsilon_k < \infty, \]

so \( \psi \) is well defined and \( |\psi| \geq 1 \). Define \( \phi \) on the unit disk by

\[ \phi(z) := \psi \left( i \frac{1 - z}{1 + z} \right) \quad (|z| < 1). \]

Then \( \phi \) is also an outer function and \( |\phi| \geq 1 \). We shall show that this function \( \phi \) satisfies the conclusion of the lemma.

For each \( n \geq 1 \), we have

\[ \log |\phi(r_n^2)/\phi(r_n)| = \log |\psi(it_n)| - \log |\psi(is_n)| \]

\[ = \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{t_n}{t_n^2 + x^2} - \frac{s_n}{s_n^2 + x^2} \right) \log |\psi^*(x)| \, dx \]

\[ = \frac{1}{\pi} \sum_{k \geq 1} \epsilon_k \frac{2t_k}{t_k} \int_{t_k} \left( \frac{t_n}{t_n^2 + x^2} - \frac{s_n}{s_n^2 + x^2} \right) \, dx. \]
Now, if \( x \geq t_n \), then
\[
\frac{t_n}{t_n^2 + x^2} - \frac{s_n}{s_n^2 + x^2} = \frac{(t_n - s_n)(x^2 - s_n t_n)}{(t_n^2 + x^2)(s_n^2 + x^2)} \geq \left( \frac{t_n}{s_n} - 1 \right) \frac{1}{t_n} \geq \frac{r_n^2}{1 - r_n}.
\]
Therefore, if \( 1 \leq k \leq n \), then
\[
\frac{\epsilon_k}{t_k} \int_{t_k}^{2t_k} \left( \frac{t_n}{t_n^2 + x^2} - \frac{s_n}{s_n^2 + x^2} \right) dx \geq \epsilon_k \frac{r_n^2}{1 - r_n}.
\]
Also, for every \( k \), we clearly have
\[
\frac{\epsilon_k}{t_k} \int_{t_k}^{2t_k} \left( \frac{t_n}{t_n^2 + x^2} - \frac{s_n}{s_n^2 + x^2} \right) dx \geq -\frac{\epsilon_k}{t_k} \int_{t_k}^{2t_k} \frac{s_n}{s_n^2 + x^2} dx \geq -\frac{\epsilon_k}{s_n} \geq -\frac{2\epsilon_k}{1 - r_n}.
\]
Putting this information together, we deduce that
\[
\log \left| \frac{\phi(r_n^2)}{\phi(r_n)} \right| \geq \frac{r_n^2}{1 - r_n} \sum_{k \leq n} \epsilon_k - \frac{2}{1 - r_n} \sum_{k > n} \epsilon_k.
\]
Since \( \sum_{k > n} \epsilon_k = o(\epsilon_n) \), it follows that \( \log |\phi(r_n^2)/\phi(r_n)| \geq C \epsilon_n/(1 - r_n) \), where \( C \) is a positive constant independent of \( n \). Hence, finally,
\[
\frac{\log \left| \frac{\phi(r_n^2)}{\phi(r_n)} \right|}{\rho(r_n)} \geq \frac{C \epsilon_n}{(1 - r_n) \rho(r_n)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.
\]
This completes the proof. \( \square \)

**Proof of Theorem 5.5.** Let \( \phi \) be the function given by Lemma 5.7, and let \( b \) be the associated element of the unit ball of \( H^\infty \). Note that \( b \) is outer and \( 1/b \) is bounded. By Lemma 5.6, applied with \( w = r \), we have
\[
\|C_r\|_{\mathcal{H}(b) \rightarrow \mathcal{H}(b)} \geq \frac{1}{2} \frac{1 + |\phi(r^2)|^2}{1 + |\phi(r)|^2} \geq \frac{1}{4} \frac{|\phi(r^2)|^2}{|\phi(r)|^2} \quad (0 < r < 1).
\]
Therefore,
\[
\limsup_{r \rightarrow 1} \frac{\log \|C_r\|_{\mathcal{H}(b) \rightarrow \mathcal{H}(b)}}{\rho(r)} \geq \limsup_{r \rightarrow 1} \frac{\log |\phi(r^2)/\phi(r)|}{\rho(r)} = \infty.
\]
Thus, there exist sequences \( r_n \rightarrow 1 \) and \( A_n \rightarrow \infty \) such that
\[
\left\| e^{-A_n \rho(r_n)} C_{r_n} \right\|_{\mathcal{H}(b) \rightarrow \mathcal{H}(b)} \rightarrow \infty.
\]
By the Banach–Steinhaus theorem, there exists $f \in \mathcal{H}(b)$ such that

$$\limsup_{n \to \infty} \|e^{-A_n \rho(r_n)} C_{r_n} f\|_b = \infty.$$ 

This gives the desired conclusion. □

If we multiply $b$ by an inner function $u$, then $a$ does not change, and the corresponding $\phi$ is also multiplied by $u$. How does multiplication by an inner function affect the star-shapedness of the corresponding de Branges–Rovnyak space?

There is one simple case: if $\mathcal{H}(b)$ is star-shaped, then so is $\mathcal{H}(z^k b)$ for every $k$. Indeed, a calculation like (8) shows that

$$\|C_r\|_{\mathcal{H}(z^k b)\to \mathcal{H}(z^k b)} \leq r^k \|C_r\|_{\mathcal{H}(b)\to \mathcal{H}(b)} \quad (0 < r < 1).$$

For general inner factors, however, the situation is very different.

**Theorem 5.8.** If $\|b\|_\infty = 1$, then there is a Blaschke product $u$ such that $\mathcal{H}(ub)$ is not star-shaped.

**Remark.** In the other case, namely when $\|b\|_\infty < 1$, the space $\mathcal{H}(ub)$ is star-shaped for every inner function $u$. Indeed, $\|T_{ub}\| < 1$, so $(I - T_{ub}^* T_{ub})^{1/2}$ is an invertible operator on $H^2$, and the inclusion $\mathcal{H}(ub) \subset H^2$ is a surjection.

To prove the theorem, we need a further lemma.

**Lemma 5.9.** Let $(\theta_n), (s_n)$ be sequences in $[0, 2\pi]$ and $(0, 1)$ respectively. Then there exist a sequence $r_n \in (s_n, 1)$ and Blaschke product $u$ such that $u(r_n e^{i\theta_n}) = 0$ for all $n$ and $\inf_n |u(r_n^2 e^{i\theta_n})| > 0$.

**Proof.** Let $\sigma$ denote the pseudo-hyperbolic metric on $\mathbb{D}$, defined by

$$\sigma(z, w) := \frac{|z - w|}{1 - \bar{z}w} \quad (z, w \in \mathbb{D}).$$

For $w$ fixed, we have $\sigma(z, w) \to 1$ as $|z| \to 1$. Thus, we may inductively choose a sequence $r_n \in (0, 1)$ so that, if $z_n := r_n e^{i\theta_n}$ and $w_n := r_n^2 e^{i\theta_n}$, then

$$\sigma(z_n, w_m) \geq \exp(-2^{-m}) \quad (m = 1, \ldots, n - 1),$$

$$\sigma(w_n, z_m) \geq \exp(-2^{-m}) \quad (m = 1, \ldots, n - 1).$$

We may further suppose that $r_n \in (s_n, 1)$ for all $n$, and that $\sum_n (1 - r_n) < \infty$. Let $u$ be the Blaschke product defined by

$$u(z) := \prod_{m=1}^{\infty} \frac{|z_m|}{z_m} \frac{z_m - z}{1 - \bar{z}_m z}.$$
Clearly \( u(z_n) = 0 \) for all \( n \). Also, for each \( n \),
\[
|u(w_n)| = \prod_{m=1}^{\infty} \sigma(z_m, w_n)
= \sigma(z_n, w_n) \prod_{1 \leq m < n} \sigma(z_m, w_n) \prod_{m > n} \sigma(z_m, w_n)
\geq \frac{r_n - r_n^2}{1 - r_n^3} \prod_{1 \leq m < n} \exp(-2^{-m}) \prod_{m > n} \exp(-2^{-m})
\geq \frac{r_n - r_n^2}{1 - r_n^3} e^{-1} \rightarrow \frac{1}{3} e^{-1} \quad \text{as } n \rightarrow \infty.
\]

This completes the proof of the lemma. \( \square \)

**Proof of Theorem 5.8.** Since \( \|b\|_{\infty} = 1 \), it follows that \( \phi \) is unbounded (in fact the two conditions are equivalent). Choose \((\theta_n)\) such that the radial limit \( \phi(e^{i\theta_n}) \) exists and satisfies \(|\phi(e^{i\theta_n})| > n \) for all \( r \in (s_n, 1) \). Let \((r_n)\) and \( u \) be as given by Lemma 5.9. By Lemma 5.6 (applied with \( w = r_n e^{i\theta_n} \)), we have
\[
\|C_{r_n}\|_{\mathcal{H}(ub) \rightarrow \mathcal{H}(ub)}^2 \geq \frac{1 + |(u\phi)(r_n^2 e^{i\theta_n})|^2}{1 + |(u\phi)(r_n e^{i\theta_n})|^2} \frac{1 - r_n^2}{1 - r_n^4}
\geq \frac{1}{2} |u(r_n^2 e^{i\theta_n})|^2 |\phi(r_n^2 e^{i\theta_n})|^2 \rightarrow \infty \quad \text{as } n \rightarrow \infty.
\]

Now apply Proposition 5.1. \( \square \)

The counterexamples in this section still leave open the possibility that, given any \( b \) and any \( f \in \mathcal{H}(b) \), there exists a sequence \( r_n \rightarrow 1 \), depending on \( b, f \), such that \( \|f - f_{r_n}\|_b \rightarrow 0 \). Can this be ruled out?

6. A transfer principle

The preceding sections demonstrate that the local Dirichlet spaces \( \mathcal{D}_\lambda \) are not typical de Branges–Rovnyak spaces. In this section we shall prove a result that points in the other direction, to the effect that de Branges–Rovnyak spaces can always be represented inside local Dirichlet spaces.

**Theorem 6.1.** Let \((b, a)\) be a pair. Set \( \psi(z) := (1 - z)b(z)/a(z) \), and define \( W : \mathcal{H}(b) \rightarrow H^2 \) by
\[
W(f) := zT_\psi f \quad (f \in \mathcal{H}(b)).
\]

Then:

(i) \( W \) is well defined on \( \mathcal{H}(b) \);
(ii) the kernel of \( W \) equals \((b_i H^2)^\perp\), where \( b_i \) is the inner factor of \( b \);
(iii) the image of $W$ is contained in the local Dirichlet space $\mathcal{D}_1$, and

$$f^+(z) = \frac{Wf(z) - Wf(1)}{z - 1} \quad (f \in \mathcal{H}(b)).$$

(9)

A great deal is known about the local Dirichlet spaces $\mathcal{D}_\lambda$. For example, there is a remarkable formula due to Richter and Sundberg (generalizing an earlier formula of Carleson in the classical Dirichlet space) expressing $\mathcal{D}_\lambda(f)$ in terms of the factorization $f = OSB$ (outer function, singular inner function, Blaschke product). For more on this see [5, Theorem 3.1]. In principle, at least, Theorem 6.1 allows us to exploit this knowledge to obtain information about general $\mathcal{H}(b)$-spaces. In practice, the success of this endeavor depends on being able to identify the operator $W$, which means understanding the Toeplitz operator $T_\Psi$.

We shall deduce Theorem 6.1 from an abstract transfer principle. To be able to state this principle, we need an alternative notation for the function $f^+$, one that indicates the dependence on $b$. Accordingly, we shall write $[f]_b := f^+$.

**Theorem 6.2.** Let $(b, a)$ and $(B, A)$ be pairs. Let $B = B_iB_o$ be the inner-outer factorization of $B$, and suppose that $1/B_o$ is bounded. Set $\psi := bA/aB_o$, and define $W : \mathcal{H}(b) \to H^2$ by

$$W(f) := B_iT_\psi f \quad (f \in \mathcal{H}(b)).$$

Then:

(i) $W$ is well defined on $\mathcal{H}(b);
(ii) the kernel of $W$ equals $(b_1H^2)\perp$, where $b_1$ is the inner factor of $b$;
(iii) the image of $W$ is contained in $\mathcal{H}(B)$, and

$$[Wf]_B = [f]_b \quad (f \in \mathcal{H}(b)).$$

**Proof.** (i) Let us begin by noting that $\psi \in N^+$, so the Toeplitz operator $T_\psi$ is defined at least on polynomials. Also, using Theorem 2.3, we have $T_\psi = T_\Lambda B_o T_{b/a}$, so the domain of $T_\psi$ includes the domain of $T_{b/a}$, which equals $\mathcal{H}(b)$. Thus $W$ is well defined on $\mathcal{H}(b)$.

(ii) Let $b = b_1b_o$ be the inner-outer factorization of $b$. By [7, II-6, p. 10], we have $\mathcal{H}(b) = \mathcal{H}(b_1) \oplus b_1\mathcal{H}(b_o)$, where the direct sum is orthogonal with respect to the inner product in $\mathcal{H}(b)$.

The first summand $\mathcal{H}(b_1)$ is just the model space $(b_1H^2)\perp$, and we shall now show that it is exactly the kernel of $W$.

Let $f \in \mathcal{H}(b)$. Then using the fact that outer functions are cyclic in $H^2$, we have

$$f \in \ker W \iff T_\psi f = 0$$

$$\iff \langle T_\psi f, aB_o h \rangle_2 = 0 \quad (h \in H^2)$$

$$\iff \langle f, bAh \rangle_2 = 0 \quad (h \in H^2)$$

$$\iff f \in (bAH^2)\perp = (b_1H^2)\perp.$$
(iii) Let \( f \in \mathcal{H}(b) \). Then, in the notation introduced just before the statement of the theorem, we have \([f]_b = T_{B/\alpha} f \in H^2\). Therefore,

\[
T_B(Wf) = T_B(B_i T_{\bar{\psi}} f) = T_B(T_{\bar{\psi}} f) = T_{\bar{\psi} B/a} f = T_{\bar{\psi} A}([f]_b).
\]

By Theorem 2.1, it follows that \( Wf \in \mathcal{H}(B) \) and that \([Wf]_B = [f]_b\). \(\square\)

**Proof of Theorem 6.1.** Let \((B, A)\) be the pair for which \(B/A = z/(1 - z)\). As we have seen in Section 3, \(\mathcal{H}(B)\) is then just the local Dirichlet space \(\mathcal{D}_1\). Thus all of Theorem 6.1 follows immediately from Theorem 6.2, except for the final formula (9). For this, we need to identify \([Wf]_B\).

Given \( g \in \mathcal{D}_1 \), we have \([g]_B = T_{B/A} g\). Hence, for \( h \in A H^2 \),

\[
\langle [g]_B, h \rangle_2 = \left( g, \frac{B}{A} h \right)_2 = \left( g, \frac{zh}{1 - z} \right)_2 = \left( \frac{g - g(1)}{z - 1}, h \right)_2.
\]

As \( (g - g(1))/(z - 1) \in H^2 \) and \( A H^2 \) is dense in \( H^2 \), it follows that

\[
[g]_B = \frac{g(z) - g(1)}{z - 1}.
\]

Taking \( g = Wf \), we obtain (9). This completes the proof. \(\square\)

7. Some open problems

(1) Do there exist measures \( \mu \) on \( T \), other than point masses, such that \( D(\mu) = \mathcal{H}(b) \) for some \( b \)? We do not assume equality of norms, though, as observed earlier, the norms must be equivalent. In these circumstances, we can no longer expect \( b \) to be determined by \( \mu \). For example, if \( \mu = 0 \), then \( D(\mu) = H^2 \), and there are many \( b \) for which the inclusion of \( \mathcal{H}(b) \) in \( H^2 \) is surjective—indeed any \( b \in H^\infty \) with \( \|b\|_\infty < 1 \) will do.

(2) A simple weak compactness argument shows that, if \( f \) is holomorphic on \( D \) and \( \lim_{r \to 1} \|f_r\|_b < \infty \), then \( f \in \mathcal{H}(b) \) and \( \|f\|_b \leq \lim_{r \to 1} \|f_r\|_b \). In the other direction, does \( f \in \mathcal{H}(b) \) imply that \( \lim_{r \to 1} \|f_r\|_b < \infty \)? If so, then do we also have \( \lim_{r \to 1} \|f_r - f\|_b = 0 \)? We have seen that the answer to both questions is ‘no’ if ‘lim inf’ is replaced by ‘lim sup’.

(3) Is it possible to characterize those \( b \) (or those \( \phi \)) for which \( \mathcal{H}(b) \) is star-shaped? As the inequality (8) makes clear, the problem boils down to being able to estimate \( |\langle f, \phi f_r g_r \rangle|_2 \) in terms of \( \|g\|_2 \) and \( \|f\|_b \).

(4) Another possible approach to problem (3) is via the \( H^2 \)-reproducing kernels \( k_w(z) := 1/(1 - \bar{w}z) \). Recall that \( T_{\bar{w}} k_w = \phi(w) k_w \) and \( C_r k_w = k_{r w} \) for all \( w \in D \) and \( r \in (0, 1) \). This remark was used in Lemma 3.6 to obtain a lower bound for \( \|C_r\|_{\mathcal{H}(b) \to \mathcal{H}(b)} \), and hence a necessary condition for \( \mathcal{H}(b) \) to be star-shaped. Since the family \( \{k_w : w \in D\} \) spans a dense subspace of \( H^2 \), it could in principle be used to determine \( \|C_r\|_{\mathcal{H}(b) \to \mathcal{H}(b)} \) exactly. However, this gives rise to a Pick-type problem which we have been unable to solve up to now.

(5) Let \( f \in \mathcal{H}(b) \). Although \( f_r \not\to f \) in \( \mathcal{H}(b) \), in general, it is always true that \( f \) can be approximated by functions holomorphic on a neighborhood of \( \overline{D} \), indeed even by polynomials. This is proved in [7, IV-3]. However, the proof given there is by duality and is not constructive. Is there a constructive scheme by which \( f \) may be approximated in \( \mathcal{H}(b) \) by functions holomorphic in a neighborhood of \( \overline{D} \)?
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References