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Singular Perturbations in Linear Control Systems with Weakly Coupled Stable and Unstable Fast Subsystems

A. L. DONTCHEV AND V. M. VELIOV

Institute of Mathematics, Bulgarian Academy of Sciences, 1090 Sofia, P.O. Box 373, Bulgaria

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A linear control system with slow and fast modes is considered, where the different dynamics is represented by a small parameter in the derivatives of the fast states. The fast subsystem consists of an asymptotically stable and an unstable part, the interaction between which contains a factor proportional to the small parameter in the derivatives. The behaviour of the set of trajectories is investigated when the small parameter tends to zero. The continuity properties of the optimal value of three classical optimal control problems with control constraints for systems of this type are studied. © 1985 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider the following control system

$$\dot{x} = A_{11}(t)x + A_{12}(t)y + A_{13}(t)z + B_1(t)u, \tag{1a}$$

$$\beta\dot{y} = A_{21}(t)x + A_{22}(t)y + \varepsilon(\beta)A_{23}(t)z + B_2(t)u, \tag{1b}$$

$$\beta\dot{z} = A_{31}(t)x + \varepsilon(\beta)A_{32}(t)y + A_{33}(t)z + B_3(t)u, \tag{1c}$$

where $x(t) \in R^n$, $y(t) \in R^p$, $z(t) \in R^q$, $u(t) \in R^m$, $A_{ij}(t)$ and $B_i(t)$ are matrices with appropriate dimensions, t represents the time, $t \in [0, T]$, β is a positive (mathematically) small parameter, and $\varepsilon(\beta)$ is a scalar function. Throughout the paper we assume that:

A1. The matrices $A_{ij}(t)$, $B_i(t)$ are continuous on $[0, T]$; the eigenvalues of the matrix $A_{22}(t)$ have negative real parts and the eigenvalues of the matrix $A_{33}(t)$ have positive real parts for all $t \in [0, T]$; $\lim_{\beta \to 0} \varepsilon(\beta) = 0$, $\varepsilon(0) = 0$.

Differential equations containing small parameters in the derivatives are often used to describe processes consisting of interacting phenomena with

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widely different speeds. Here the vector $x(t)$ designates the "slow" states and $(y(t), z(t))$ are the "fast" states. The scalar β may represent small physical parameters as time constants, masses, and parasitic inductances and capacitances, the suppression of which results in a reduction of the number of states. If we take $\beta = 0$ the order of the system (1) reduces from $n + p + q$ to n, that is, (1) becomes

$$\dot{x} = A_{11}(t)x + A_{12}(t)y + A_{13}(t)z + B_1(t)u, \tag{2a}$$

$$0 = A_{21}(t)x + A_{22}(t)y + B_2(t)u, \tag{2b}$$

$$0 = A_{31}(t)x + A_{33}(t)z + B_3(t)u. \tag{2c}$$

Clearly, this order reduction leads to an essential simplification of the original modes, at least because the differential equation (1) is "stiff" for computations. The change of the state space, however, may be accompanied by various pathological effects as boundary layers, discontinuity of the system performance, etc. Therefore, the perturbation represented by a small parameter in the derivative is called *singular*.

Recently, a number of papers have developed a variety of asymptotic methods for solving singularly perturbed optimal control problems, see the surveys in Kokotovic *et al.* [9] and Vasil'eva and Dmitriev [10]. As collateral results, some of these methods give conditions under which the optimal solution (the optimal value) is a continuous function of the parameter β at $\beta = 0$, i.e., the problem considered is well posed with respect to singular perturbations. The asymptotic methods, however, use essentially the representation of the optimal control as an explicit function of the adjoint state. In general, such representation exists only for unconstrained optimal control problems.

This paper presents a qualitative study of the order reduction for optimal control problems with control constraints. For such problems the continuity properties of the multivalued mapping "singular parameter $\rightarrow$ set of trajectories" play a crucial role. In contrast to the case when the perturbed parameter is not in the derivative (regular perturbations) this mapping turns out to be really singular, namely, its pointwise Hausdorff limit is a larger set than the limit in the L_2-weak topology. This effect was noted first in Dontchev and Veliov [4], for some extensions see Dontchev [5, Chap. 3]. Here we develop the approach of these works relaxing in the same time the assumption that the system which reduces is asymptotically stable. In our case the fast subsystem contains an asymptotically stable (1b) and an unstable (1c) part which are "weakly coupled" by the factor $\varepsilon(\beta)$. Such systems are often called *conditionally stable*.

Section 2 studies the behavior of the trajectories of the system (1) when the parameter β tends to zero. We present four rather technical lemmas, which, however, provide a basis for our principal results. The proofs of these lemmas are given in the Appendix. In the next three sections we consider three classical optimal control problems for the system (1): Mayer's problem, a time-optimal control problem, and a Lagrange problem. Defining properly the corresponding limit problems we develop conditions under which the optimal values of these problems are continuous with respect to the singular parameter β at $\beta = 0$.

We refer here to the earlier papers by Dmitriev [2] (linear Mayer's problem), Binding [1] (nonlinear Mayer's problem independent of the fast states), Javid and Kokotovic [8] (a decomposition of time-optimal control), Gičev and Dontchev [6] (time-optimal problem), and Dontchev and Gičev [3] (integral functional and terminal constraints). The technique from the last two papers was applied in Gičev [7] to a special class of conditionally stable systems without constraints. Here we extend and generalize the corresponding results of these papers.

2. Convergence of the Set of Trajectories

Throughout the paper $|\cdot|$ denotes the euclidean norm. The norm of the space X will be denoted as $\|\cdot\|_X$, and the L_2-norm will be simply $\|\cdot\|$.

Let β_k be an arbitrary sequence, $\beta_k > 0$, $\lim_{k \to +\infty} \beta_k = 0$, and $u_k(\cdot)$ be a sequence of controls, $k = 1, 2, \ldots$. We denote by $(x_k(\cdot), y_k(\cdot), z_k(\cdot))$ the solution of the system (1) on $[0, T]$ with fixed initial conditions

$$x(0) = x^0, \tag{3a}$$

$$y(0) = y^0, \qquad z(0) = z^0, \tag{3b}$$

corresponding to β_k and $u_k(\cdot)$.

The proofs of the lemmas in this section are given in the Appendix.

Lemma 1. *Suppose that the sequence* $z_k(\cdot)$ *satisfies*

C1. $\limsup_{k \to +\infty} \|z_k\| < +\infty$,

where $< +\infty$ *means boundedness. Let the sequence* $u_k(\cdot)$ *converge* L_2-*weakly to* $u_0(\cdot) \in L_2^{(m)}[0, T]$ *and* $(x_0(\cdot), y_0(\cdot), z_0(\cdot))$ *be the solution of the reduced system* (2) *with initial condition* (3a). *Then*

$$\begin{aligned} x_k(\cdot) &\to x_0(\cdot) && \textit{strongly in } C^{(n)}[0, T], \\ y_k(\cdot) &\to y_0(\cdot) && \textit{weakly in } L_2^{(p)}[0, T], \\ z_k(\cdot) &\to z_0(\cdot) && \textit{weakly in } L_2^{(q)}[0, T], \end{aligned}$$

as $k \to +\infty$.

The following assumption

C2. $\limsup_{k \to +\infty} |z_k(T)| < +\infty$

turns out to be stronger than C1, *in context of Lemma* 1.

LEMMA 2. *Suppose that* C2 *holds. Let* $u_k(\cdot)$ *be a bounded sequence in* $L_2^{(m)}[0, T]$ *and* $(x_k(\cdot), y_k(\cdot), z_k(\cdot))$ *be as in Lemma* 1. *Then the sequences* $\|x_k\|_C$, $\|y_k\|$, *and* $\|z_k\|$ *are bounded when* $k \to +\infty$, *i.e.*, C1 *holds.*

From Lemmas 1 and 2 we conclude that

COROLLARY 1. *Let* C2 *be fulfilled and the sequence* $u_k(\cdot)$ *converge* L_2-*weakly to* $u_0(\cdot)$. *Then Lemma* 1 *holds.*

COROLLARY 2. *Suppose that* C2 *holds and the sequence* $\|u_k\|_{L_\infty}$ *is bounded. Then the corresponding sequences* $y_k(\cdot)$ *and* $z_k(\cdot)$ *are uniformly bounded, that is,*

$$\limsup_{k \to +\infty} (\|y_k\|_C + \|z_k\|_C) < +\infty.$$

COROLLARY 3. *Suppose that* C2 *holds and* $u_k(\cdot)$ *converges strongly in* $L_2^{(m)}[0, T]$ *to* $u_0(\cdot)$. *Then*

$$\lim_{k \to +\infty} (\|y_k - y_0\| + \|z_k - z_0\|) = 0.$$

Remark 1. Observe that all the above results hold when, instead of the initial conditions (3), one imposes fixed boundary conditions

$$x(0) = x^0, \qquad y(0) = y^0, \qquad z(T) = z^T,$$

for the system (1). Clearly, in this case C2, and hence C1, is trivially satisfied.

We continue the analysis of the perturbed system (1) on the assumption A1, supposing that the admissible set of controls is

$$U(T) = \{u(\cdot), u(t) \in V \text{ for a.e. } t \in [0, T], u(\cdot) \in L_1^{(m)}[0, T]\},$$

where V is an arbitrary closed set in R^m.

Solving Eqs. (2b) and (2c) with respect to y and z and substituting in (2a) we obtain the following low-order system

$$\dot{x} = A_0(t)x + B_0(t)u, \qquad x(0) = x^0, \tag{4}$$

where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{13}A_{33}^{-1}A_{31}$, $B_0 = B_1 - A_{12}A_{22}^{-1}B_2 - A_{13}A_{33}^{-1}B_3$.

In the sequel we shall denote by $P_x(T)$ the reachable set on $[0, T]$ of the system (4) with controls from $U(T)$, that is, the set of all points in R^n, which can be achieved at the time $t = T$ by means of feasible controls starting from the point x^0 at the time $t = 0$.

We introduce the set

$$P_y(T) = \left\{ y,\ y = \int_0^{+\infty} f(t)\,dt,\ f(t) \in \exp(A_{22}(T)t)\,B_2(T)V,\ f(\cdot) \in L_1^{(r)}[0, +\infty) \right\}.$$

This definition is related to the standard definition of a multivalued mapping. One can easily show that the set $P_y(T)$ can be defined equivalently as

$$P_y(T) = \{ y, \forall \varepsilon > 0\ \exists t_\varepsilon\ \forall t > t_\varepsilon\ \exists y_\varepsilon \in K_y(T, t) \text{ such that } |y - y_\varepsilon| < \varepsilon \},$$

where $K_y(T, t)$ is the reachable set at the time t of the system

$$\dot{y} = A_{22}(T)\,y + B_2(T)u, \qquad y(0) = 0, \tag{5}$$

that is,

$$K_y(T, t) = \left\{ y,\ y = \int_0^t \exp(A_{22}(T)s)\,B_2(T)\,u(s)\,ds,\ u(\cdot) \in L_1^{(m)}[0, t],\ u(s) \in V,\ s \in [0, t] \right\},$$

or

$$P_y(T) = \operatorname{cl} \bigcup_{r > 0} \left\{ y,\ y = \int_0^{+\infty} \exp(A_{22}(T)t)\,B_2(T)\,u(t)\,dt,\ u(t) \in V \cap B_r,\ t \in [0, +\infty),\ u(\cdot)\text{-measurable} \right\},$$

where B_r is the closed ball in R^m centered at zero with radius r. In the second definition $P_y(T)$ consists of all points being limits of sequences from $K_y(T, t)$ when $t \to +\infty$. The last definition is most convenient for our further analysis.

If $0 \in B_2 V$ then $P_y(T)$ is exactly the closure of the reachable set of (5), that is, the set of all points y for which there exist $s \geqslant 0$ and a control feasible on $[0, s]$ driving the state of (5) from 0 at $t=0$ to y at $t=s$. If V is compact then $P_y(T)$ is compact and convex.

Analogously we define a set P_z for the system

$$\dot{z} = -A_{33}(0)z - B_3(0)u, \qquad z(0)=0, \tag{6}$$

that is,

$$P_z = \left\{ z,\ z = -\int_0^{+\infty} f(t)\,dt,\ f(t) \in \exp(-A_{33}(0)t)\,B_3(0)V,\ t \in [0, +\infty), \right.$$
$$\left. f(\cdot) \in L_1^{(q)}[0, +\infty) \right\}.$$

If int $V \neq \varnothing$ and the pair $(A_{33}(0), B_3(0))$ is controllable, i.e.,

$$\operatorname{rank}[B_3(0), A_{33}(0)\,B_3(0), \ldots, A_{33}^{q-1}(0)\,B_3(0)] = q,$$

then int $P_z \neq \varnothing$.

Lemma 3. *Let β_k be an arbitrary sequence, $\beta_k > 0$, $\lim_{k \to +\infty} \beta_k = 0$, and $u_k(\cdot)$ be a sequence from $U(T)$ such that $\limsup_{k \to +\infty} \|u_k\|_{L_\infty} < +\infty$. Denote by $(x_k(\cdot), y_k(\cdot), z_k(\cdot))$ the solution of (1) corresponding to $u_k(\cdot)$ and β_k with initial conditions $x(0)=x^0$, $y(0)=y^0$, $z(0)=z^0$. Let the sequence $z_k(T)$ be bounded. Then the initial conditions x^0 and z^0 satisfy*

$$z^0 \in A_{33}^{-1}(0)\,A_{31}(0)\,x^0 + P_z.$$

Moreover, the sequence $(x_k(T), y_k(T))$ is bounded and every condensation point of this sequence satisfies

$$x \in P_x(T), \qquad y \in -A_{22}^{-1}(T)\,A_{21}(T)x + P_y(T).$$

Remark 2. Observe that $z^0 \notin A_{33}^{-1}(0)\,A_{31}(0)\,x^0 + P_z$ implies that $\lim_{\beta \to 0} |z_\beta(T)| = +\infty$.

In the next lemma we consider the perturbed system (1) on the assumption A1 with controls from $U(T)$ and with initial conditions (3), which satisfy the relation

A2. $z^0 - A_{33}^{-1}(0)\,A_{31}(0)\,x^0 \in \operatorname{int} P_z$;

Lemma 4. *Let $u_0(\cdot) \in U(T) \cap L_\infty^{(m)}[0, T]$ be given and $x_0(\cdot)$ be the corresponding solution of the system (4). Let X^1 and Y^1 be subsets of R^n and R^p, respectively, and $x_0(\cdot)$ satisfies*

$$x_0(T) \in X^1, \qquad x_0(T) \notin \partial P_x(T) \cap \partial X^1,$$

where ∂X^1 *denotes the boundary of the set* X^1. *Suppose that the point* $y \in R^p$ *fulfills*

$$y \in (-A_{22}^{-1}(T)\, A_{21}(T)\, x_0(T) + P_y(T)) \cap Y^1,$$

$$y \notin (-A_{22}^{-1}(T)\, A_{21}(T)\, x_0(T) + \partial P_y(T)) \cap \partial Y^1,$$

and let z *be an arbitrary point in* R^q.

Then for every sufficiently small $\beta > 0$ *there exists a control* $u_\beta(\cdot) \in L_\infty^m[0, T]$, $u_\beta(t) \in \mathrm{co}\, V$ *for* a.e. $t \in [0, T]$ *such that*

$$\lim_{\beta \to 0} u_\beta(t) = u_0(t) \qquad \text{for} \quad \text{a.e. } t \in [0, T]$$

and the corresponding trajectory $(x_\beta(\cdot), y_\beta(\cdot), z_\beta(\cdot))$ *of the perturbed system* (1) *with initial conditions* (3) *satisfies*

$$\lim_{\beta \to 0} (x_\beta(T), y_\beta(T)) = (x_0(T), y), \qquad z_\beta(T) = z$$

and

$$x_\beta(T) \in X^1, \qquad y_\beta(T) \in Y^1.$$

The above results can be restated in terms of convergence of the set of trajectories as a multifunction of the parameter β. For simplicity, let us consider only the stable fast subsystem

$$\beta \dot{y} = A_{22}(t)\, y + B_2(t) u, \qquad y(0) = y^0, \qquad t \in [0, T], \tag{7}$$

assuming that the control takes values from a compact set V in R^m. For fixed $\beta > 0$ denote by Σ_β the set of the trajectories of (7) on $[0, T]$, that is, the set of absolutely continuous functions $y(\cdot)$, every element of which is a solution of (7) for some feasible control. Let Σ_0 be the set of "trajectories" of the reduced system

$$0 = A_{22}(t)\, y + B_2(t) u,$$

that is,

$$\Sigma_0 = \{ y(\cdot),\ y(t) = -A_{22}^{-1}(t)\, B_2(t)\, u(t),\ u(t) \in V,\ t \in [0, T],$$
$$u(\cdot)\text{-measurable} \}.$$

Let

$$P_y(t) = \int_0^{+\infty} \exp(A_{22}(t) s)\, B_2(t) V\, ds$$

and let the set of functions Σ_1 be defined as

$$\Sigma_1 = \{ y(\cdot),\ y(t) \in P_y(t),\ t \in [0, T] \}.$$

Observe that for each $t \in [0, T]$ the set of values of Σ_1 contains, but may be essentially larger than, the set of values of Σ_0 (for an example see Dontchev and Veliov [4]).

We say that the sequence of sets A_k is M-convergent to A_0 as $k \to +\infty$ if: (1) for every $a_0 \in A_0$ there exists a sequence a_k, $a_k \in A_k$ such that $a_k \to a_0$ as $k \to +\infty$; (2) if $a_k \in A_k$ and $a_k \to a_0$ as $k \to +\infty$ then $a_0 \in A_0$.

From the above lemmas we conclude that:

(1) For every sequence $\beta_k \to 0$ the set Σ_{β_k} is M-convergent to Σ_0 in the L_2-weak topology;

(2) For every $t > 0$ the set Σ_β is pointwise convergent to Σ_1 when $\beta \to 0$ in the sense of Hausdorff.

The proof of the first statement follows from Lemma 1. Let $y_k(\cdot) \in \Sigma_{\beta_k}$, $y_k(\cdot) \to y_0(\cdot)$ as $k \to +\infty$ L_2-weakly. Since V is compact, from the corresponding sequence of controls one can extract a L_2-weakly convergent subsequence. Then, the corresponding subsequence of solutions of (7) will converge L_2-weakly to an element of Σ_0. Conversely, if $y_0(\cdot) \in \Sigma_0$ then if $u_0(\cdot)$ is the corresponding control one can apply $u_0(\cdot)$ to the perturbed system (7) and get a sequence from Σ_{β_k} convergent even L_2-strongly to $y_0(\cdot)$, see Corollary 3.

The second statement is a consequence of Lemmas 3 and 4, where T should be replaced by an arbitrary $t > 0$.

3. Singular Perturbations in Mayer's Problem

In this and in the next sections we apply the above lemmas to three classical optimal control problems. As a result we get conditions under which the optimal value of these problems is continuous with respect to the singular perturbation parameter.

We consider the singularly perturbed linear system (1) from the Introduction, assuming that the matrices A_{ij} and B_i and the function $\varepsilon(\cdot)$ satisfy the conditions in A1, and the initial conditions x^0 and z^0 satisfy A2. The reduced system corresponding to $\beta = 0$ is given in (4).

Let the time interval $[0, T]$ be fixed. Consider the problem

$$g(x(T), y(T), z(T)) \to \inf \qquad (M_\beta)$$

subject to (1) and

$$u(\cdot)\in U(T)=\{u(\cdot),\, u(t)\in V \text{ for a.e. } t\in[0,T],\, u(\cdot)\text{-measurable}\},$$
$$x(T)\in X^1, \qquad y(T)\in Y^1, \qquad z(T)\in Z^1,$$

for given constrained sets $V\subset R^m$, $X^1\subset R^n$, $Y^1\subset R^p$, $Z^1\subset R^q$.

Let $P_x(T)$ and $P_y(T)$ be the reachable sets introduced in the previous section. We shall prove that the optimal value of the problem (M_β) converges to the optimal value of the following problem

$$g(x,y,z)\to\inf \tag{M_0}$$

subject to

$$x\in P_x(T)\cap X^1, \qquad y\in(-A_{22}^{-1}(T)\,A_{21}(T)x+P_y(T))\cap Y^1, \qquad z\in Z^1.$$

This problem can be restated as

$$g_0(x(T))\to\inf$$

subject to

$$\dot{x}=A_0(t)x+B_0(t)u, \qquad x(0)=x^0$$
$$u(\cdot)\in U(T), \qquad x(T)\in X^1,$$

where

$$g_0(x)=\inf\{g(x,y,z),\, y\in(-A_{22}^{-1}(T)\,A_{21}(T)x+P_y(T))\cap Y^1,\, z\in Z^1\}.$$

THEOREM 1. *Suppose that the following conditions hold:*

M1. *The set V is compact, X^1, Y^1, Z^1 are closed, and the function $g(\cdot)$ is continuous.*

M2. *The set Z^1 is compact or*

$$\inf\{g(x,y,z),\, x\in X^1,\, y\in Y^1,\, z\in Z^1,\, |z|\geqslant n\}\to+\infty \qquad \textit{as} \quad n\to+\infty.$$

M3. *The reduced problem (M_0) has a solution $\hat{u}_0(\cdot)$, $\hat{x}_0$, $\hat{y}_0$, $\hat{z}_0$ such that*

(i) $\hat{x}_0\notin\partial P_x(T)\cap\partial X^1$ (*i.e.*, $\hat{x}_0\in\operatorname{int}P_x(T)\cap X^1$ *or* $\hat{x}_0\in\operatorname{int}X^1\cap P_x(T)$),

(ii) $\hat{y}_0\notin(-A_{22}^{-1}(T)\,A_{21}(T)\,\hat{x}_0+\partial P_y(T))\cap\partial Y^1$.

Then for sufficiently small β the perturbed problem (M_β) has a solution and if $\hat{g}_\beta$ is the optimal value then

$$\lim_{\beta \to 0} \hat{g}_\beta = \hat{g}_0 = g(\hat{x}_0, \hat{y}_0, \hat{z}_0).$$

Proof. According to Lemma 4, for sufficiently small β there exists a control $u_\beta(\cdot) \in L_\infty^{(m)}[0, T]$, $u_\beta(t) \in \operatorname{co} V$ for a.e. $t \in [0, T]$ such that the corresponding solution $(x_\beta(\cdot), y_\beta(\cdot), z_\beta(\cdot))$ of (1) satisfies

$$\lim_{\beta \to 0} (x_\beta(T), y_\beta(T)) = (\hat{x}_0, \hat{y}_0)$$

and

$$x_\beta(T) \in X^1, \qquad y_\beta(T) \in Y^1, \qquad z_\beta(T) = \hat{z}_0.$$

Hence, for small β, the intersection of the reachable set of the perturbed system (1) with $X^1 \times Y^1 \times Z^1$ is nonempty. (Taking co V instead of V does not change the reachable set.) Since this intersection is compact, for small β the perturbed problem (M_β) has a solution. Moreover,

$$\limsup_{\beta \to 0} \hat{g}_\beta \leqslant \lim_{\beta \to 0} g(x_\beta(T), y_\beta(T), z_\beta(T)) = \hat{g}_0. \tag{8}$$

Let $\hat{u}_\beta(\cdot)$ be an optimal control for (M_β) and $(\hat{x}_\beta(\cdot), \hat{y}_\beta(\cdot), \hat{z}_\beta(\cdot))$ be the corresponding optimal trajectory. From M2 and (8) it follows that $|z_\beta(T)|$ is bounded when $\beta \to 0$. Then one can apply Lemma 3 obtaining that every condensation point (x_0, y_0) of $(\hat{x}_\beta(T), \hat{y}_\beta(T))$ satisfies

$$x_0 \in P_x(T) \cap X^1, \qquad y_0 \in (-A_{22}^{-1}(T)\, A_{21}(T)\, x_0 + P_x(T)) \cap Y^1.$$

Let z_0 be a condensation point of $\hat{z}_\beta(T)$. Then, choosing properly (x_0, y_0, z_0) we obtain

$$\hat{g}_0 \leqslant g(x_0, y_0, z_0) = \liminf_{\beta \to 0} \hat{g}_\beta.$$

This inequality, combined with (8), gives us the desired result.

The following examples show that the assumptions of Theorem 1 are essential for the obtained results:

EXAMPLE 1 (M2 doesn't hold).

$$x(1) \to \inf,$$

$$\dot{x} = z, \qquad x(0) = 0, \qquad u(t) \in V = [-1, 1],$$

$$\dot{z} = z + u, \qquad z(0) = 0, \qquad X^1 = R^1, \qquad Z^1 = R^1.$$

The reduced system is

$$\dot{x} = u$$

and $\hat{g}_0 = -1$. For $u(t) \equiv -1$ we have

$$x_\beta(1) = 1 - \beta \exp(1/\beta).$$

Hence

$$\hat{g}_\beta \leqslant g(x_\beta(1)) \to -\infty \qquad \text{as} \quad \beta \to 0,$$

that is, the problem is not well posed.

EXAMPLE 2 (M3(i) doesn't hold).

$$(x(1)-1)^2 + (y(1)-1)^2 \to \inf,$$

$$\dot{x} = y, \qquad x(0) = 0,$$

$$\beta \dot{y} = -y + u, \qquad y(0) = 0,$$

$$u(t) \in V = [-1, 1], \qquad X^1 = [1, 2], \qquad Y^1 = R^1.$$

Here

$$P_x(1) = [-1, 1], \qquad \hat{u}_0(t) \equiv 1, \qquad \hat{x}_0(1) \in \partial X^1 \cap \partial P_x(1).$$

For $\beta > 0$ we have

$$y_\beta(t) \leqslant 1 - \exp(-t/\beta)$$

and

$$x_\beta(1) < 1.$$

This means that the problem (M_β) has no solution.

EXAMPLE 3 (M3(ii) doesn't hold). The system of Example 2 with $X^1 = R^1$, $Y^1 = [1, 2]$, $V = [-1, 1]$, and with the same functional. We have $\hat{y}_0 \in \partial P_y(1) \cap \partial Y^1$, and for $\beta > 0$

$$y_\beta(1) \leqslant 1 - \exp(-1/\beta) < 1,$$

that is, (M_β) has no solution.

4. Time-Optimal Problem

Let X^1, Y^1, Z^1 be given sets in R^n, R^p, and R^q, respectively, and let

$$U = \bigcup_{T>0} U(T),$$

where $U(T)$ is defined as in the previous section. For fixed $\beta > 0$ we consider the following problem (T_β): find a control $\hat{u}_\beta(\cdot)$ from U such that the corresponding trajectory $(\hat{x}_\beta(\cdot), \hat{y}(\cdot), \hat{z}_\beta(\cdot))$ of (1) starting from the point (x^0, y^0, z^0) at $t=0$ reaches the target set $X^1 \times Y^1 \times Z^1$ in minimal time $\hat{T}_\beta$. The reduced problem (T_0) consists in finding a control $\hat{u}_0(\cdot)$ from U which drives the state of (4) from x^0 at $t=0$ to X^1 in minimal time $\hat{T}_0$.

For simplicity, we suppose that all the conditions in the general assumption A1 hold on $[0, +\infty)$.

Theorem 2. *Suppose that the following conditions hold:*

T1. *The sets V and Z^1 are compact, X^1 is closed.*

T2. *The reduced problem (T_0) has a solution $\hat{u}_0(\cdot)$, $\hat{x}_0(\cdot)$, $\hat{T}_0$ such that*

$$(\text{int}\, Q \cap Y^1) \cup (Q \cap \text{int}\, Y^1) \neq \varnothing,$$

where $Q = -A_{22}^{-1}(\hat{T}_0)\, A_{21}(\hat{T}_0)\, \hat{x}_0(\hat{T}_0) + P_y(\hat{T}_0)$.

T3. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$*

$$\hat{x}_0(\hat{T}_0) \in \text{int}\, P_x(\hat{T}_0 + \varepsilon).$$

Then for small $\beta > 0$ the perturbed problem (T_β) has a solution and

$$\lim_{\beta \to 0} \hat{T}_\beta = \hat{T}_0.$$

Proof. Let $\varepsilon \in (0, \varepsilon_0]$. Then $\hat{x}_0 = \hat{x}_0(\hat{T}_0) \in \text{int}\, P_x(\hat{T}_0 + \varepsilon)$ implies that there exists a control $u_\varepsilon(\cdot) \in U(\hat{T}_0 + \varepsilon)$ such that the corresponding solution $x_\varepsilon(\cdot)$ of the reduced system (4) satisfies $x_\varepsilon(\hat{T}_0 + \varepsilon) = \hat{x}_0$. Since $A_{22}(T)$, $A_{21}(T)$, and $P_y(T)$ depend continuously on T at $T = \hat{T}_0$, then for small $\varepsilon \in (0, \varepsilon_0)$ there exists y_0 such that

$$y_0 \in (-A_{22}^{-1}(\hat{T}_0 + \varepsilon)\, A_{21}(\hat{T}_0 + \varepsilon)\, x_0 + \text{int}\, P_y(\hat{T}_0 + \varepsilon)) \cap Y^1$$

or

$$y_0 \in (-A_{22}^{-1}(\hat{T}_0 + \varepsilon)\, A_{21}(\hat{T}_0 + \varepsilon)\, \hat{x}_0 + P_y(\hat{T}_0 + \varepsilon)) \cap \text{int}\, Y^1.$$

Choosing an arbitrary $z \in Z^1$ and applying Lemma 4 we conclude that one can find a control $u_\beta(\cdot) \in L_\infty^m[0, \hat{T}_0 + \varepsilon]$, $u_\beta(t) \in \text{co}\, V$ for a.e. $t \in$

$[0, \hat{T}_0+\varepsilon]$, which drives the perturbed system to a point from $X^1 \times Y^1 \times Z^1$ in time $[0, \hat{T}_0+\varepsilon]$. Thus,

$$\limsup_{\beta \to 0} \hat{T}_\beta \leqslant T_0+\varepsilon,$$

and since ε could be arbitrary small, then

$$\limsup_{\beta \to 0} \hat{T}_\beta \leqslant \hat{T}_0.$$

Denote $T_0 = \liminf_{\beta \to 0} \hat{T}_\beta$. For some small $\varepsilon > 0$ we extend the optimal control $\hat{u}_\beta(\cdot)$ (for instance, by a constant) on $[0, T_0+\varepsilon]$. Choosing a L_2-weakly convergent sequence of $\hat{u}_\beta(\cdot)$ and applying Lemmas 1 and 2 we conclude that the corresponding sequence of the optimal trajectories $\hat{x}_\beta(\cdot)$ converges uniformly to a trajectory $x_0(\cdot)$ of the reduced system. Since X^1 is closed, $x_0(T_0) \in X^1$, hence $\hat{T}_0 \leqslant T_0$. This means that

$$\hat{T}_0 \leqslant \liminf_{\beta \to 0} \hat{T}_\beta,$$

which completes the proof.

Remark 3. The condition T3 holds, for example, when $X^1 = \{0\}$, $0 \in \operatorname{int} V$ and the pair $(A_0(\hat{T}_0), B_0(\hat{T}_0))$ is controllable.

EXAMPLE 4 (T2 doesn't hold). The system of Example 2 with $V = [-1, 1]$, $X^1 = Y^1 = \{1\}$. Here $P_y(1) = [-1, 1]$, $\hat{T}_0 = 1$. For $\beta > 0$

$$y(t) = \frac{1}{\beta}\int_0^t \exp\left(-\frac{t-s}{\beta}\right) u(s)\, ds < 1,$$

hence, the perturbed problem has no solution.

EXAMPLE 5 (T3 doesn't hold).

$$\begin{aligned}
&\dot{x}_1 = u, && u(t) \in [-1, 1], && \\
&\dot{x}_2 = y+u, && x_2(0) = 0, && X^1 = \{(1, 2)\}, \\
&\beta\dot{y} = -y+u, && y(0) = 0, && Y^1 = \{1\}.
\end{aligned}$$

For some $T < +\infty$ one should have

$$\int_0^T u(t)\, dt = 1, \qquad \frac{1}{\beta}\int_0^T \exp\left(-\frac{T-t}{\beta}\right) u(t)\, dt = 1,$$

$$\int_0^T u(t)\, dt + \frac{1}{\beta}\int_0^T \int_0^t \exp\left(-\frac{t-s}{\beta}\right) u(s)\, ds\, dt = 2.$$

This means that some $u(\cdot)$ should satisfy

$$\int_0^T \exp\left(-\frac{T-t}{\beta}\right) u(t)\,dt = 0, \qquad \int_0^T \exp\left(-\frac{T-t}{\beta}\right) u(t)\,dt = \beta,$$

which is impossible. Hence, the perturbed problem has no solution.

5. Lagrange Problem

For fixed $\beta > 0$ consider the problem (L_β): minimize the functional

$$J_\beta(u(\cdot)) = \int_0^T f(x(t), y(t), z(t), u(t), t)\,dt \tag{9}$$

subject to (1) and

$$u(\cdot) \in U(T) = \{u(\cdot),\ u(t) \in V \text{ for a.e. } t \in [0, T],\ u(\cdot) \in L_1^{(m)}[0, T]\},$$
$$x(T) \in X^1, \qquad y(T) \in Y^1, \qquad z(T) \in Z^1,$$

where X^1, Y^1, Z^1, and V are given sets in R^n, R^p, R^q, and R^m, respectively, and the final time T is fixed. We will compare this problem with the following reduced problem (L_0): minimize (9) subject to (4), $u(\cdot) \in U(T)$, and $x(T) \in X^1$.

Theorem 3. *Suppose that the following conditions hold:*

L1. *The set X^1 is closed, V is convex and closed; the function $f(\cdot, t)$ is continuous in $R^{n+p+q+m}$ for all $t \in [0, T]$; the integral (9) is lower semicontinuous in the uniform topology for $x(\cdot)$ and in the L_2-weak topology for $(y(\cdot), z(\cdot), u(\cdot))$.*

L2. *One of the following conditions holds:*

(i) *Z^1 and V are compact;*

(ii) *Z^1 is compact and there exists $c > 0$ such that*

$$f(x, y, z, u, t) \geq c\,|u|^2$$

for all $x \in R^n$, $y \in R^p$, $z \in R^q$, $u \in V$, $t \in [0, T]$;

(iii) *there exists $c > 0$ such that*

$$f(x, y, z, u, t) \geq c(|u|^2 + |z|^2)$$

for all $x \in R^n$, $y \in R^p$, $z \in R^q$, $u \in V$, $t \in [0, T]$.

L3. *The reduced problem* (L_0) *has a solution* $\hat{u}_0(\cdot)$, $\hat{x}_0(\cdot)$, $\hat{y}_0(\cdot)$, $\hat{z}_0(\cdot)$ *such that* $\hat{u}_0(\cdot)\in L^m_\infty[0, T]$ *and* $\hat{x}_0=\hat{x}_0(T)$ *satisfies*

$$\hat{x}_0 \notin \partial X^1 \cap \partial P_x(T)$$

and

$$(\operatorname{int} Q \cap Y^1) \cup (Q \cap \operatorname{int} Y^1) \neq \emptyset,$$

where $Q = -A_{22}^{-1}(T)\, A_{21}(T)\, \hat{x}_0 + P_y(T)$.

Then for small β *the perturbed problem* (L_β) *has a solution and if* $\hat{J}_\beta$ *is the optimal value, then*

$$\lim_{\beta\to 0} \hat{J}_\beta = \hat{J}_0 = \hat{J}_0(\hat{u}_0(\cdot)).$$

Proof. Since $(\operatorname{int} Q \cap Y^1) \cup (Q \cap \operatorname{int} Y^1) \neq \emptyset$ one can apply Lemma 4 and find a control $u_\beta(\cdot)$, $u_\beta(t)\in V$ for a.e. $t\in[0, T]$, $\limsup_{\beta\to 0} \|u\|_{L_\infty} < +\infty$, and $\lim_{\beta\to 0} u_\beta(t)=\hat{u}_0(t)$ for a.e. $t\in[0, T]$ such that the corresponding solution $(x_\beta(\cdot), y_\beta(\cdot), z_\beta(\cdot))$ of the perturbed system satisfies $x_\beta(T)\in X^1$, $y_\beta(T)\in Y^1$, $z_\beta(T)=z$ for an arbitrary $z\in Z^1$. From Lemma 1 and Corollary 3 we deduce that

$$\lim_{\beta\to 0} (\|x_\beta - \hat{x}_0\|_C + \|y_\beta - \hat{y}_0\| + \|z_\beta - \hat{z}_0\|) = 0.$$

Moreover, by Lemma 2

$$\limsup_{\beta\to 0} (\|y_\beta\|_C + \|z_\beta\|_C) < +\infty.$$

Then, choosing a pointwise convergent sequence of $(y_\beta(\cdot), z_\beta(\cdot))$ in an appropriate way and using the continuity of $f(\cdot, t))$, we conclude that

$$\limsup_{\beta\to 0} \hat{J}_\beta \leqslant \lim_{\beta\to 0} J_\beta(u_\beta(\cdot)) = \hat{J}_0. \tag{10}$$

This relation, combined with L1 and L2, implies existence of an optimal solution of the problem (L_β) for sufficiently small β.

Now, let $\hat{u}_\beta(\cdot)$ be an optimal control for (L_β) and $(\hat{x}_\beta(\cdot), \hat{y}_\beta(\cdot), \hat{z}_\beta(\cdot))$ be the corresponding trajectory. From L2 and (10) it follows that $\limsup_{\beta\to 0} \|\hat{u}_\beta\| < +\infty$, moreover $\|\hat{z}_\beta\|$ or $|\hat{z}_\beta(T)|$ are bounded when $\beta\to 0$. Choosing a L_2-weakly convergent sequence of controls $\hat{u}_\beta(\cdot)$ one can apply Lemma 1 and get that the corresponding sequence $(\hat{x}_\beta(\cdot), \hat{y}_\beta(\cdot), \hat{z}_\beta(\cdot))$ of the trajectories converges (uniformly for $\hat{x}_\beta(\cdot)$ and L_2-weakly for

$(\hat{y}_\beta(\cdot), \hat{z}_\beta(\cdot)))$ to $(x_0(\cdot), y_0(\cdot), z_0(\cdot))$, which solves (2) for the weak limit $u_0(\cdot)$. Since X^1 is closed, $x_0(T) \in X^1$ and, therefore,

$$\hat{J}_0 \leqslant J_0(u_0(\cdot)) \leqslant \liminf_{\beta \to 0} \hat{J}_\beta.$$

This inequality, together with (10), completes the proof.

We note that the condition L1 is standard for existence of optimal controls of both the reduced and the original problems. The requirements in L2 give us boundedness of $\|\hat{z}_\beta\|$ or of $|\hat{z}_\beta(T)|$ when $\beta \to 0$. Z^1 is not compact or the grouth condition in L2(ii) does not hold, then, in general, the remaining conditions do not imply continuity of the optimal value. For such an example, see Example 1 with

$$J(u(\cdot)) = \int_0^1 (x^2(t) + u^2(t))\, dt.$$

The assumption L3 is related to the reachability of the target set by the perturbed system and, hence, to the existence of a solution of the perturbed problem. This condition is also essential, see Examples 2 and 3 with a functional

$$J(u(\cdot)) = \int_0^1 ((x(t) - 1)^2 + (y(t) - 1)^2)\, dt.$$

APPENDIX

We start the presentation of the proofs of the lemmas with some preliminary results.

In the sequel c denotes a generic constant which does not depend on the time t and on the parameter β but may change in different relations. Denote by $Y(t, \tau, \beta)$, $t \geqslant \tau$, and by $Z(t, \tau, \beta)$, $t \leqslant \tau$, the fundamental matrix solutions of the equations

$$\beta \dot{y} = A_{22}(t)\, y, \qquad \beta \dot{z} = A_{33}(t) z,$$

so that $Y(\tau, \tau, \beta) = I^p$, $Z(t, t, \beta) = I^q$ (the identity). From the assumption A1 it follows that there exist constants σ, $\sigma_0 > 0$ such that

$$|Y(t, \tau, \beta)| \leqslant \sigma_0 \exp(-\sigma(t - \tau)/\beta), \qquad t \geqslant \tau, \tag{A1}$$

$$|Z(t, \tau, \beta)| \leqslant \sigma_0 \exp(\sigma(t - \tau)/\beta), \qquad t \leqslant \tau. \tag{A2}$$

We use further the following standard result: Let $p(\cdot)\in L_1[0, T]$, $q(\cdot)\in L_2[0, T]$, and the function $r(\cdot)$ be defined as

$$r(t)=\int_0^t p(t-s)\, q(s)\, ds.$$

Then

$$\|r\| \leqslant \|p\|_{L_1} \|q\|. \tag{A3}$$

The same inequality holds when

$$r(t)=\int_t^T p(s-t)\, q(s)\, ds.$$

Using (A1) and (A2) one can easily show that if $f_1^\beta(\cdot)\in L_\infty^{(p)}[0, T]$ and $f_2^\beta(\cdot)\in L_\infty^{(q)}[0, T]$ then

$$\left\|\frac{1}{\beta}\int_0^{\cdot} Y(\cdot, \tau, \beta)\, f_1^\beta(\tau)\, d\tau\right\|_C \leqslant c\, \|f_1^\beta\|_{L_\infty}, \tag{A4}$$

$$\left\|\frac{1}{\beta}\int_{\cdot}^T Z(\cdot, \tau, \beta)\, f_2^\beta(\tau)\, d\tau\right\|_C \leqslant c\, \|f_2^\beta\|_{L_\infty}. \tag{A5}$$

Moreover, from (A1–3), if $f_1^\beta(\cdot)\in L_2^{(p)}[0, T]$, $f_2^\beta(\cdot)\in L_2^{(q)}[0, T]$, then

$$\left\|\frac{1}{\beta}\int_0^{\cdot} Y(\cdot, \tau, \beta)\, f_1^\beta(\tau)\, d\tau\right\| \leqslant c\, \|f_1^\beta\|, \tag{A6}$$

$$\left\|\frac{1}{\beta}\int_{\cdot}^T Z(\cdot, \tau, \beta)\, f_2^\beta(\tau)\, d\tau\right\| \leqslant c\, \|f_2^\beta\|. \tag{A7}$$

In Dontchev [5, p. 64] it is proved that if $\varphi_1(\cdot)\in L_2^{(p)}[0, T]$ then

$$\lim_{\beta\to 0}\left\|\frac{1}{\beta}\int_0^{\cdot} Y(\cdot, \tau, \beta)\, \varphi_1(\tau)\, d\tau + A_{22}^{-1}(\cdot)\, \varphi_1(\cdot)\right\| = 0. \tag{A8}$$

Similarly, if $\varphi_2(\cdot)\in L_2^{(q)}[0, T]$ then

$$\lim_{\beta\to 0}\left\|\frac{1}{\beta}\int_{\cdot}^T Z(\cdot, \tau, \beta)\, \varphi_2(\tau)\, d\tau - A_{33}^{-1}(\cdot)\, \varphi_2(\cdot)\right\| = 0. \tag{A9}$$

Let $f_1^\beta(\cdot)\in L_2^{(p)}[0, T]$. Consider the integral

$$I_\beta^1(t)=\frac{1}{\beta}\int_0^t\int_0^\tau Y(\tau, s, \beta)\, f_1^\beta(s)\, ds\, d\tau = \frac{1}{\beta}\int_0^t\int_s^t Y(\tau, s, \beta)\, d\tau\, f_1^\beta(s)\, ds.$$

Using the technique of the proof of Lemma 3.1 in Dontchev [5, p. 62], one can get that for every $t \in (0, T]$

$$\lim_{\beta \to 0} \left\| \frac{1}{\beta} \int^{t} Y(\tau, \cdot, \beta)\, d\tau + A_{22}^{-1}(\cdot) \right\|_{L_2^{(p^2)}[0,t]} = 0.$$

Hence, applying Hoelder inequality we have

$$I_\beta^1(t) = \delta_\beta \, \| f_1^\beta(\cdot) \| + \left| \int_0^t A_{22}^{-1}(\tau) f_1^\beta(\tau)\, d\tau \right|, \tag{A10}$$

where $\delta_\beta \to 0$ as $\beta \to 0$ uniformly in $[0, T]$.

Consider now the integral

$$\begin{aligned} I_\beta^2(t) &= \frac{1}{\beta} \int_0^t \int_\tau^T Z(\tau, s, \beta) f_2^\beta(s)\, ds\, d\tau \\ &= \frac{1}{\beta} \int_t^T \int_0^t Z(\tau, s, \beta)\, d\tau\, f_2^\beta(s)\, ds + \frac{1}{\beta} \int_0^t \int_0^s Z(\tau, s, \beta)\, d\tau\, f_2^\beta(s)\, ds. \end{aligned}$$

where $f_2^\beta(\cdot) \in L_2^{(q)}[0, T]$. For the first summand, using (A2), by simple integration we get

$$\begin{aligned} \left| \frac{1}{\beta} \int_t^T \int_0^t Z(\tau, s, \beta)\, d\tau\, f_2^\beta(s)\, ds \right| &\leqslant \frac{\sigma_0}{\beta} \int_t^T \int_0^t \exp(\sigma(\tau - s)/\beta)\, d\tau\, |f_2^\beta(s)|\, ds \\ &\leqslant c \sqrt{\beta}\, \| f_2^\beta \| \end{aligned}$$

for all $t \in [0, T]$ and for small β. As before

$$\lim_{\beta \to 0} \left\| \int_0 Z(\tau, \cdot, \beta)\, d\tau - A_{33}^{-1}(\cdot) \right\|_{L_2^{(q^2)}[0,t]} = 0.$$

Finally, we obtain

$$|I_\beta^2(t)| \leqslant \delta_\beta \, \| f_2^\beta(\cdot) \| + \left| \int_0^t A_{33}^{-1}(s) f_2^\beta(s)\, ds \right|, \tag{A11}$$

where $\delta_\beta \to 0$ as $\beta \to 0$ uniformly in $t \in [0, T]$.

Consider the operators

$$C_\beta \colon L_2^{(q)}[0, T] \to C^{(p)}[0, T],$$
$$D_\beta \colon L_2^{(p)}[0, T] \to C^{(q)}[0, T],$$

defined as

$$(C_\beta f_1)(t) = \frac{1}{\beta^2} \int_0^t Y(t, \tau, \beta)\, A_{23}(\tau) \int_\tau^T Z(\tau, s, \beta)\, f_1(s)\, ds\, d\tau,$$

$$(D_\beta f_2)(t) = \frac{1}{\beta^2} \int_t^T Z(t, \tau, \beta)\, A_{32}(\tau) \int_0^\tau Y(\tau, s, \beta)\, f_2(s)\, ds\, d\tau.$$

Using (A1) and (A2) one can get the estimates

$$\|C_\beta f_1^\beta\|_C \leqslant c\, \|f_1^\beta\|_{L_\infty}, \qquad \|D_\beta f_2^\beta\|_C \leqslant c\, \|f_2^\beta\|_{L}. \tag{A12}$$

Moreover, applying (A3) twice we obtain

$$\|C_\beta f_1^\beta\| \leqslant c\, \|f_1^\beta\|, \qquad \|D_\beta f_2^\beta\| \leqslant c\, \|f_2^\beta\|. \tag{A13}$$

In the sequel we shall designate by δ_k a generic sequence convergent to zero as $k \to +\infty$ uniformly in $[0, T]$.

Proof of Lemma 1. We show first that the sequences $\|x_k\|_C$ and $\|y_k\|$ are bounded. To this end we use the following relations: From (A1) we get

$$\|Y(\cdot, 0, \beta_k)\, y^0\| = \delta_k. \tag{A14}$$

Furthermore, by (A4)

$$\left\| \frac{\varepsilon(\beta_k)}{\beta_k} \int_0^\cdot Y(\cdot, \tau, \beta_k)\, A_{23}(\tau)\, z_k(\tau)\, d\tau \right\| \leqslant \varepsilon(\beta_k) c\, \|z_k\|, \tag{A15}$$

and

$$\left\| \frac{1}{\beta_k} \int_0^\cdot Y(\cdot, \tau, \beta_k)\, B_2(\tau)\, u_k(\tau)\, d\tau \right\| \leqslant c\, \|u_k\|. \tag{A16}$$

Similarly as in (A10) one has

$$\left| \frac{1}{\beta_k} \int_0^t A_{12}(\tau) \int_0^\tau Y(\tau, s, \beta_k)\, A_{21}(s)\, x_k(s)\, ds\, d\tau \right|$$

$$\leqslant \delta_k\, \|x_k\|_C + \left| \int_0^t A_{12}(\tau)\, A_{22}^{-1}(\tau)\, A_{21}(\tau)\, x_k(\tau)\, d\tau \right|. \tag{A17}$$

Using (A14–17) in the Cauchy formula

$$y_k(t) = Y(t, 0, \beta_k)\, y^0 + \frac{1}{\beta_k} \int_0^t Y(t, \tau, \beta_k)(A_{21}(\tau)\, x_k(\tau)$$

$$+\ \varepsilon(\beta_k)\, A_{23}(\tau)\, z_k(\tau) + B_2(\tau)\, u_k(\tau))\, d\tau$$

we obtain that

$$\left|\int_0^t A_{12}(\tau)\, y_k(\tau)\, d\tau\right| \leqslant \delta_k + c \int_0^t |x_k(\tau)|\, d\tau.$$

This inequality, applied to

$$|x_k(t)| \leqslant |x^0| + c\left(\int_0^t |x_k(\tau)|\, d\tau + \left|\int_0^t A_{12}(\tau)\, y_k(\tau)\, d\tau\right| + \|z_k\| + \|u_k\|\right),$$

gives us uniform boundedness of the sequence $x_k(\cdot)$, on the basis of the Gronwall lemma. Then, the boundedness of $\|y_k\|$ follows from (A14–16) and from the assumption C1.

Consider Eq. (1c). Since $\|x_k\|$, $\|y_k\|$, and $\|u_k\|$ are bounded, using Hoelder inequality and (A2) we get

$$|z_k(T)| \leqslant |z^0| + \frac{c}{\beta_k}\left(\int_0^T |z_k(t)|\, dt + \delta_k\right) \leqslant \frac{c}{\beta_k}. \tag{A18}$$

Denote $\Delta x_k = x_k - x_0$, $\Delta y_k = y_k - y_0$, $\Delta z_k = z_k - z_0$, $\Delta u_k = u_k - u_0$. In the sequel we use the following relations:

$$\begin{aligned}\Delta y_k(t) = {} & Y(t, 0, \beta_k)\, y^0 - \varepsilon^2(\beta_k)(C_{\beta_k} A_{32}\, \Delta y_k)(t) \\ & + \frac{1}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\, A_{21}(\tau)\, \Delta x_k(\tau)\, d\tau - \varepsilon(\beta_k)(C_{\beta_k} A_{31}\, \Delta x_k)(t) \\ & + \frac{1}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\, B_2(\tau)\, \Delta u_k(\tau)\, d\tau - \varepsilon(\beta_k)(C_{\beta_k} B_3\, \Delta u_k)(t) \\ & + \frac{\varepsilon(\beta_k)}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\, A_{23}(\tau)\, Z(\tau, T, \beta_k)\, z_k(T)\, d\tau \\ & - \frac{1}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\, A_{22}(\tau)\, y_0(\tau)\, d\tau - y_0(t),\end{aligned} \tag{A19}$$

$$\begin{aligned}\Delta z_k(t) = {} & Z(t, T, \beta_k)\, z_k(T) - \varepsilon^2(\beta_k)(D_{\beta_k} A_{23}\, \Delta z_k)(t) \\ & - \frac{1}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, A_{31}(\tau)\, \Delta x_k(\tau)\, d\tau - \varepsilon(\beta_k)(D_{\beta_k} A_{21}\, \Delta x_k)(t) \\ & - \frac{1}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, B_3(\tau)\, \Delta u_k(\tau)\, d\tau - \varepsilon(\beta_k)(D_{\beta_k} B_2\, \Delta u_k)(t) \\ & - \frac{\varepsilon(\beta_k)}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, A_{32}(\tau)\, Y(\tau, 0, \beta_k)\, y^0\, d\tau \\ & + \frac{1}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, A_{33}(\tau)\, z_0(\tau)\, d\tau - z_0(t).\end{aligned} \tag{A20}$$

We will prove that

$$\left|\int_0^t A_{12}(\tau)\,\Delta y_k(\tau)\,d\tau\right| \leqslant \delta_k + c\int_0^t |\Delta x_k(\tau)|\,d\tau, \tag{A21}$$

$$\left|\int_0^t A_{13}(\tau)\,\Delta z_k(\tau)\,d\tau\right| \leqslant \delta_k + c\int_0^t |\Delta x_k(\tau)|\,d\tau. \tag{A22}$$

Having this done, the uniform convergence of $\Delta x_k(\cdot)$ to zero will follow from the inequality

$$|\Delta x_k(t)| \leqslant c\int_0^t |\Delta x_k(\tau)|\,d\tau + \left|\int_0^t (A_{12}(\tau)\,\Delta y_k(\tau) + A_{13}(\tau)\,\Delta z_k(\tau) + B_1(\tau)\,\Delta u_k(\tau))\,d\tau\right|$$

and from the Gronwall lemma.

Clearly,

$$\left|\int_0^t A_{12}(\tau)\,Y(\tau, 0, \beta_k)\,y^0\,d\tau\right| = \delta_k. \tag{A23}$$

Since $\|\Delta y_k\|$ is bounded, then, from (A13)

$$\varepsilon^2(\beta_k)\left|\int_0^t A_{12}(\tau)(C_{\beta_k}A_{32}\,\Delta y_k)(\tau)\,d\tau\right| \leqslant c\varepsilon^2(\beta_k)\,\|\Delta y_k\| = \delta_k. \tag{A24}$$

In view of (A10) we have

$$\frac{1}{\beta_k}\left|\int_0^t A_{12}(\tau)\int_0^\tau Y(\tau, s, \beta_k)\,A_{21}(s)\,\Delta x_k(s)\,ds\,d\tau\right|$$
$$\leqslant \delta_k\,\|x_k\| + \left|\int_0^t A_{12}(\tau)\,A_{22}^{-1}(\tau)\,\Delta x_k(\tau)\,d\tau\right|. \tag{A25}$$

By (A12) and the boundedness of $\|\Delta x_k\|_C$ we obtain

$$\varepsilon(\beta_k)\left|\int_0^t A_{12}(\tau)(C_{\beta_k}A_{31}\,\Delta x_k)(\tau)\,d\tau\right| \leqslant c\varepsilon(\beta_k)\,\|x_k\|_C = \delta_k. \tag{A26}$$

Furthermore, by (A10)

$$\left|\frac{1}{\beta_k}\int_0^t A_{12}(\tau)\int_0^\tau Y(\tau, s, \beta_k)\,B_2(s)\,\Delta u_k(s)\,d\tau\right|$$
$$\leqslant \delta_k\,\|u_k\| + \left|\int_0^t A_{12}(\tau)\,A_{22}^{-1}(\tau)\,B_2(\tau)\,\Delta u_k(\tau)\,d\tau\right| = \delta_k, \tag{A27}$$

since $\Delta u_k(\cdot)$ is L_2-weakly convergent to zero.

As in (A24) we obtain

$$\varepsilon(\beta_k)\left|\int_0^t A_{12}(\tau)(C_{\beta_k}B_3\,\Delta u_k)(\tau)\,d\tau\right| \leqslant c\varepsilon(\beta_k)\,\|u_k\| = \delta_k. \qquad \text{(A28)}$$

Using (A1), (A2), and (A18), an integration gives us

$$\frac{\varepsilon(\beta_k)}{\beta_k}\left|\int_0^t A_{12}(\tau)\int_0^\tau Y(\tau, s, \beta_k)\,A_{23}(s)\,Z(s, T, \beta_k)\,z_k(T)\,ds\,d\tau\right| \leqslant c\varepsilon(\beta_k). \qquad \text{(A29)}$$

From (A8) we get immediately

$$\left|\int_0^t A_{12}(\tau)\left(\frac{1}{\beta_k}\int_0^\tau Y(\tau, s, \beta_k)\,A_{22}(s)\,y_0(s)\,ds + y_0(\tau)\right)d\tau\right| = \delta_k. \quad \text{(A30)}$$

Thus, applying (A23–30) to (A19) we obtain (A21). The inequality (A22) can be obtained in the same manner from (A2), (A9), (A11–13) applied to (A20).

In order to prove L_2-weak convergence of $\Delta y_k(\cdot)$ and $\Delta z_k(\cdot)$ to zero, it is sufficient to observe that the sequence of norms $\|\Delta y_k\|$ is bounded ($\|\Delta z_k\|$ is bounded by assumption) and to prove that

$$\lim_{k\to\infty}\left|\int_0^t \Delta y_k(\tau)\,d\tau\right| = 0,$$

and

$$\lim_{k\to\infty}\left|\int_0^t \Delta z_k(\tau)\,d\tau\right| = 0$$

for almost all $t\in[0, T]$. We have already obtained these two relations in (A21) and (A22), where $\Delta x_k(\cdot)$ converges uniformly to zero (the presence of the matrices $A_{12}(t)$ and $A_{13}(t)$ is not essential). The proof of Lemma 1 is complete.

Proof of Lemma 2. The fast trajectories $y_k(\cdot)$ and $z_k(\cdot)$ satisfy the equations

$$\begin{aligned} y_k(t) = {} & Y(t, 0, \beta_k)\,y^0 - \varepsilon^2(\beta_k)(C_{\beta_k}A_{32}\,y_k)(t) \\ & + \frac{1}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\,A_{21}(\tau)\,x_k(\tau)\,d\tau - \varepsilon(\beta_k)(C_{\beta_k}A_{31}x_k)(t) \\ & + \frac{1}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\,B_2(\tau)\,u_k(\tau)\,d\tau - \varepsilon(\beta_k)(C_{\beta_k}B_3u_k)(t) \\ & + \frac{\varepsilon(\beta_k)}{\beta_k}\int_0^t Y(t, \tau, \beta_k)\,A_{23}(\tau)\,Z(\tau, T, \beta_k)\,z_k(T)\,d\tau, \end{aligned} \qquad \text{(A31)}$$

$$z_k(t) = Z(t, T, \beta_k)\, z_k(T) - \varepsilon^2(\beta_k)(D_{\beta_k} A_{23} z_k)(t)$$
$$- \frac{1}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, A_{31}(\tau)\, x_k(\tau)\, d\tau - \varepsilon(\beta_k)(D_{\beta_k} A_{21} x_k)(t)$$
$$- \frac{1}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, B_3(\tau)\, u_k(\tau)\, d\tau - \varepsilon(\beta_k)(D_{\beta_k} B_2 u_k)(t)$$
$$- \frac{\varepsilon(\beta_k)}{\beta_k}\int_t^T Z(t, \tau, \beta_k)\, A_{32}(\tau)\, Y(\tau, 0, \beta_k)\, y^0\, d\tau. \tag{A32}$$

By repeating the arguments in (A23–29) we get

$$\left|\int_0^t A_{12}(\tau)\, y_k(\tau)\, d\tau\right| \leqslant \delta_k + c\left(\int_0^t |x_k(\tau)|\, d\tau + \varepsilon^2(\beta_k)\, \|y_k\| + \|u_k\|\right).$$

Similarly

$$\left|\int_0^t A_{13}(\tau)\, z_k(\tau)\, d\tau\right| \leqslant \delta_k + c\left(\int_0^t |x_k(\tau)| + \varepsilon^2(\beta_k)\, \|z_k\| + \|u_k\|\right).$$

These two inequalities, applied to

$$|x_k(t)| \leqslant |x^0| + \left|\int_0^t (A_{11}(\tau)\, x_k(\tau) + A_{12}(\tau)\, y_k(\tau) + A_{13}(\tau)\, z_k(\tau) + B_1(\tau)\, u_k(\tau))\, d\tau\right|,$$

give us

$$\|x_k\|_C \leqslant c(1 + \varepsilon^2(\beta_k)(\|y_k\| + \|z_k\|) + \|u_k\|). \tag{A33}$$

Consider again Eq. (A31). Using consequently (A14), (A13) (for $y_k(\cdot)$), (A6), (A12) (for $x_k(\cdot)$), again (A6), again (A13) (for $u_k(\cdot)$), and (A1–2) ($z_k(T)$ is bounded), we have

$$\|y_k\| \leqslant c(1 + \varepsilon^2(\beta_k)\, \|y_k\| + \|u_k\| + \|x_k\|_C). \tag{A34}$$

For $z_k(\cdot)$, using (A32), (A1–2), (A7), (A11), (A13) we obtain that

$$\|z_k\| \leqslant c(1 + \varepsilon^2(\beta_k)\, \|z_k\| + \|u_k\| + \|x_k\|_C). \tag{A35}$$

Combining (A31 35) we complete the proof.

Corollary 2 follows immediately from the above proof if we replace (A6), (A7) by (A4) and (A5).

If we use additionally (A8), (A9) to Eqs. (A19) and (A20) for the differences Δx_k, Δy_k, Δz_k, Δu_k one can obtain Corollary 3.

Proof of Lemma 3. Since C2 holds, the uniform boundedness of $(x_k(\cdot), y_k(\cdot), z_k(\cdot))$ follows from Lemma 2 and Corollary 2. Denote by $\bar{z}_k(\cdot)$ the solution of the equation

$$\beta_k \dot{z} = A_{31}(0)\, x^0 + A_{33}(0) z + B_3(0)\, u_k(t), \qquad z(T) = 0.$$

Then, if $\Delta z_k = z_k - \bar{z}_k$, we have

$$\beta_k\, \Delta\dot{z}_k = \Delta A_{31}(t)\, x_k(t) + A_{31}(t)\, \Delta x_k(t) + \varepsilon(\beta_k)\, A_{32}(t)\, y_k(t) + \Delta A_{33}(t)\, \bar{z}_k(t) + A_{33}(t)\, \Delta z_k + \Delta B_3(t)\, u_k(t), \qquad \Delta z_k(T) = z_k(T),$$

where $\Delta A_{31}(t) = A_{31}(t) - A_{31}(0)$, $\Delta x_k(t) = x_k(t) - x^0$, $\Delta A_{33}(t) = A_{33}(t) - A_{33}(0)$, $\Delta B_3(t) = B_{33}(t) - B_{33}(0)$. This means that the difference $\Delta z_k(0)$ satisfies

$$\begin{aligned}\Delta z_k(0) = Z(0, T, \beta_k)\, z_k(T) &- \frac{1}{\beta_k}\int_0^{\sqrt{\beta_k}} Z(0, \tau, \beta_k)(\Delta A_{31}(\tau)\, x_k(\tau) \\ &+ A_{31}(\tau)\, \Delta x_k(\tau) + \varepsilon(\beta_k)\, A_{32}(\tau)\, y_k(\tau) \\ &+ \Delta A_{33}(\tau)\, \bar{z}_k(\tau) + \Delta B_k(\tau)\, u_k(\tau))\, d\tau \\ &- \frac{1}{\beta_k}\int_{\beta_k}^{T} Z(0, \tau, \beta_k)(\Delta A_{31}(\tau)\, x_k(\tau) + A_{31}(\tau)\, \Delta x_k(\tau) \\ &+ \varepsilon(\beta_k)\, A_{32}(\tau)\, y_k(\tau) + \Delta A_{33}(\tau)\, \bar{z}_k(\tau) + \Delta B_k(\tau)\, u_k(\tau))\, d\tau.\end{aligned}$$

Since $z_k(T)$ is bounded, by (A2) we get that the first summand tends to zero as $k \to +\infty$. The first integral can be estimated by the expression

$$\frac{c}{\beta_k}\int_0^{\sqrt{\beta_k}} \exp(-\sigma t/\beta_k)\, dt \underset{0 \leq t \leq \sqrt{\beta_k}}{\text{vraisup}} (|\Delta A_{31}(t)|\, |x_k(t)| + |A_{31}(t)|\, |\Delta x_k(t)| + \varepsilon(\beta_k)\, |A_{32}(t)\, y_k(t)| + |\Delta A_{33}(t)|\, |\bar{z}_k(t)| + |\Delta B_3(t)|\, |u_k(t)|),$$

which tends to zero as $k \to +\infty$ since $A_{31}(t)$, $A_{32}(t)$, $B_3(t)$ are continuous, $x_k(\cdot)$ is continuous uniformly in k, and $x_k(\cdot)$, $y_k(\cdot)$, $\bar{z}_k(\cdot)$, $u_k(\cdot)$ are uniformly bounded.

From the uniform boundedness of $x_k(\cdot)$, $y_k(\cdot)$, $\bar{z}_k(\cdot)$, and $u_k(\cdot)$ we get that the expression

$$\frac{c}{\beta_k}\int_{\sqrt{\beta_k}}^{T} \exp(-\sigma t/\beta_k)\, dt \to 0 \qquad \text{as} \quad k \to +\infty,$$

estimates the second integral. Hence

$$\lim_{k \to +\infty} \Delta z_k(0) = 0. \tag{A36}$$

Therefore, it is sufficient to prove that a subsequence of $\bar{z}_k(0)$ tends to a point from $A_{33}^{-1}(0)\, A_{31}(0)\, x^0 + P_z$.

We have

$$\begin{aligned}\bar{z}_k(0) &= -\frac{1}{\beta_k}\int_0^T \exp(-A_{33}(0)t/\beta_k)(A_{31}(0)\, x^0 + B_3(0)\, u_k(t))\, dt\\ &= A_{33}^{-1}(0)\, A_{31}(0)\, x^0 + q_k + \delta_k,\end{aligned}$$

where, as before, $\delta_k \to 0$ as $k \to +\infty$, and, denoting

$$\begin{aligned} v_k(t) &= u_k(\beta_k t), \qquad && t \in [0, T/\beta_k),\\ &= v, && t \geqslant T/\beta_k,\end{aligned}$$

for some $v \in V$, then

$$q_k = -\int_0^{+\infty} \exp(-A_{33}(0)t)\, B_3(0)\, v_k(t)\, dt.$$

The sequence $v_k(\cdot)$ is uniformly bounded, hence there exists a ball $B_r \subset R^m$ such that $v_k(t) \in V \cap B_r$ for a.e. $t \in [0, +\infty)$. Let us denote by P'_z the closure of the reachable set of the system (6) with feasible controls being locally integrable functions with values from $V \cap B_r$. Clearly, $P'_z \subset P_z$ and $q_k \in P'_z$. Moreover, P'_z is compact. Let q^+ be a condensation point of the sequence q_k. Then a subsequence of $\bar{z}_k(0)$ converges to $A_{33}^{-1}(0)\, A_{31}(0)\, x^0 + q^+$. This, combined with (A36), means that $z^0 \in A_{33}^{-1}(0)\, A_{31}(0)\, x^0 + P_z$.

Denote by $P'_x(T)$ the reachable set of the system (4) for controls with values from $V \cap B_r$. If we take $\mathrm{co}(V \cap B_r)$ instead of $V \cap B_r$, the set $P'_x(T)$ does not change. This implies that, from the sequence $u_k(\cdot)$, one can extract a L_2-weakly convergent subsequence which tends to some $u(\cdot)$, $u(t) \in \mathrm{co}(V \cap B_r)$ for a.e. $t \in [0, T]$. The corresponding sequence of states $x_k(\cdot)$ will converge uniformly to the solution of (4) with the control $u(\cdot)$ (see Corollary 1), hence all condensation points of $x_k(T)$ will lie in $P'_x(T)$. Since $P'_x(T) \subset P_x(T)$, we have $x \in P_x(T)$.

In order to prove the second inclusion, it is sufficient to repeat the argument in the first part of the proof, but for the state y. Denoting by $\bar{y}_k(\cdot)$ the solution of the equation

$$\beta_k\, \dot{y} = A_{21}(T)\, x_k(T) + A_{22}(T)\, y + B_2(T)\, u_k(t), \qquad y(0) = 0,$$

one can prove that

$$\lim_{k \to \infty} |\bar{y}_k(T) - y_k(T)| = 0.$$

On the other hand, if $x_k(T)$ converges to x then a subsequence of $y_k(T)$ will converge to $-A_{22}^{-1}(T) A_{21}(T)x + p^*$, where $p^* \in P_y(T)$. This completes the proof of Lemma 3.

Proof of Lemma 4. We first prove our lemma on the additional assumption that $X^1 = R^n$ and $Y^1 = R^p$.

For simplicity, denote $D = -A_{22}^{-1}(T) A_{21}(T)$. Let $\varepsilon > 0$ be fixed. The condition

$$y \in Dx_0(T) + P_y(T)$$

implies that there exist y_ε and a sufficiently large number $r(\varepsilon)$ such that

$$|y - y_\varepsilon| < \varepsilon$$

and

$$y_\varepsilon = Dx_0(T) + \int_0^{+\infty} \exp(A_{22}(T)t) B_2(T) u_\varepsilon(t)\, dt,$$

where $u_\varepsilon(t) \in V \cap B_{r(\varepsilon)}$, $t \in [0, +\infty)$, $u_\varepsilon(\cdot) \in L_1^{(m)}[0, +\infty)$.

For every point $z \in \operatorname{int} P_z$ there exists a feasible control driving the system (6) from 0 to z with values contained in a certain ball in R^m. From A2 we conclude that there exist $z^i \in A_{33}^{-1}(0) A_{31}(0) x^0 + \operatorname{int} P_z$, $i = 1,..., i_0$, such that $z^0 \in \operatorname{intco}\{z^i\}$. Then there exist $r > 0$ and functions $w^i(\cdot) \in L_\infty^{(m)}[0, +\infty)$, $w^i(t) \in V \cap B_r$ for a.e. $t \in [0, +\infty)$ such that

$$z^i = A_{33}^{-1}(0) A_{31}(0) x^0 - \int_0^{+\infty} \exp(-A_{33}^{-1}(0)t) B_3(0) w^i(t)\, dt.$$

Introduce the control

$$\begin{aligned} u_\beta^{i,\varepsilon}(t) &= w^i(t/\beta), && t \in [0, \sqrt{\beta}),\\ &= u_0(t), && t \in [\sqrt{\beta}, T - \sqrt{\beta}),\\ &= u_\varepsilon((T-t)/\beta), && t \in [T - \sqrt{\beta}, T]. \end{aligned}$$

For every $\varepsilon > 0$ we have $u_\beta^{i,\varepsilon}(t) \in V$ for a.e. $t \in [0, T]$. Moreover

$$\limsup_{\beta \to 0} \operatorname*{vraisup}_{0 \le t \le T} |u_\beta^{i,\varepsilon}(t)| < +\infty$$

and for every $t \in (0, T)$

$$\lim_{\beta \to 0} u_\beta^{i,\varepsilon}(t) = u_0(t).$$

Denote by $(x_\beta^{i,\varepsilon}(\cdot), y_\beta^{i,\varepsilon}(\cdot), z_\beta^{i,\varepsilon}(\cdot))$ the solution of the perturbed system (1) corresponding to $u_\beta^{i,\varepsilon}(\cdot)$ with boundary conditions

$$x(0) = x^0, \qquad y(0) = y^0, \qquad z(T) = z.$$

By repeating the argument in the proof of Lemma 1, see Remark 1, one can obtain that

$$\lim_{\beta \to 0} \|x_\beta^{i,\varepsilon} - x_0\|_C = 0.$$

We will prove that for every $\varepsilon > 0$

$$\lim_{\beta \to 0} y_\beta^{i,\varepsilon}(T) = y_\varepsilon \tag{A37}$$

and

$$\lim_{\beta \to 0} z_\beta^{i,\varepsilon}(0) = z^i. \tag{A38}$$

If this is so, then, for small β there exist $\alpha_\beta^i \geq 0$, $\sum_{i=1}^{i_0} \alpha_\beta^i = 1$, such that

$$\sum_{i=1}^{i_0} \alpha_\beta^i z_\beta^{i,\varepsilon}(0) = z^0.$$

Define

$$u_\beta^\varepsilon(t) = \sum_{i=1}^{i_0} \alpha_\beta^i u_\beta^{i,\varepsilon}(t).$$

Obviously, for every $t \in (0, T)$

$$\lim_{\beta \to 0} u_\beta^\varepsilon(t) = u_0(t).$$

Furthermore, if $(x_\beta^\varepsilon(\cdot), y_\beta^\varepsilon(\cdot), z_\beta^\varepsilon(\cdot))$ is the solution of (1) with boundary conditions $x(0) = x^0$, $y(0) = y^0$, $z(T) = z$, then

$$\lim_{\beta \to 0} x_\beta^\varepsilon(T) = x_0(T), \qquad \lim_{\beta \to 0} y_\beta^\varepsilon(T) = y_\beta, \qquad z_\beta^\varepsilon(0) = z^0.$$

In order to complete the proof of the first part of the lemma it is sufficient to choose a diagonal sequence of controls for $\beta \to 0$ and $\varepsilon \to 0$.

It remains to prove that (A37) and (A38) hold. Clearly, $\|y_\beta^{i,\varepsilon}(\cdot)\|_C$ is bounded when $\beta \to 0$, see Corollary 2 and Remark 1. Using the preliminary results one can show that the expression

$$Y(T, 0, \beta)\, y^0 + \frac{1}{\beta}\int_0^{T-\sqrt{\beta}} Y(T, t, \beta)(A_{21}(t)\, x_\beta^{i,\varepsilon}(t) + B_2(t)\, u_\beta^{i,\varepsilon}(t))\, dt$$

$$+ \frac{\varepsilon(\beta)}{\beta^2}\int_0^{T-\sqrt{\beta}} Y(T, t, \beta)\, A_{23}(t) \int_t^T Z(t, s, \beta)(A_{31}(s)\, x_\beta^{i,\varepsilon}(s)$$

$$- \varepsilon(\beta)\, A_{32}(s)\, y_\beta^{i,\varepsilon}(s))\, ds\, dt + \frac{\varepsilon(\beta)}{\beta}\int_0^{T-\sqrt{\beta}} Y(T, t, \beta)\, A_{23}(t)\, Z(t, T, \beta) z\, dt$$

tends to zero as $\beta \to 0$. Furthermore, an integration by parts gives us

$$\lim_{\beta \to 0} \frac{1}{\beta}\int_{T-\sqrt{\beta}}^{T} Y(T, t, \beta)\, A_{21}(t)\, x_\beta^{i,\varepsilon}(t)\, dt = -A_{22}^{-1}(T)\, A_{21}(T)\, x_0(T).$$

We have

$$\left|\frac{1}{\beta}\int_{T-\sqrt{\beta}}^{T} Y(T, t, \beta)\, B_2(t)\, u_\beta^{i,\varepsilon}(t)\, dt - \int_0^{+\infty} \exp(A_{22}(T)t)\, B_2(T)\, u_\varepsilon(t)\, dt\right|$$

$$\leqslant c \max_{T-\sqrt{\beta} \leqslant t \leqslant T} (|A_{22}(t) - A_{22}(T)| + |B_2(t) - B_2(T)|)\, r(\varepsilon) + \delta_\beta.$$

where $\delta_\beta \to 0$ as $\beta \to 0$. The last three observations, applied to the Cauchy formula for $y_\beta^{i,\varepsilon}(\cdot)$ (see (A31)), give us (A37). The relation (A38) can be obtained analogously.

Let us take now arbitrary sets X^1 and Y^1 from R^n and R^p, respectively, such that $x_0(T)$, y, and z satisfy the statement of Lemma 4.

Obviously $x_0(T) \in X^1 \cap P_x(T)$. Suppose first that

$$x_0(T) \in \operatorname{int} P_x(T) \cap X^1 \qquad \text{and} \qquad y \in (Dx_0(T) + \operatorname{int} P_y(T)) \cap Y^1.$$

Clearly, there exists $\varepsilon_0 > 0$ such that if $|x - x_0(T)| < \varepsilon_0$ then $y \in Dx + \operatorname{int} P_y(T)$. Furthermore, there exist $\varepsilon_1 > 0$ and $x^i \in P_x(T)$, $i = 1, \ldots, i_0$, such that $|x^i - x_0(T)| < \varepsilon_0$ and if $|\tilde{x}^i - x^i| < \varepsilon_1$ then $x_0(T) \in \operatorname{co}\{\tilde{x}^i\}$. Next, there exist $\varepsilon_2 > 0$ and y^j, $j = 1, \ldots, j_0$, such that $y^j - Dx^i \in P_y(T)$, $i = 1, \ldots, i_0$, $j = 1, \ldots, j_0$, and if $|\tilde{y}^j - y^j| < \varepsilon_2$ then $y \in \operatorname{co}\{\tilde{y}_j\}$.

According to the first part of the proof, for sufficiently small β there exists a control $u_\beta^{ij}(\cdot) \in L_\infty^{(m)}[0, T]$, $u_\beta^{ij}(t) \in V$ for a.e. $t \in [0, T]$, such that

$$\lim_{\beta \to 0} u_\beta^{ij}(t) = u_0(t) \qquad \text{for a.e. } t \in [0, T]$$

and if $(x_\beta^{ij}(\cdot), y_\beta^{ij}(\cdot), z_\beta^{ij}(\cdot))$ is the corresponding trajectory of the perturbed system, then

$$\lim_{\beta \to 0} (x_\beta^{ij}(T), y_\beta^{ij}(T)) = (x^i, y^j), \qquad z_\beta^{ij}(T) = z.$$

Hence, for small β we shall have

$$|x_\beta^{ij}(T) - x^i| < \varepsilon_1, \qquad |y_\beta^{ij}(T) - y^j| < \varepsilon_2$$

for $i = 1,..., i_0$, $j = 1,..., j_0$. This implies that there exist $\alpha_\beta^{ij} \geq 0$, $\sum_{i=1}^{i_0} \alpha_\beta^{ij} = 1$ such that

$$\sum_{i=1}^{i_0} \alpha_\beta^{ij} x_\beta^{ij}(T) = x_0(T), \qquad j = 1,..., j_0.$$

Obviously

$$\left| \sum_{i=1}^{i_0} \alpha_\beta^{ij} y_\beta^{ij}(T) - y^j \right| < \varepsilon_2.$$

Then there exist $\gamma_\beta^j \geq 0$, $\sum_{j=1}^{j_0} \gamma_\beta^j = 1$, such that

$$\sum_{j=1}^{j_0} \sum_{i=1}^{i_0} \gamma_\beta^j \alpha_\beta^{ij} y_\beta^{ij}(T) = y.$$

Moreover

$$\sum_{j=1}^{j_0} \sum_{i=1}^{i_0} \gamma_\beta^j \alpha_\beta^{ij} x_\beta^{ij}(T) = x_0(T)$$

and

$$\sum_{j=1}^{j_0} \sum_{i=1}^{i_0} \gamma_\beta^j \alpha_\beta^{ij} z_\beta^{ij}(T) = z.$$

The control

$$u_\beta(t) = \sum_{j=1}^{j_0} \sum_{i=1}^{i_0} \gamma_\beta^j \alpha_\beta^{ij} u_\beta^{ij}(t) \in \text{co } V$$

gives us a trajectory, for which

$$x_\beta(T) = x_0(T) \in X^1, \qquad y_\beta(T) = y \in Y^1, \qquad z_\beta(T) = z.$$

This proves the lemma in the case considered.

Suppose now that

$$x_0(T) \in \operatorname{int} X^1, \qquad y \in \operatorname{int} Y^1 \cap (Dx_0(T) + P_y(T)).$$

The first part of the proof implies that there exists a control $u_\beta(\cdot) \in L_\gamma^{(m)}[0, T]$, $u_\beta(t) \in V$ for a.e. $t \in [0, T]$ such that

$$\lim_{\beta \to 0} u_\beta(t) = u_0(t) \qquad \text{for} \quad \text{a.e. } t \in [0, T],$$

and the corresponding trajectory $(x_\beta(\cdot), y_\beta(\cdot), z_\beta(\cdot))$ of the perturbed system satisfies

$$\lim_{\beta \to 0} (x_\beta(T), y_\beta(T)) = (x_0(T), y), \qquad z_\beta(T) = z.$$

Then for β sufficiently small we shall have

$$x_\beta(T) \in X^1, \qquad y_\beta(T) \in Y^1,$$

and the statement of the lemma holds as well. The remaining two cases are combinations of the considered cases.

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