

NOTE

Enumeration of Lozenge Tilings of Punctured Hexagons

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We present a combinatorial solution to the problem of determining the number of lozenge tilings of a hexagon with sides $a$, $b+1$, $b$, $a+1$, $b$, $b+1$, with the central unit triangle removed. For $a=b$, this settles an open problem posed by Propp [7].

Let $a$, $b$, $c$ be positive integers, and denote by $H$ the hexagon whose side-lengths are (in cyclic order) $a$, $b$, $c$, $a$, $b$, $c$ and all whose angles have 120 degrees. The lozenge tilings (i.e., tilings by unit rhombi) of $H$ can be regarded as plane partitions contained in an $a \times b \times c$ box (cf. [2]), and therefore their number is given by the simple product formula [5]

$$
\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.
$$

Motivated by this, Propp [7] considered the problem of enumerating the lozenge tilings of a hexagon whose sides are alternately $a$ and $a+1$, from which the central unit triangle has been removed (removal of a suitable unit triangle is necessary for the remaining region to have lozenge tilings). Based on numerical evidence, he conjectured that there exists a simple product formula for the number of tilings of these regions.

The more general question of finding the number of lozenge tilings of a hexagon with sides $a$, $b+1$, $c$, $a+1$, $b$, $c+1$, with the central unit triangle removed—denote it by $N(a, b, c)$—appeared in work of Kuperberg [4] concerning certain weighted enumerations of plane partitions. This general question has been recently settled by Okada and Krattenthaler [6], who proved that $N(a, b, c)$ is equal to the product of four factors of type (1) (their proof relies on a new Schur function identity they prove using the minor summation formula of Ishikawa and Wakayama [3]).
The purpose of this paper is to give a simple product formula (with a simple combinatorial proof) for \( N(a, b, b) \) (this settles in particular Propp’s original question; Fig. 1 shows the region corresponding to \( a = 2, b = 4 \)).

Let \( SC(a, b, c) \) be the number of self-complementary pane partitions that fit in an \( a \times b \times c \) box (see [8] for the definition). In [8] it is given a simple product formula for \( SC(a, b, c) \).

**Theorem 1.**

\[
N(a, b, b) = SC(a + 1, b, b) SC(a, b + 1, b + 1).
\]

**Proof.** Let \( G \) be the graph dual to the region of the triangular lattice obtained from a hexagon of size \( a \times (b + 1) \times b \times (a + 1) \times b \times (b + 1) \) by removing the central unit triangle (Fig. 2(a) illustrates this for \( a = 2, b = 4 \)). Any lozenge tiling of our region can be identified with a perfect matching of \( G \). Therefore, \( N(a, b, b) \) is just the number \( M(G) \) of perfect matchings of \( G \).

The graph \( G \) has a symmetry axis; let \( v_1, v_2, \ldots, v_{2b} \) be the vertices of \( G \) on this axis, as they occur from left to right. It is immediate to check that all the conditions in the hypothesis of the Factorization Theorem of [1] are met. Applying this to \( G \) we obtain that

\[
M(G) = 2^b M(G^+) M(G^-),
\]

where \( G^+ \) (resp., \( G^- \)) is the top (resp., bottom) connected component of the subgraph of \( G \) obtained by removing the edges incident to the \( v_i \)'s from above, for \( 1 \leq i \leq b \), the edges incident to the \( v_i \)'s from below, for \( b + 1 \leq i \leq 2b \), and finally by weighting by \( 1/2 \) the edges of these two subgraphs along the symmetry axis of \( G \) (see Fig. 2(b)).

Consider now the \( a \times (b + 1) \times (b + 1) \) honeycomb graph \( H \) (the case \( a = 2, b = 4 \) is pictured in Fig. 3(a)); the matchings of this graph are in

**FIGURE 1**
bijection with the plane partitions fitting in an $a \times (b + 1) \times (b + 1)$ box. According to this bijection, $SC(a, b + 1, b + 1)$ is equal to the number of matchings of $H$ that are invariant under rotation by 180 degrees.

Let $H_1$ be the subgraph of $H$ induced by the vertices on or above its horizontal symmetry axis $\ell$ (the boundary of $H_1$ is shown in thick solid lines in Fig. 3(a)). Label the vertices of $H_1$ on $\ell$ according to their distance to the center of $H$ (the two closest vertices are labeled 1, the next two closest 2, and so on). Denote by $H_2$ the graph obtained from $H_1$ by identifying vertices with the same label (if two edges have both endpoints identified they are considered identical; note that the edge whose endpoints are labeled 1 gives rise to a loop). The matchings of $H$ invariant under rotation by 180 degrees can be identified with the matchings of $H_2$. Therefore,

$$M(H_2) = SC(a, b + 1, b + 1).$$

The graph $H_2$ can be symmetrically embedded in the plane. The symmetry axis contains precisely $b + 1$ of its vertices. Therefore, if $b$ is even, all perfect matchings of $H_2$ contain the loop at the vertex labeled 1 (henceforth referred to simply as the loop), while for odd $b$ none of them contains it.
Suppose \( b \) is even (the case \( b \) odd is treated similarly). Since all matchings of \( H_2 \) contain the loop, we may remove it (together with the vertex labeled 1) without changing the number of matchings of our graph; for the sake of notational simplicity, denote the resulting graph still by \( H_2 \).

Even though \( H_3 \) is not “separated” by its symmetry axis in the sense of [1], the variant of the Factorization Theorem in [1, Section 7] is applicable and yields

\[
M(H_3) = 2^{b/2} M(H_1),
\]

(4)

where \( H_1 \) is the graph obtained from \( H_1 \) by removing the edges incident from above to the leftmost \( b + 2 \) vertices on \( \ell \) and then weighting the edges along \( \ell \) of the remaining subgraph by \( 1/2 \). However, remarkably, the graph obtained from \( H_3 \) by removing the vertices matched by forced edges is isomorphic to \( G^+ \) (see Fig. 3(b)). We obtain therefore from (3) and (4) that

\[
M(G^+) = 2^{-b/2} SC(a, b + 1, b + 1).
\]

(5)

To determine \( M(G^-) \), take \( H \) to be the \((a + 1) \times b \times b\) honeycomb graph. Construct the graphs \( H_1 \) and \( H_2 \) as before (see Fig. 4(a)). Since the symmetry axis of \( H_2 \) contains now \( b \) vertices (and \( b \) is even), no perfect matching of \( H_2 \) contains the loop, and therefore we may replace \( H_2 \) by its subgraph obtained by removing this loop (and keeping the vertex labeled 1). Applying the variant of the Factorization Theorem in [1, Section 7] we obtain

\[
M(H_2) = 2^{b/2} M(H_1),
\]

(6)

where \( H_3 \) is the graph obtained from \( H_1 \) by removing the edges incident from above to the leftmost \( b \) vertices on \( \ell \) and then weighting the edges along \( \ell \) of the remaining subgraph by \( 1/2 \). However, again, the graph obtained from \( H_3 \) by removing the vertices matched by forced edges is

![Figure 4](image-url)
isomorphic to the subgraph of $G^-$ left after removing its vertices matched by forced edges (see Fig. 4(b)). Since now $M(H_2) = SC(a + 1, b, b)$, (6) implies

$$M(G^-) = 2^{-b/2}SC(a + 1, b, b).$$

(7)

The statement of the theorem follows from (2), (5) and (7).

REFERENCES