JOURNAL OF ALGEBRA **208,** 216–261 (1998)
ARTICLE NO. JA987422

Actions of Algebraic Groups on the Spectrum of Rational Ideals, II

Nikolaus Vonessen*

Department of Mathematical Sciences, University of Montana, Missoula,

Communicated by Susan Montgomery

Received August 12, 1997

We study rational actions of a linear algebraic group G on an algebra V, and the induced actions on $\mathrm{Rat}(V)$, the spectrum of rational ideals of V (a subset of $\mathrm{Spec}(V)$ which often includes all primitive ideals). This work extends results of Moeglin and Rentschler to prime characteristic, often also relaxing their finiteness assumptions on V. In particular, we relate properties of a rational ideal J and its *orb*, the ideal $(J:G) = \bigcap_{\gamma \in G} \gamma(J)$. The rational ideals of V containing the orb of J are precisely those in the Zariski-closure X of the orbit of J in $\mathrm{Rat}(V)$. The *G-stratum* of J consists of all rational ideals in X whose orbit is dense in X (i.e., whose orb is equal to the orb of J). We show that the G-stratum of a rational ideal consists of exactly one G-orbit, and that rational ideals are maximal in their strata in a strong sense. These results are useful for studying prime and primitive spectra of certain algebras, cf. recent work by Goodearl and Letzter. We further show that the orbit of J is open in its closure in $\mathrm{Rat}(V)$, provided that J is locally closed. Among other results, we show that the semiprime ideal $(J:G)$ is Goldie, and we relate the uniform and Gelfand–Kirillov dimensions of V/J and $V/(J:G)$. © 1998 Academic Press

Contents

*Partially supported by NSA Grant MDA 904-96-1-0060.

0021-8693/98 $25.00

1. INTRODUCTION

In determining and classifying primitive ideals, rational actions of linear algebraic groups have been quite useful. In the series of papers [MR_1–MR_4], Moeglin and Rentschler developed a theory for such actions dealing with *rational ideals*, which often include all primitive ideals (see below). They applied it to the classification of primitive ideals in the enveloping algebra of a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. Recently, actions of tori have been used to describe prime and primitive spectra for certain function algebras on quantum groups, see Joseph [J_1, J_2, J_3] and Hodges, Levasseur, and Toro [HLT]. Goodearl and E. Letzter [GL] considered iterated skew polynomial algebras. Continuing our investigation in [V_1], we study in this paper rational actions of linear algebraic groups on the spectrum of rational ideals of an algebra in general.

We follow the conventions, and use the definitions and the notation introduced in [V_1, 2.1–2.4] (which in turn is based on [MR_3]); a summary of the notation appears in Subsection 3.1. In particular, k is throughout an algebraically closed field, G is a (not necessarily connected) linear algebraic group over k, and V is an associative k-algebra on which G acts rationally by k-algebra automorphisms. We sometimes denote the action of G on V by β.

The main difference with the situation studied by Moeglin and Rentschler is that we work over an algebraically closed field of arbitrary characteristic, not characteristic zero, and that we are able to avoid certain finiteness assumptions: many results in their work require that every semiprime ideal of the algebra V be Goldie, or even that V be Noetherian. In writing this paper, it seemed unavoidable for the sake of clarity and intelligibility, to follow (at times quite literally) parts of their work. I have tried to point this out in all important instances.

The remainder of the Introduction consists of a survey of the main results, leaving detailed statements and comments (and in particular references to the work of Moeglin and Rentschler) to the following section.

We begin with a few words about rational ideals. A prime left Goldie ideal J of V is called *rational* if the only central elements in the total ring of fractions of V/J are the scalars in the field k (which we assume to be algebraically closed). We denote by $\mathrm{Rat}(V)$ the set of all rational ideals of V. A profusion of important examples shows that rational ideals are worth studying. For example, if k is uncountable and V countably dimensional over k, then every primitive left Goldie ideal is rational, see, e.g., [BGR, p. 25]. In characteristic zero, the rational and primitive ideals of the enveloping algebra of a finite-dimensional Lie algebra coincide. This is part of the Dixmier–Moeglin equivalence; see [Re, 1.9]. The Dixmier–Moeglin equivalence has been generalized to other classes of algebras, see [GL]. Finally, for affine (i.e., finitely generated) k-algebras satisfying a polynomial identity, the rational ideals are just the maximal ideals (see, e.g., [V_1, 2.6]).

Given a rational ideal J of V, our goal is to study the orbit of J, and the relationship between J and $(J : G) = \bigcap_{\gamma \in G} \gamma(J)$, which we call, following Farkas [Fa], the *orb* of J.

As a first result, we prove that $(J : G)$ is always left Goldie; this is a special case of Theorem 2.1, which allows us to avoid some of the finiteness assumptions necessary in Moeglin and Rentschler's work.

The rational ideals containing $(J : G)$ are precisely those in the Zariski closure X of the orbit of J in $\mathrm{Rat}(V)$. The *G-stratum* of J in $\mathrm{Rat}(V)$ consists of all rational ideals whose orb is equal to $(J : G)$. Said differently, the G-stratum of J in $\mathrm{Rat}(V)$ consists of all rational ideals whose orbit is dense in X. One of the main results of this paper is that G acts transitively on each stratum in $\mathrm{Rat}(V)$, see Theorem 2.2.

One way to think of this transitivity result is that the G-orbit of J is uniquely determined by $(J : G)$ (which *a priori* determined only the G-stratum of J). This raises the question of which ideals can be orbs of rational ideals. The answer is that an ideal is the orb of a rational ideal if and only if it is "G-rational," see Theorem 2.10. Thus the G-orbits in $\mathrm{Rat}(V)$ are in bijection with the G-rational ideals of V.

We show also that the rational ideals are maximal in their strata in the following strong sense: If $P \supseteq J$ is any ideal with $(P : G) = (J : G)$, then $P = J$, see Theorem 2.3.

A prime ideal J is called *locally closed* if the intersection of all prime ideals strictly containing J contains J strictly, i.e., if the point $\{J\}$ is open in its closure in $\mathrm{Spec}(V)$. Such ideals are quite common; we will review this in the next section. We prove that a rational ideal is locally closed if and only if its orb is "G-locally closed," see Theorem 2.6. As a corollary, we show that if J is a locally closed rational ideal of V, then the orbit of J in $\mathrm{Rat}(V)$ is open in its closure, see Corollary 2.7.

Our investigations enable us to relate V/J and $V/(J : G)$ through a series of central ring extensions, see Diagrams 2.13 and 7.13. As an application, we relate the Gelfand–Kirillov dimensions and the uniform dimensions of V/J and $V/(J : G)$, see Theorem 2.8 and Proposition 2.9.

Given a rational ideal J of V, further results relate the center C of the total ring of fractions of $V/(J:G)$ to $Q(G)$, the algebra of rational functions defined on dense open subsets of G, see Theorems 2.11 and 2.12. These theorems, which extend our work in [V_1], are important ingredients in the proofs of some of the results mentioned before.

2. STATEMENTS OF MAIN RESULTS

Orbs of Rational Ideals Are Goldie. Let J be a rational ideal of V. Its orb $(J:G)$ is clearly a semiprime ideal. Moeglin and Rentschler's techniques depend heavily on the total ring of fractions of $V/(J:G)$. They ensure its existence in [MR_1] by assuming that V is Noetherian. In their later paper [MR_4], they work under the more general hypothesis that every semiprime ideal of V is Goldie. Using results of Bell and Ferrero (see Proposition 5.2), we are able to prove the following theorem.

2.1. THEOREM. *A prime ideal J of V is left Goldie if and only if its orb, the semiprime ideal $(J:G)$, is left Goldie.*

This theorem will be proved in 5.4. Many results which follow are based on it. Even in characteristic zero, it enables us to significantly strengthen many of Moeglin and Rentschler's results. In particular, Theorem 2.1 ensures that the total ring of fractions of $V/(J:G)$ exists for every rational ideal J of V. As a first indication of how this allows us to relate V/J and $V/(J:G)$, we remark that the total rings of fractions of both algebras embed into the total ring of fractions of $V/J \otimes Q(G)$ (Lemma 7.3). Here $Q(G)$ denotes the algebra of rational functions which are defined on dense open subsets of G.

The Stratum of a Rational Ideal Is Its Orbit. Recall that the G-stratum of J in $\mathrm{Rat}(V)$ consists of all rational ideals whose orb is equal to $(J:G)$, the orb of J. One of the main results of this paper is that G acts transitively on each stratum in $\mathrm{Rat}(V)$:

2.2. THEOREM. *Each G-stratum in* $\mathrm{Rat}\, V$ *consists of a single G-orbit*.

This result follows from Theorem 7.10. It was proved in characteristic zero in [MR_4, Théorème 2], under the assumption that every semiprime ideal of V is Goldie.

We also prove that rational ideals are maximal in their strata, in the following strong sense:

2.3. THEOREM. *Let J be a rational ideal of V. If $P \supseteq J$ is any ideal with $(P:G) = (J:G)$, then $P = J$. In other words, rational ideals are maximal in their G-strata in* $\mathrm{Spec}\, V$.

This result has no counterpart in the work of Moeglin and Rentschler. It generalizes work by Goodearl and Letzter [GL] for certain classes of Noetherian algebras. Theorem 2.3 is proved in 7.7. It immediately implies the following result, which was obtained in [MR_4] in characteristic zero, under the (unstated) assumption that every prime ideal of V is Goldie.

2.4. COROLLARY. *Let J be a rational ideal of V which is G-stable. Then the G-stratum of J in* Spec V *consists of only one element, namely J itself.*

Note the difference with Theorem 2.2: that result deals with G-strata in the smaller space Rat V.

The G-stratum of a rational ideal J in Spec(V) will usually contain many prime ideals not in the orbit of J. For example, any prime P with $(J:G) \subseteq P \subsetneq J$ belongs to the stratum of J but is not rational by Theorem 2.3. A natural question is the following: If P is an ideal of V maximal with respect to $(P:G) = (J:G)$, is then P necessarily rational? In general, Theorem 2.1 implies easily that such a P is at least a prime ideal which is left Goldie, see Corollary 5.5. Under suitable hypotheses, we can give a positive answer. Recall that an algebra is called *Jacobson* if every prime ideal is an intersection of primitive ideals.

2.5. THEOREM. *Let J be a rational ideal of V. Assume that J is locally closed, that $V/(J:G)$ is Jacobson, and that every primitive ideal of $V/(J:G)$ is rational. If P is an ideal of V maximal with respect to $(P:G) = (J:G)$, then P is rational* (*and thus contained in the G-orbit of J*).

This theorem depends on results on locally closed ideals, which we will discuss in the next subsection. Theorem 2.5 is a special case of Theorem 8.15. The hypotheses of Theorem 2.5 are in particular satisfied if V is a finitely generated k-algebra and V/J is a finite-dimensional k-vector space, see Corollary 8.16. (Note that one could simplify the statement of Theorem 2.5 by not requiring that J is rational, as this is implied by the other hypotheses: a locally closed ideal in a Jacobson ring is trivially primitive, and we assume that all primitive ideals containing $(J:G)$ are rational.)

Locally Closed Ideals. A prime ideal J is called *locally closed* if the intersection of all prime ideals strictly containing J contains J strictly, i.e., if the point $\{J\}$ is open in its closure in Spec(V). (One could define this notion for semiprime ideals, but one easily sees that a semiprime locally closed ideal is necessarily prime.) Maximal ideals are clearly locally closed. Often, rational ideals are also locally closed. In particular, the already mentioned Dixmier–Moeglin equivalence states that in characteristic zero, the rational, primitive, and locally closed ideals of the enveloping algebra of a finite-dimensional Lie algebra all coincide. The same is true for finitely generated PI-algebras over k. (We already remarked that in such an algebra V, the rational ideals coincide with the maximal ones, which are locally closed for trivial reasons; conversely, since V is Jacobson, locally closed ideals are primitive and hence maximal by Kaplansky's theorem.)

A closely related notion is the following. A G-stable semiprime ideal I is called *G-locally closed* if the intersection of all G-stable semiprime ideals strictly containing I contains I strictly. One checks easily that a G-locally closed ideal I is G-prime.

2.6. THEOREM. *Let J be a rational ideal of V. The following are equivalent*:

(a) *J is locally closed*.

(b) *$(J:G)$ is G-locally closed*.

This result was proved by Moeglin and Rentschler in characteristic zero under additional finiteness assumptions. They proved (a) ⇒ (b) for Noetherian V [MR_1, Théorème 3.8], and (b) ⇒ (a) in case that every semiprime ideal of V is Goldie [MR_4, Théorème 3]. Having established that rational ideals are maximal in their strata in Spec(V) (Theorem 2.3), our proof of (b) ⇒ (a) is much easier, see 8.1. Preparation for the proof of the other direction (given in 8.13) requires a large part of Section 8.

Combining the previous result with the fact that G acts transitively on each stratum in Rat(V), one obtains the following nice corollary, whose importance comes from the fact that rational ideals are often locally closed.

2.7. COROLLARY. *Let J be a rational ideal of V which is locally closed. Then the orbit of J in* Rat(V) *is open in its closure*.

We will prove this result in 8.14. It is crucial in proving Theorem 2.5 above.

Relating Dimensions of V/J and $V/(J:G)$. Given a rational ideal J of V, we now consider the Gelfand–Kirillov dimension and the uniform dimension (Goldie rank) of the algebras V/J and $V/(J:G)$. We denote these dimensions by GKdim and udim, respectively.

2.8. THEOREM. *Let J be a rational ideal of V, and denote by H the stabilizer of J in G. Then*

$$\operatorname{GKdim}(V/(J:G)) = \operatorname{GKdim}(V/J) + \dim(G/H).$$

In particular, if V/J has finite dimension over k, then

$$\operatorname{GKdim}(V/(J:G)) = \dim(G/H).$$

Here $\dim(G/H) = \dim G - \dim H$ denotes the dimension of the homogeneous space G/H. This result is proved in 9.5. It extends [MR_1, 3.12], which deals with enveloping algebras of finite-dimensional Lie algebras in characteristic zero.

2.9. PROPOSITION. *Let J be a rational ideal of V. Then*

$$\operatorname{udim}(V/(J:G)) \leq |G/G^\circ| \cdot \operatorname{udim}(V/J).$$

We will prove this result in 9.3; it extends the inequality in [MR_4, Proposition 4]. Example 9.4 shows that the inequality can be strict, even for connected groups both in characteristic zero and in prime characteristic (see also Example 9.1). Moeglin and Rentschler give in [MR_4] two necessary conditions for equality. We will discuss them in Section 9, following 9.3. (Note that in [MR_4], the inequality $\operatorname{udim}(V/(J:G)) \leq \operatorname{udim}(V/J)$ is proved (in characteristic zero) without mentioning the additional necessary assumption that G be connected. To give an example of the necessity of this assumption, let G be an affine algebraic group. Denote by Γ the action of G on its coordinate ring $V = A(G)$ induced by left multiplication. Then for every maximal ideal J of V, $(J:G) = 0$ so that $V = V/(J:G)$. But $\operatorname{udim}(V/J) = 1$ while $\operatorname{udim} V = |G/G^\circ|$.)

G-Rational Ideals. A G-stable, semiprime left Goldie ideal I is called *G-rational* if the only G-stable elements in the center of the total ring of fractions of V/I are the scalars in k, i.e., if $(Z(Q(V/I)))^G = k$. One verifies easily that G-rational ideals are G-prime, see Lemma 3.4(b). The next result relates rational and G-rational ideals.

2.10. THEOREM. *The assignment $J \mapsto (J:G)$ defines a surjection from* $\mathrm{Rat}(V)$ *onto the set of G-rational ideals of V; the fibers of this surjection are the G-orbits in* $\mathrm{Rat}(V)$. *Thus the G-orbits in* $\mathrm{Rat}(V)$ *are in one-to-one correspondence with the set of G-rational ideals of V. In particular*:

(a) *Let J be a rational ideal of V. Then $I = (J:G)$ is G-rational.*

(b) *Let I be a G-rational ideal of V. Then there is a rational ideal J of V with $(J:G) = I$.*

The statement that the fibers of this surjection are the G-orbits in $\mathrm{Rat}(V)$ is the content of Theorem 2.2. Theorem 2.10 extends [MR_4, Théorème 2; V_1, Theorem 5.1]. New here is that we do not have to assume that every semiprime ideal of V is left Goldie.

Using Theorem 2.1 (which ensures that I is left Goldie), part (a) follows immediately from a result of Braun (see [V_1, Lemma 4.1]). We will prove part (b) in 6.7; the argument in the proof of [V_1, Theorem 5.1] goes through, if one uses at a crucial point the main result of Section 6, which asserts that certain semiprime ideals are left Goldie (see Theorem 6.6).

Relating the Center of $V/(J:G)$ to $Q(G)$. We now turn to some more technical, but very important results which are used in proving several of the above theorems. The first theorem is an improvement of [V_1, 3.1]; it generalizes characteristic zero results of Moeglin and Rentschler, in particular [MR_3, I.29]. Concerning the terminology: the precise definition of a *purely inseparable extension* of commutative semisimple algebras is given in Definition 4.1. If H is a closed subgroup of G, we denote by $Q(G)$ and $Q(G/H)$ the rational functions defined on dense open subsets of G and G/H, respectively. Left multiplication of G on itself induces actions on $Q(G)$ and $Q(G/H)$, which we denote by Γ. The actions induced by right multiplication are denoted by Δ.

2.11. THEOREM. *Assume that V is semiprime and left Goldie, with total ring of fractions $Q(V)$. Let L be a commutative semisimple subalgebra of $Q(V)$ which is stable under the action of G. Suppose that $L^G = k \cdot 1_L$. Then there is a closed subgroup H of G and an injective, G-equivariant homomorphism ν from L into $Q(G/H)$ such that $Q(G/H)$ is a finite purely inseparable extension of $\nu(L)$. Moreover*:

(a) *The group H is unique up to conjugation, and the embedding ν is unique up to automorphisms of $Q(G)$ induced by Δ. That is, if K is a closed subgroup of G, and $\tilde{\nu}$ a G-equivariant embedding of L into $Q(G/K)$ such that $Q(G/K)$ is purely inseparable over $\tilde{\nu}(L)$, then there is some $z \in G$ such that*

$$K = zHz^{-1} \qquad \text{and} \qquad \tilde{\nu} = \Delta(z) \circ \nu.$$

(b) *The group H is uniquely determined by the G-equivariant embedding ν of L into $Q(G)$. In fact,*

$$H = \{\gamma \in G | \Delta(\gamma) f = f \text{ for all } f \in \nu(L)\}.$$

That ν is G-equivariant means that ν intertwines the action on V with the action Γ on $Q(G)$ induced by left multiplication of G on itself. The existence of H and ν, and the uniqueness of H up to conjugation, was shown in [V_1, 3.1]. Parts (a) and (b) are proved in Section 4; they follow from Propositions 4.2 and 4.5, respectively.

Now let J be a rational ideal of V. Then $(J:G)$ is G-rational by Theorem 2.10(a). That is, if we denote by C the center of $Q(V/(J:G))$, the total ring of fractions of $V/(J:G)$, then $C^G = k$. Hence Theorem 2.11 applies to the semisimple commutative subalgebra C of the algebra $Q(V/(J:G))$. Thus there is a closed subgroup H of G, and a G-equivariant homomorphism ν: $C \to Q(G/H)$ such that $Q(G/H)$ is a finite purely inseparable extension of $\nu(C)$. Moreover, H is unique up to conjugation. Our next result tells us how closely ν, H, and J are related. In particular, H turns out to be a conjugate of the stabilizer of J in G. In Section 7, the technical heart of this paper, we will—using techniques of Moeglin and Rentschler−explicitly construct such an equivariant embedding ν. We call it ν_J, and our next result is phrased in terms of that map. Note that by Theorem 2.11(a), $\nu = \Delta(\gamma)\circ \nu_J$ for some $\gamma \in G$.

2.12. THEOREM. *Let J be a rational ideal of V, and set $I = (J:G)$. Denote by C the center of the total ring of fractions $Q(V/I)$ of V/I. Let ν_J be the G-equivariant map $C \to Q(G)$ defined in Subsection* 7.5. *Then the stabilizer of J in G is the unique closed subgroup H of G such that $Q(G/H)$ is a finite purely inseparable extension of $\nu_J(C)$. Explicitly,*

$$H = \{\gamma \in G | \Delta(\gamma) f = f \text{ for all } f \in \nu_J(C)\}.$$

Regarding $Q(G/H)$ as a C-module via ν_J, denote by π the natural function from V to $Q(V/I) \otimes_C Q(G/H)$. Then

$$J = \pi^{-1}\big(\pi(V) m_{(H)}\big),$$

where $m_{(H)}$ denotes the set of those rational functions in $Q(G/H)$ defined on a dense open subset of G/H containing the coset H, and vanishing at H.

So J (and its stabilizer H), are uniquely determined by, and can be recovered from, the equivariant embedding ν_J (provided $I = (J:G)$ is known). In characteristic zero, this result is due to Moeglin and Rentschler, see [MR_3, I.19; MR_4, Théorème 2]. Our characteristic-free version is proved in 7.11.

Relating V/J and $V/(J:G)$ by Central Extensions. Let J be a rational ideal of V. Using the map ν_J just introduced, one can relate V/J and $V/(J:G)$ by a series of central extensions. This relationship, which is very useful for applications, was discovered and heavily exploited by Moeglin and Rentschler (though extension (3) did not occur in their work in characteristic zero). To simplify notation, assume that $(J:G) = 0$. As before, denote by C the center of the total ring of fractions of $V = V/(J:G)$, and by VC the subalgebra generated by V and C. If H is the stabilizer of J, then $Q(G/H)$ is a C-module via ν_J. Here is an abbreviated diagram of the central ring extensions relating V/J and $V/(J:G)$; details will be given in Subsection 7.12 and Diagram 7.13.

$$\begin{array}{ccc}
(V/J) \otimes Q(G) & \cong & T \otimes_{Q(G/H)} Q(G) \\
\Big|\,{\scriptstyle(1)} & & \Big|\,{\scriptstyle(2)} \\
& & T \cong VC \otimes_C Q(G/H) \\
& & \Big|\,{\scriptstyle(3)} \\
V/J & & VC \\
& & \Big|\,{\scriptstyle(4)} \\
& & V = V/(J:G)
\end{array} \tag{2.13}$$

Using this diagram gives a powerful technique to relate V/J and $V/(J:G)$. As a simple illustration of this, we refer to Proof 9.5 of Theorem 2.8, which asserts that $\mathrm{GKdim}(V/(J:G)) = \mathrm{GKdim}(V/J) + \dim(G/H)$.

3. PRELIMINARIES

In this section, we introduce our notation and present some preliminary results. The highlight is a short new proof of a result of Chin, see Proposition 3.6.

3.1. The following table summarizes our notation and conventions.

k	The algebraically closed base field. All algebras are k-algebras, and all tensor products are over k, unless otherwise indicated.
V	A k-algebra.
G	A linear algebraic group acting rationally on V.
G°	The connected component of G.
$(J:G)$	$= \bigcap_{\gamma \in G} \gamma(J)$ is the *orb* of a k-subspace J of V.
β	Denotes (where necessary) the action of G on V.
μ	The rationality of the action of G on V is equivalent to the existence of a certain $Q(G)$-linear automorphism μ of $V \otimes Q(G)$, see [MR$_3$, 0.4] or [V$_1$, 2.2, 2.3]; in particular, recall the *intertwining properties* of μ stated there. See also Subsection 3.2 below.
Γ, Δ	All actions of G (e.g., those on G and $Q(G)$) induced by left (resp., right) multiplication of G on itself are denoted by Γ (resp., Δ), see [V$_1$, 2.1].
$Q(G/H)$	For a closed subgroup H of G, we identify $Q(G/H)$ with $Q(G)^{\Delta(H)}$.
$\natural$	Given a closed subgroup H of G, $\natural$ is the one-to-one correspondence between the H-stable subspaces J of V and the $(\beta \otimes \Gamma)(G)$-stable subspaces of $V \otimes Q(G/H)$, see [MR$_3$, I.4] or [V$_1$, 2.4]. By definition, $\natural$ sends J to $J^\natural = \mu^{-1}(J \otimes Q(G)) \cap (V \otimes Q(G/H))$.
$\sharp$	The correspondence inverse to $\natural$.

For an algebra R:

$Q(R)$	The total ring of fractions of R (if it exists).
$Z(R)$	The center of R.

For a variety X, and a point x on X:

$A(X)$	The algebra of regular functions on X.
$Q(X)$	The algebra of rational functions defined on dense open subsets of X.
$\mathcal{O}_x$	The algebra of rational functions defined on open neighborhoods of x, sometimes also denoted by $\mathcal{O}_{x,X}$.
$\mathcal{O}_{(x)}$	The algebra of rational functions defined on dense open neighborhoods of x, sometimes also denoted by $\mathcal{O}_{(x),X}$.

3.2. *A few additional remarks about* μ. One can see as follows that the map μ defined in [V_1, 2.2] is an automorphism of $V \otimes Q(G)$. In analogy to the map μ_0: $V \to V \otimes A(G)$ in [V_1, 2.2], define a map $\tilde{\mu}_0$: $V \to V \otimes A(G)$ as follows. Given $x \in V$, $\tilde{\mu}_0(x) = \sum x_i \otimes f_i$ iff $\sum x_i f_i(\gamma) = \beta(\gamma)x$ for all $\gamma \in G$. Both μ_0 and $\tilde{\mu}_0$ extend $A(G)$-linearly to endomorphisms of $V \otimes A(G)$. A straightforward calculation shows that these two endomorphisms are inverse to each other. It follows that μ is an automorphism.

In this context, it is worth noting (although this observation will not be used in the sequel) that $\tilde{\mu}_0$, the restriction of μ^{-1} to a map $V \to V \otimes A(G)$, makes V into an $A(G)$-comodule algebra, see [M] (cf. also [W, Sect. 3.2]).

We next recall a basic result of Moeglin and Rentschler (see [MR_3, I.5]), which will be important in the sequel. Its proof goes through in prime characteristic.

3.3. Proposition. *Let J be an ideal of V, and let H be a closed subgroup of V stabilizing J. Then* $(J : G) = J^{\natural} \cap V$.

We now discuss some elementary results which will be useful in later sections.

3.4. Lemma.

(a) *Let I be a G-prime ideal of V, and P an ideal of V such that $(P : G) = I$. Then there is an ideal $J \supseteq P$ maximal with respect to the property $(J : G) = I$, and any such J is a prime ideal.*

(b) *Let I be a G-rational ideal of V. Then I is G-prime.*

Proof. (a) Since the action of G on V is rational, a Zorn's lemma argument shows the existence of such a maximal J (see [V_2, 2.18]). It is easy to show that any such J is prime.

(b) Suppose that I is G-rational, so in particular semiprime and left Goldie. Since $(Z(Q(V/I)))^G = k$, the action of G permutes the minimal prime ideals of $Q(V/I)$ transitively. Thus if P is a prime ideal of V minimal over I, then $I = \bigcap_{\gamma \in G} \gamma P$. It follows that I is G-prime. ∎

3.5. LEMMA. *Let J be a prime ideal of V.*

(a) *$J \otimes Q(G^\circ)$ is a prime ideal of $V \otimes Q(G^\circ)$.*

(b) *$J \otimes Q(G)$ is a semiprime ideal of $V \otimes Q(G)$. In fact, it is the intersection of finitely many prime ideals of $V \otimes Q(G)$.*

Note that $J \otimes Q(G)$ is certainly not prime if G is not connected.

Proof. (a) This follows immediately from the fact that $Q(G^\circ)$ is unirational over k, see [Bo, Theorem 18.2]. We remark that this can also be seen without using the unirationality of $Q(G^\circ)$, see Proposition 6.9. Part (b) follows from (a), as $Q(G)$ is a finite direct sum of fields all k-isomorphic to $Q(G^\circ)$. ∎

The next result is due to Chin [Ch, Corollary 1.3]; his proof uses Hopf algebra techniques and the hyperalgebra action on V as higher derivations. We give a short new proof of his result, and note some corollaries.

3.6. PROPOSITION (Chin). *Every G°-prime ideal of V is prime.*

It follows from Lemma 3.4(a) that every G°-prime ideal I is semiprime. And if I is the intersection of finitely many prime ideals, then elementary arguments show that I is prime (cf. [V_2, 2.6]). So Chin's result is of particular interest in situations where semiprime ideals are not necessarily intersections of finitely many prime ideals.

Proof. To simplify the notation, assume that G is connected. Let I be a G-prime ideal of V. By Lemma 3.4(a), there is a prime ideal J of V such that $I = (J : G)$. Let H be the trivial subgroup of G. By Lemma 3.5(a), $J \otimes Q(G)$ is prime. Hence also $J^\natural = \mu^{-1}(J \otimes Q(G))$ is prime. By Proposition 3.3, $I = (J : G) = J^\natural \cap V$. Since $V \otimes Q(G)$ is a central extension of V, and since $J^\natural$ is prime, $I = J^\natural \cap V$ is also prime. ∎

We mention some immediate consequences of Proposition 3.6.

3.7. COROLLARY. *Let J be a prime ideal of V.*

(a) *$(J : G^\circ)$ is prime.*

(b) *$(J : G)$ is the intersection of finitely many, G°-stable prime ideals (which are in fact the G-conjugates of $(J : G^\circ)$).*

Proof. (a) This follows from Proposition 3.6 since $(J:G^\circ)$ is G°-prime if J is prime.

(b) $(J:G)$ is the intersection of the finitely many G-conjugates of $(J:G^\circ)$. As G° is connected, the latter are all G°-stable. Finally, $(J:G^\circ)$ is prime by (a). ∎

Of independent interest is the next corollary.

3.8. COROLLARY. *If I is a semiprime, G-stable ideal of V, then every prime ideal of V minimal over I is G°-stable. In particular, the minimal prime ideals of V are G°-stable.*

Proof. If J is a prime ideal minimal over I, then $I \subseteq (J:G^\circ) \subseteq J$. As $(J:G^\circ)$ is prime, $J = (J:G^\circ)$, i.e., J is G°-stable. ∎

We will need the following easy consequence of a theorem of Azumaya and Nakayama [Co_3, 7.1.2]. It should be well known, and a proof is only included for lack of a reference.

3.9. LEMMA. *Let R be a finite direct sum of simple rings. Denote the center of R by C, and let S be a C-algebra. Then intersection gives a one-to-one correspondence between the ideals of $R \otimes_C S$ and S.*

We identify here S with $C \otimes_C S \subseteq R \otimes_C S$; this makes sense since S is a flat C-module (see [Co_3, 6.6.5]).

Proof. Say $R = \oplus_i R_i$, where each R_i is a simple ring with center C_i. Set $S_i = C_i S$. Then $C = \oplus_i C_i$ and $S = \oplus_i S_i$. There is a natural isomorphism $R \otimes_C S \cong \oplus_i (R_i \otimes_{C_i} S_i)$ (cf. Subsection 6.1) which maps S onto $\oplus_i (C_i \otimes_{C_i} S_i)$. By the theorem of Azumaya and Nakayama, intersection gives a one-to-one correspondence between the ideals of $R_i \otimes_{C_i} S_i$ and $C_i \otimes_{C_i} S_i$. The lemma follows now from the elementary fact that the ideals of a finite direct sum of rings $\oplus_i A_i$ are just the direct sums of the ideals of the A_i. ∎

4. $\Gamma(G)$-STABLE SUBALGEBRAS OF $Q(G)$

In this section, we prove parts (a) and (b) of Theorem 2.11; they follow from Propositions 4.2 and 4.5, respectively. Combining the latter two results, we then prove that G acts transitively on certain sets of equivariant homomorphisms, see Theorem 4.7. This will be a crucial ingredient in the proof of Theorem 2.2, which asserts that G acts transitively on each stratum in $\mathrm{Rat}(V)$; see Theorem 7.10.

It is worth noting that Proposition 4.5 gives rise to a Galois type correspondence between the closed subgroups of the linear algebraic group G and certain subalgebras of $Q(G)$, see Remark 4.6.

We begin by defining formally the notion of a purely inseparable extension of commutative semisimple rings.

4.1. DEFINITION. Let $R = R_1 \oplus \cdots \oplus R_n$ be a commutative semisimple algebra where the R_i are fields. We say that R is *purely inseparable* over a semisimple subalgebra L of R if there are subfields L_i of R_i such that R_i is purely inseparable over L_i, and such that $L = L_1 \oplus \cdots \oplus L_n$.

4.2. PROPOSITION. *Let H and K be closed subgroups of a (not necessarily connected) linear algebraic group G over k. Let L be a $\Gamma(G)$-stable (commutative) semisimple subalgebra of $Q(G/H)$ such that $Q(G/H)$ is purely inseparable over L. Suppose there is a $\Gamma(G)$-equivariant homomorphism ψ^* from L into $Q(G/K)$ such that $Q(G/K)$ is purely inseparable over $\psi^*(L)$. Then there is some $z \in G$ such that ψ^* is induced by right multiplication by z. That is, $\psi^* = \Delta(z)|_L$. Moreover, $K = zHz^{-1}$.*

This is somewhat similar to [V_1, 3.6 and 3.7]. Note that this result proves part (a) of Theorem 2.11: if $\psi^* = \tilde{\nu} \circ \nu^{-1}$: $\nu(L) \to Q(G/K)$, then for some $z \in G$, $K = zHz^{-1}$ and $\psi^* = \Delta(z)|_{\nu(L)}$. Hence $\tilde{\nu} = \psi^* \circ \nu = \Delta(z) \circ \nu$.

Proof. Note that ψ^* is injective since $\Gamma(G)$ permutes the minimal idempotents of $Q(G/H)$ transitively. Denote by π^* the inclusion of L into $Q(G/H)$. For $\gamma \in G$, define

$$A_{\overline{\gamma H}} = (\pi^*)^{-1}(\mathscr{O}_{(\gamma H)}) \qquad \text{and} \qquad A_{\overline{\gamma K}} = (\psi^*)^{-1}(\mathscr{O}_{(\gamma K)}).$$

Here $\mathscr{O}_{(\gamma H)} = \mathscr{O}_{(\gamma H), G/H}$, see Subsection 3.1. We apply the proof of [V_1, Lemma 3.5] simultaneously to the embeddings π^*: $L \to Q(G/H)$ and ψ^*: $L \to Q(G/K)$. Thus there is an affine variety Y with $Q(Y) = L$, and there are affine dense open subsets U and U' of G/H and G/K, respectively, such that π^* and ψ^* induce epimorphisms π: $U \to Y$ and ψ: $U' \to Y$. Moreover, as the proof of [V_1, Lemma 3.5] shows, we can choose U and U' in such a way that

$$A_{\overline{\gamma'\gamma H}} = \Gamma(\gamma') A_{\overline{\gamma H}} \qquad \text{for all } \gamma, \gamma' \in G,$$

$$A_{\overline{\gamma'\gamma K}} = \Gamma(\gamma') A_{\overline{\gamma K}} \qquad \text{for all } \gamma, \gamma' \in G,$$

$$A_{\overline{\gamma H}} = \mathscr{O}_{(\pi(\gamma H))} \qquad \text{for all } \gamma \in G \text{ with } \gamma H \in U,$$

$$A_{\overline{\gamma K}} = \mathscr{O}_{(\psi(\gamma K))} \qquad \text{for all } \gamma \in G \text{ with } \gamma K \in U'.$$

Here $\mathscr{O}_{(\pi(\gamma H))} = \mathscr{O}_{(\pi(\gamma H)), Y}$, see Subsection 3.1.

Since π and ψ are both surjective, we can find $x, y \in G$ such that $xH \in U$, $yK \in U'$, and $\pi(xH) = \psi(yK)$. Set $z = y^{-1}x$.

4.3. LEMMA. *$K = zHz^{-1}$. Consequently, $\Delta(z)$ induces an isomorphism of varieties $G/H \to G/K$ defined by $\Delta(z)(\gamma H) = \gamma z^{-1}K$.*

Proof. We show that xHx^{-1} is the stabilizer in G of $A_{\overline{xH}}$. Since

$$A_{\overline{xH}} = \mathscr{O}_{(\pi(xH))} = \mathscr{O}_{(\psi(yK))} = A_{\overline{yK}},$$

it follows then by symmetry that $xHx^{-1} = yKy^{-1}$, i.e., $K = zHz^{-1}$.

In characteristic zero, π^* is an isomorphism of L onto $Q(G/H)$. Thus in this case, $\pi^*(A_{\overline{xH}}) = \mathscr{O}_{(xH)}$. The stabilizer of $\mathscr{O}_{(xH)}$ is equal to the stabilizer of $xH \in G/H$ under the action of $\Gamma(G)$, which is xHx^{-1}, as required.

Assume now that char $k = p$ is prime. Since $Q(G/H)$ is purely inseparable over $L = \pi^*(L)$, there is some n such that $\pi^*(L)$ contains $Q(G/H)^{p^n}$. Denote by f the Frobenius map of $Q(G/H)$ which sends a to a^{p^n}. It is a ring (not algebra) isomorphism of $Q(G/H)$ onto $Q(G/H)^{p^n}$. Note that for any $\gamma \in G$,

$$\mathscr{O}_{(\gamma H)} \cap Q(G/H)^{p^n} = f(\mathscr{O}_{(\gamma H)}).$$

Suppose that $A_{\overline{\gamma xH}} = A_{\overline{xH}}$, i.e., $\pi^*(A_{\overline{\gamma xH}}) = \pi^*(A_{\overline{xH}})$. Then $\mathscr{O}_{(\gamma xH)}$ and $\mathscr{O}_{(xH)}$ have the same intersection with $\pi^*(L)$, so also the same intersection with $Q(G/H)^{p^n}$, and are hence equal, implying that $\gamma xH = xH$. Hence for $\gamma \in G$,

$$\begin{aligned}\Gamma(\gamma)A_{\overline{xH}} = A_{\overline{xH}} &\Leftrightarrow \pi^*(A_{\overline{\gamma xH}}) = \pi^*(A_{\overline{xH}}) \\ &\Leftrightarrow \gamma xH = xH \Leftrightarrow \gamma \in xHx^{-1}.\end{aligned}$$

Thus xHx^{-1} is the stabilizer in G of $A_{\overline{xH}}$.

Finally, since $zHz^{-1} = K$, $\Delta(z)$ induces a well-defined map $G/H \to G/K$: indeed, $\Delta(z)(\gamma H) = \gamma Hz^{-1} = \gamma z^{-1}K$. ∎

4.4. LEMMA. *$\psi \circ \Delta(z) = \pi$ as rational maps of varieties $G/H \to Y$.*

Proof. Set $U_0 = \{\gamma \in G | \gamma xH \in U\}$. Then U_0 is open in G, being the inverse image in G of the open subset U of G/H under the morphism $\gamma \mapsto \gamma xH$. By definition of G/H, the quotient map $G \to G/H$ is open, i.e., maps open sets to open sets. Hence also the morphism $\gamma \mapsto \gamma xH$ is open. Since U is a dense open subset of G/H it follows that its inverse image, the open set U_0, is dense in G. Similarly, also $U_0' = \{\gamma \in G | \gamma yK \in U'\}$ is dense and open in G. Hence also $V = U_0 \cap U_0'$ is dense and open in G, implying that VxH is dense and open in G/H. Fix a $\gamma \in V$. Since $z = y^{-1}x$,

$$[\psi \circ \Delta(z)](\gamma xH) = \psi(\gamma xz^{-1}K) = \psi(\gamma yK).$$

So it suffices to show that $\pi(\gamma xH) = \psi(\gamma yK)$, i.e., that $\mathcal{O}_{(\pi(\gamma xH))} = \mathcal{O}_{(\psi(\gamma yK))}$. But

$$\mathcal{O}_{(\pi(\gamma xH))} \overset{(1)}{=} A_{\overline{\gamma xH}} = \Gamma(\gamma) A_{\overline{xH}} \overset{(2)}{=} \Gamma(\gamma)\mathcal{O}_{(\pi(xH))}$$
$$\overset{(3)}{=} \Gamma(\gamma)\mathcal{O}_{(\psi(yK))} \overset{(4)}{=} \Gamma(\gamma) A_{\overline{yK}} = A_{\overline{\gamma yK}} \overset{(5)}{=} \mathcal{O}_{(\psi(\gamma yK))}.$$

Here (1) is true since $\gamma xH \in U$; (2) because $xH \in U$; (3) because $\pi(xH) = \psi(yK)$; (4) because $yK \in U'$; and (5) because $\gamma yK \in U'$. ∎

We now finish the proof of Proposition 4.2. By the previous lemma, $\Delta(z)^* \circ \psi^* = \pi^*$. Once checks readily that $\Delta(z)^* = \Delta(z^{-1})$. Indeed, for $\gamma, \delta \in G$ and $f \in \mathcal{O}(G)$, $[\Delta(\gamma)^*(f)](\delta) = [f \circ \Delta(\gamma)](\delta) = f(\delta\gamma^{-1})$, while $[\Delta(\gamma)(f)](\delta) = f(\delta\gamma)$. Consequently, $\Delta(z^{-1}) \circ \psi^* = \pi^*$, i.e., $\psi^* = \Delta(z) \circ \pi^*$. Since π^* is the inclusion of L into $Q(G/H)$, it follows that $\psi^* = \Delta(z)|_L$. This concludes the proof of Proposition 4.2. ∎

4.5. Proposition. *Let L be a (commutative) semisimple subalgebra of $Q(G)$ which is $\Gamma(G)$-stable. Then*

$$H = \{\gamma \in G | \Delta(\gamma) f = f \text{ for all } f \in L\}$$

is the unique closed subgroup of G such that L is contained in $Q(G/H)$ and such that $Q(G/H)$ is purely inseparable over L.

Note that this proves part (b) of Theorem 2.11.

Proof. We will show below that H is closed, and that $Q(G/H)$ is purely inseparable over L. Assuming this, we now show that H is unique. Suppose K is another closed subgroup of G such that L is contained in $Q(G/K)$ and such that $Q(G/K)$ is purely inseparable over L. Then H and K are conjugate by Proposition 4.2. Moreover, since $L \subseteq Q(G/K) = Q(G)^{\Delta(K)}$, it follows that $K \subseteq H$. These two facts easily imply $H = K$.

We now show that H is closed, and that $Q(G/H)$ is purely inseparable over L. In case that L is a field, this is the content of [V_1, 3.4]; we will use that result below.

Say $L = L_1 \oplus \cdots \oplus L_n$, where the L_i are fields. We denote the unit element of L_1 by e. Let G' be the stabilizer of e for the action Γ of G. Then G' is also the stabilizer of $L_1 = Le$, and the index of G' in G is n. The unit element of L is the sum of certain primitive idempotents of $Q(G)$ which are permuted transitively by $\Gamma(G)$. Consequently, L and $Q(G)$ have the same unit element. Moreover, G' is closed and contains G° by [MR_3, I.22] (see also [V_1, 2.5]).

We may assume that e contains as a summand the primitive idempotent of $Q(G)$ which is the rational function which is identically one on G° and identically zero on the other irreducible components of G. Note that this primitive idempotent belongs to $Q(G')$ since $G^{\circ} \subseteq G'$. We next show that under this assumption on e, $L_1 \subseteq Q(G')$. Denote by m the number of irreducible components of G. This is also the number of (orthogonal) primitive idempotents in $Q(G)$. The number of primitive idempotents of $Q(G)$ which are summands of e is m/n; this is also the number of primitive idempotents of $Q(G)$ contained in $Q(G')$. Note that $\Gamma(G')$ permutes transitively the primitive idempotents of $Q(G')$ (which are also primitive idempotents of $Q(G)$). Since $\Gamma(G')$ permutes also the primitive idempotents which are summands of e, it follows that all m/n primitive idempotents of $Q(G')$ are summands of e. Consequently, e is the unit element of $Q(G')$, and $L_1 \subseteq Q(G)e = Q(G')$.

Choose $\gamma_i \in G$ such that $\Gamma(\gamma_i)L_1 = L_i$. Then the $\gamma_i G'$ are the distinct left cosets of G' in G, and $L_i \subseteq \Gamma(\gamma_i)Q(G') = Q(\gamma_i G')$.

Note that $H \subseteq G'$. Denote by H' the set of all $\gamma \in G'$ such that $\Delta(\gamma)f = f$ for all $f \in L_1$. We show next that $H = H'$. Clearly, $H \subseteq H'$. Suppose now that $\gamma \in H'$ and $f \in L_i$. We have to show that $\Delta(\gamma)f = f$. Now for some $g \in L_1$, $f = \Gamma(\gamma_i)g$. Hence $\Delta(\gamma)f = \Delta(\gamma)\Gamma(\gamma_i)g = \Gamma(\gamma_i)\Delta(\gamma)g = \Gamma(\gamma_i)g = f$, since the actions Γ and Δ commute. Thus $H = H'$. It follows by [V_1, 3.4] that H is a closed subgroup of G' (and thus of G), and that $Q(G'/H)$ is a finite, purely inseparable field extension of L_1.

Now $Q(G) = \oplus_{i=1}^{n} Q(\gamma_i G')$. Since the $\gamma_i G'$ are $\Delta(H)$-stable (as H is a subgroup of G'), we have

$$Q(G/H) = Q(G)^{\Delta(H)} = \bigoplus_{i=1}^{n} Q(\gamma_i G')^{\Delta(H)}.$$

Since $\Gamma(\gamma_i)$ induces a $\Delta(H)$-equivariant isomorphism of $Q(G')$ onto $Q(\gamma_i G')$, it maps $Q(G'/H) = Q(G')^{\Delta(H)}$ isomorphically onto $Q(\gamma_i G')^{\Delta(H)}$. Hence $Q(\gamma_i G')^{\Delta(H)}$ is a field which is a finite, purely inseparable extension of $L_i = \Gamma(\gamma_i)L_1$. ∎

4.6. *Remark*. Proposition 4.5 gives rise to a Galois type correspondence between the closed subgroups H of a linear algebraic group G and those $\Gamma(G)$-stable semisimple subalgebras L of $Q(G)$ for which no element of $Q(G) \setminus L$ is purely inseparable over L. For connected linear algebraic groups G, this recovers a result of Abe and Kanno [AK] (although their correspondence is phrased slightly differently). We omit the verification of the details.

Combining Propositions 4.2 and 4.5, we can now prove the following theorem which will be important in the sequel, in particular in the proofs of Theorems 7.10 and 2.2.

4.7. THEOREM. *Let L be a (commutative) semisimple subalgebra of $Q(G)$ which is $\Gamma(G)$-stable. Denote by $\mathrm{Hom}_G(L, Q(G))$ the set of $\Gamma(G)$-equivariant k-algebra homomorphisms from L into $Q(G)$. Then the action of G on $\mathrm{Hom}_G(L, Q(G))$ induced by Δ is transitive.*

To be precise, the group G acts on $\mathrm{Hom}_G(L, Q(G))$ as follows: given $\gamma \in G$ and $f \in \mathrm{Hom}_G(L, Q(G))$, γ sends f to $\Delta(\gamma) \circ f$.

Proof. Let $\psi^* \in \mathrm{Hom}_G(L, Q(G))$. By Proposition 4.5, there are closed subgroups H and K of G such that $Q(G/H)$ and $Q(G/K)$ are finite purely inseparable extensions of L and $\psi^*(L)$, respectively. So by Proposition 4.2, there is some $z \in G$ such that $\psi^* = \Delta(z)|_L$. Then $\psi^* = \Delta(z) \circ f$, where f is the embedding of L into $Q(G)$. ∎

5. GOLDIE CONDITIONS FOR PRIME IDEALS AND THEIR ORBS

Recall that a semiprime ideal is left Goldie if modulo that ideal, the ring has finite left uniform dimension and satisfies the ascending chain condition on left annihilators. In this section we prove Theorem 2.1, which states that a prime ideal of V is left Goldie if and only if its orb is left Goldie. We begin with a well-known elementary lemma.

5.1. LEMMA. *Let $R[x]$ be the polynomial ring in one central indeterminate over a prime ring R. If $R[x]$ is left Goldie, then so is R.*

Proof. Note that $R[x]$ is a prime ring. Its subring R satisfies the ascending chain condition on left annihilators. If $\oplus I_j$ is a direct sum of left ideals of R contained in R, then $\oplus I_j[x]$ is a direct sum of left ideals of $R[x]$ contained in $R[x]$. Hence the left uniform dimension of R is bounded above by the left uniform dimension of $R[x]$, which we assume to be finite. Thus R is left Goldie. ∎

5.2. PROPOSITION (Bell, Ferrero). *Let $R[x]$ be the polynomial ring in one central indeterminate over a ring R. Let P be a prime ideal of $R[x]$. Then P is left Goldie if and only if the prime ideal $P \cap R$ is left Goldie.*

The implication $\Leftarrow$ is due to Bell [Be, Proposition 2.4], see also [FP, Corollary 10]. The other implication is due to Ferrero, who kindly permitted its inclusion in this paper. We remark that one can prove $\Leftarrow$ also along the lines of Ferrero's proof of the other implication, which we present next.

Proof of $\Rightarrow$. Factoring out by $(P \cap R)[x]$, we may assume that $P \cap R = 0$. So R is now a prime ring. If $P = 0$, we are done by Lemma 5.1. So assume that $P \neq 0$. Denote by T the left Martindale quotient ring of R, see [P, Sect. 10]. The extended centroid C of R is by definition the center of T. We need to know only the following fact about T: given finitely many non-zero elements $q_i \in T$, there is a non-zero ideal A of R such that $0 \neq Aq_i \subseteq R$ for all i, see [P, 10.2]. By [Fe, Corollary 2.8], there is a prime ideal P' of $T[x]$ lying over P, i.e., with $P' \cap R[x] = P$. Since $P \cap R = 0$, $P' \cap T = 0$. By [Fe, Corollary 2.7 and Remark 2.6], P' is generated as an ideal by a monic irreducible polynomial with coefficients in C. Consequently, $T \subseteq T[x]/P'$ is a finite centralizing extension of prime rings.

Since $R[x]/P$ is left Goldie, its subring R satisfies the ascending chain condition on left annihilators. We have to show that R has finite left uniform dimension. We show first that $T[x]/P'$ has finite left uniform dimension. For that, it suffices to show that every non-zero left ideal of $T[x]/P'$ has non-zero intersection with $R[x]/P$. To prove this, it suffices to show that if $f(x) \in T[x] \setminus P'$, then $rf(x) \in R[x] \setminus P$ for some $r \in R$. There is certainly a non-zero ideal A of R such that $0 \neq Af(x) \subseteq R[x]$. Suppose $Af(x) \subseteq P$. Then also $AR[x]f(x) \subseteq P$. But as $P \cap R = 0$ and $f(x) \notin P'$, this is impossible. Consequently, $T[x]/P'$ has finite left uniform dimension. Since $T[x]/P'$ is a finite centralizing extension of T, also T has finite left uniform dimension, see [McR, 10.1.9]. It follows easily that also R has finite left uniform dimension and is thus left Goldie. Indeed, let $\oplus I_j$ be a finite direct sum of left ideals of R contained in R. Then $\sum TI_j$ is a direct sum of left ideals of T contained in T: suppose not. Then there are $a_j \in TI_j$, not all zero, such that $\sum a_j = 0$. For some non-zero ideal A of R, $Aa_j \subseteq I_j$ for all j, and $Aa_j \neq 0$ if $a_j \neq 0$. This is a contradiction to the assumption that the sum of the I_j is direct. Hence R has finite left uniform dimension and is thus left Goldie. ∎

5.3. PROPOSITION. *Let L be a finitely generated field extension of a field K, and let R be a K-algebra. Let P be a prime ideal of $R \otimes_K L$. Then P is left Goldie if and only if the prime ideal $P \cap R$ is left Goldie.*

Proposition 6.5 below gives a partial extension of this result for semiprime ideals.

Proof. Let L_0 be a subfield of L which is a purely transcendental extension of K, and over which L is algebraic. Then $R \otimes_K L_0 \subseteq R \otimes_K L$ is a finite centralizing extension. Hence P is left Goldie if and only if the prime ideal $P \cap (R \otimes_K L_0)$ is left Goldie, see [RS, Theorem 5.6]. We may thus assume that L is purely transcendental over K. By induction, we may assume that $L = K(x)$ is a rational function field in one variable x over K.

Note that $R \otimes_K K[x] = R[x]$ is a polynomial ring in one central indeterminate over R. Moreover, $R_2 = (R \otimes_K K(x))/P$ is the localization of $R_1 = R[x]/(P \cap R[x])$ at a set of regular central elements. By [McR, 2.2.12], R_1 and R_2 have the same left uniform dimension. If R_2 is left Goldie, its subring R_1 satisfies the ascending chain condition on left annihilators and is thus also left Goldie. Conversely, if R_1 is left Goldie, then $R_2 \subseteq Q(R_1)$, so that R_2 satisfies the ascending chain condition on left annihilators and is thus left Goldie. Consequently, P is left Goldie if and only if $P_0 = P \cap R[x]$ is left Goldie. By Proposition 5.2, P_0 is left Goldie if and only if $P_0 \cap R = P \cap R$ is left Goldie. This proves the proposition. ∎

5.4. *Proof of Theorem* 2.1. Recall that the intersection of a finite number of minimal prime ideals is left Goldie if and only if all the minimal prime ideals involved are left Goldie, see [McR, 3.2.5]. It suffices thus by Corollary 3.7 to show that J is left Goldie if and only if $(J : G^\circ)$ is left Goldie. Hence we may assume that G is connected.

Denote by H the trivial subgroup of G. By Lemma 3.5, $P = J \otimes Q(G)$ is prime, and so is $J^\natural = \mu^{-1}(P)$. By Proposition 3.3, $(J : G) = J^\natural \cap V$. Applying Proposition 5.3 twice to the ring extension $V \subseteq V \otimes Q(G)$ shows that $J = P \cap V$ is left Goldie iff P is left Goldie iff $J^\natural = \mu^{-1}(P)$ is left Goldie iff $(J : G) = J^\natural \cap V$ is left Goldie. ∎

As an immediate corollary of Theorem 2.1, we obtain the next result which applies in particular to the G-rational ideals I of V (see Lemma 3.4(b)).

5.5. COROLLARY. *Let I be a G-prime ideal of V which is left Goldie. Let J be an ideal of V maximal with respect to the property that $(J : G) = I$. Then J is a prime ideal which is left Goldie.*

Proof. By Lemma 3.4(a), such a J exists and is a prime ideal. By Theorem 2.1, J is left Goldie. ∎

6. THE CORRESPONDENCE $\natural$ AND GOLDIE CONDITIONS

In the sequel, we sometimes denote the action of G on V by β. Let H be a closed subgroup of the linear algebraic group G. Recall the correspondences $\natural$ and $\sharp$ between the H-stable subspaces of V and the $(\beta \otimes \Gamma)(G)$-stable subspaces of $V \otimes Q(G/H)$, see Subsection 3.1. The main result of this section is that $\natural$ and $\sharp$ preserve the property that an ideal is semiprime, and the property that an ideal is semiprime left Goldie, see Theorem 6.6. This enables us to prove Theorem 2.10(b), see 6.7. At the end of this section, we also study to what extent $\natural$ and $\sharp$ preserve the property that an ideal is prime, see Theorem 6.6(c) (proved in 6.10). Example 6.11 shows what can go wrong.

6.1. It will be helpful to quickly review an elementary property of tensor products over commutative semisimple rings. Say $K = K_1 \oplus \cdots \oplus K_n$ is a commutative semisimple ring, where the K_i are fields. Let M and N be K-modules. Denote by $M_i = MK_i$ and $N_i = NK_i$ their respective isotypic components; so $M = \oplus M_i$ and $N = \oplus N_i$. Then $M_i \otimes_K N_j = 0$ if $i \neq j$, and $M_i \otimes_K N_i \cong M_i \otimes_{K_i} N_i$. Consequently, $M \otimes_K N \cong \oplus_{i=1}^{n} M_i \otimes_{K_i} N_i$. In this way, many assertions involving tensor products over commutative semisimple rings can be reduced to statements involving tensor products over fields.

6.2. *Hypotheses for Lemmas* 6.3, 6.4, *and Proposition* 6.5. Let $K \subseteq L$ be commutative semisimple rings such that $K = L^{\Omega}$ for some group Ω acting on L. Let M be a K-module. We identify M with its isomorphic image in $M \otimes_K L$. Then Ω acts on $M \otimes_K L$ by acting trivially on M. In particular, Ω acts in this way on $R \otimes_K L$ for every K-algebra R.

6.3. LEMMA. *Let K, L, Ω, and M be as in Subsection* 6.2.

(a) *If X is an Ω-stable L-submodule of $M \otimes_K L$, then*

$$X = (X \cap M)L \cong (X \cap M) \otimes_K L.$$

(b) *If N is a K-submodule of M, then $N = (NL) \cap M$.*

We will apply this lemma as well as Lemma 6.4 and Proposition 6.5 mostly in the following two situations:

- $L = Q(G)$, $\Omega = \Delta(G)$, and $K = L^{\Omega} = k$; or
- $L = Q(G)$, $\Omega = \Delta(H)$, and $K = L^{\Omega} = Q(G/H)$.

Lemma 6.3 is similar to [MR_3, I.2]; it is classical in case L is a field. Not aware of a suitable reference, we include its proof.

Proof. (a) Here we identify M with $M \otimes 1 \subseteq M \otimes_K L$, and $(X \cap M)L$ denotes the L-submodule of $M \otimes_K L$ generated by $X \cap M$.

Since K is semisimple, L is a flat K-module [Co_3, 6.6.5]. Tensoring the inclusion $X \cap M \hookrightarrow M$ with L over K shows that $(X \cap M) \otimes_K L \cong (X \cap M)L$. Moreover, $(M \otimes_K L)/(X \cap M)L \cong (M/(X \cap M)) \otimes_K L$. In order to prove that $X = (X \cap M)L$, we may hence replace M by $M/(X \cap M)$, and assume that $X \cap M = 0$. We have now to show that $X = 0$. We reduce first to the case that K is a field. Say K is the direct sum of finitely many fields K_i. For any K-module Y, denote by Y_i its K_i-isotypic component. So $Y_i = YK_i$, and $Y = \oplus_i Y_i$. Clearly $X = 0$ if and only if all $X_i = 0$. Moreover, $L_i = LK_i$ is Ω-stable, and $K_i \subseteq (L_i)^{\Omega} \subseteq L^{\Omega} \cap L_i = K \cap L_i =$

K_i, showing that $(L_i)^{\Omega} = K_i$. Also, X_i is an Ω-stable L_i-submodule of $(M \otimes_K L)_i = M_i \otimes_K L_i \cong M_i \otimes_{K_i} L_i$. So in order to prove (a), we may assume that K is a field.

Suppose now that $X \neq 0$. Choose a non-zero $x \in X$ such that $x = \sum_{i=1}^{n} m_i \otimes a_i \in M \otimes_K L$ with n as small as possible. Denote by e_j the finitely many minimal primitive idempotents of L. Then $L_j = Le_j$ is a field, and L is the direct sum of the L_j. Since $1 = \sum e_j$, $xe_j \neq 0$ for some j, say $j = 1$. Replacing x by xe_1, we may assume that all $a_i = a_i e_1$ belong to L_1. Since $L^{\Omega} = K$, Ω permutes the e_j transitively. Hence there are $\omega_j \in \Omega$ such that $\omega_j(e_1) = e_j$. Set $b_i = \sum_j \omega_j(a_i) = \sum_j \omega_j(a_i)e_j$. Note that $b_i e_j = \omega_j(a_i) \neq 0$ for all j, so that b_i is invertible in L. Set $y = \sum_i m_i \otimes b_i = \sum_j \omega_j(x)$. Since $ye_j = \omega_j(x) \neq 0$, $y \neq 0$. Replacing x by y, we may hence assume that all a_i are invertible elements of L. Replacing x by $x(a_1)^{-1}$, we may furthermore assume that $a_1 = 1$. Now for every $\omega \in \Omega$, $x - \omega(x) = \sum_{i=2}^{n} m_i \otimes (a_i - \omega(a_i))$, so it must be zero. By the minimality of n, the m_i are K-linearly independent, implying that $a_i = \omega(a_i)$ for all i. Consequently, all $a_i \in L^{\Omega} = K$. It follows that $x = m \otimes 1$ for some $m \in M$. But then $x \in X \cap M = 0$, a contradiction. This proves (a).

(b) Set $N_1 = NL \cap M$. Then $N \subseteq N_1$, and $NL = N_1 L$. Denote by φ and ψ the inclusions $N \hookrightarrow N_1$ and $N_1 \hookrightarrow M$, respectively. Since L is flat over K, the maps $\varphi \otimes_K \mathrm{id}_L$ and $\psi \otimes_K \mathrm{id}_L$ are injections. Since $(\psi \otimes_K \mathrm{id}_L) \circ (\varphi \otimes_K \mathrm{id}_L)$ and $(\psi \otimes_K \mathrm{id}_L)$ have the same image in $M \otimes_K L$, namely $NL = N_1 L$, $\varphi \otimes_K \mathrm{id}_L$ must be surjective, so an isomorphism. Since L is flat over K, this implies that $(N_1/N) \otimes_K L = 0$. Hence $N_1/N = 0$, since L is faithfully flat over K, see [AM, p. 45, Exercise 16]. ∎

6.4. LEMMA. *Let K and L be as in Subsection* 6.2, *and let J be an ideal of a K-algebra R. Then J is a semiprime ideal of R if and only if $J \otimes_K L$ is a semiprime ideal of $R \otimes_K L$.*

Proof. Since $R \subseteq R \otimes_K L$ is a centralizing extension, $J = R \cap (J \otimes_K L)$ is semiprime if $J \otimes_K L$ is. Suppose now that J is semiprime. Factoring out by J, we may assume that J is zero. Denote by N the intersection of all prime ideals of $R \otimes_K L$. It is Ω-stable, so by Lemma 6.3(a) generated by its intersection with R. Now let p be a prime ideal of R. Then $P_0 = p \otimes_K L$ is an ideal of $R \otimes_K L$ lying over p, i.e., P_0 is an ideal with $P_0 \cap R = p$. By a Zorn's lemma argument, there is an ideal P of $R \otimes_K L$ which is maximal with respect to lying over p. One checks easily that P is a prime ideal, so contains N. Hence p contains $N \cap R$. So every prime ideal of the semiprime ring R contains $N \cap R$, implying that $N \cap R = 0$. Consequently $N = (N \cap R)L = 0$, so that $R \otimes_K L$ is semiprime. ∎

6.5. PROPOSITION. *Let K and L be as in Subsection* 6.2, *with the additional property that each direct summand L_0 of L is a finitely generated field extension of the image of K in L_0. Let R be a K-algebra.*

(a) *Let J be a semiprime ideal of R which is left Goldie. Then $J \otimes_K L$ is a semiprime ideal of $R \otimes_K L$ which is left Goldie.*

(b) *Let J be a semiprime ideal of $R \otimes_K L$ which is left Goldie. Then $J \cap R$ is a semiprime ideal of R which is left Goldie.*

For the proof of part (b), we will not use the fact that $L^{\Omega} = K$. Note that some finiteness condition is needed in part (a). To give an example, let $R = L = \overline{\mathbb{Q}}$ be the algebraic closure of the rational numbers $K = \mathbb{Q}$. As one can check, the semiprime ring $R \otimes_K L = \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ has an infinite number of minimal prime ideals, so is not Goldie.

Proof. (a) Factoring out by J, we may assume that $J = 0$. By Lemma 6.4, $R \otimes_K L$ is semiprime. Since L is flat over K (see, e.g., [Co$_3$, Theorem 6.6.5, p. 242]), $R \otimes_K L$ embeds into $Q(R) \otimes_K L$. By the finiteness assumption on L and a generalization of Hilbert's basis theorem (e.g., [McR, 1.2.10 and 2.1.16(iii)]), the latter algebra is Noetherian. Consequently, its subalgebra $R \otimes_K L$ satisfies the ascending chain condition on left annihilators. Note that the set S of regular elements of R consists of regular elements of $R \otimes_K L$. We show next that S satisfies the left Ore condition for $R \otimes_K L$. Indeed, if $s \in S$ and $\sum a_i \otimes l_i \in R \otimes_K L$, we can bring the fractions $a_i s^{-1}$ to a common left denominator. So there are $b_i \in R$ and $t \in S$ such that $ta_i = b_i s$ for all i. Then $(\sum b_i \otimes l_i)s = t(\sum a_i \otimes l_i)$, and $(R \otimes_K L)s \cap S(\sum a_i \otimes l_i) \neq \emptyset$. Consequently, $Q(R) \otimes_K L$ is a localization of $R \otimes_K L$ at a left Ore set consisting of regular elements. By [McR, 2.2.12], these two rings have the same left uniform dimension, which is finite since the larger ring is Noetherian.

(b) Because $R \subseteq R \otimes_K L$ is a central extension, $J \cap R$ is semiprime. Since J is a finite intersection of prime left Goldie ideals J_i, and since $J \cap R = \bigcap (J_i \cap R)$, we may assume that J is actually prime, see [McR, 3.2.5]. Say $K = K_1 \oplus \cdots \oplus K_n$, where the K_i are fields. Set $R_i = RK_i$ and $L_i = LK_i$. As in Subsection 6.1, $R \cong \oplus R_i$ and $R \otimes_K L \cong \oplus (R_i \otimes_{K_i} L_i)$. Replacing R by a suitable R_i, and $R \otimes_K L$ by its corresponding direct summand $R_i \otimes_{K_i} L_i$, we may assume that K is a field.

Now write $L = L_1 \oplus \cdots \oplus L_n$, where the L_i are field extensions of K. Then $R \otimes_K L \cong \oplus (R \otimes_K L_i)$, and under this isomorphism, $R = R \otimes_K 1$ corresponds to the "diagonal": $r \in R$ maps to $(r \otimes 1, \ldots, r \otimes 1)$. We may assume that J corresponds under this isomorphism to $P \oplus [\oplus_{i=2}^{n} (R \otimes_K L_i)]$, where P is a prime left Goldie ideal of $R \otimes_K L_1$. Note that $J \cap R \subseteq R \otimes_K L$ is left Goldie if and only if $P \cap R \subseteq R \otimes_K L_1$ is. Hence we may assume that L is a field as well. Now we are done by Proposition 5.3. ∎

We are ready for the main result of this section.

6.6. THEOREM. *Let H be a closed subgroup of G. The correspondences $\natural$ and $\sharp$ preserve the property that an ideal is semiprime, and the property that* *an ideal is semiprime and left Goldie. Under certain connectedness assumptions, these correspondences also preserve the property that an ideal is prime. To be more precise, let J be an ideal of V which is* $\beta(H)$*-stable, and consider the corresponding ideal* $J^\natural$ *of* $V \otimes Q(G/H)$.

(a) *J is semiprime if and only if* $J^\natural$ *is semiprime.*

(b) *J is semiprime left Goldie if and only if* $J^\natural$ *is semiprime left Goldie.*

(c) *If J is prime and G connected,* $J^\natural$ *is prime. If* $J^\natural$ *is prime and both G and H are connected, then J is prime.*

Part (a) extends [MR_3, I.37], which asserts that J is $\beta(H)$-(semi-)prime if and only if $J^\natural$ is $(\beta \otimes \Gamma)(G)$-(semi-)prime. We remark that part (c), though interesting in its own right, will not be used in the sequel. We will see in Example 6.11 that the conclusions in part (c) can be false if the connectedness assumptions are not satisfied.

Proof of Parts (*a*) *and* (*b*). The proof of (a) is similar to the proof of (b), but easier, using Lemma 6.4 instead of Proposition 6.5. Therefore we omit it, and turn to the proof of (b). Assume first that J is semiprime and left Goldie. Then so is $J \otimes Q(G)$ by Proposition 6.5(a) with $\Omega = \Delta(G)$. Hence also $\mu^{-1}(J \otimes Q(G))$ is semiprime and left Goldie. Using the fact that $V \otimes Q(G) = (V \otimes Q(G/H)) \otimes_{Q(G/H)} Q(G)$, Proposition 6.5(b) with $\Omega = \Delta(H)$ implies that $J^\natural = \mu^{-1}(J \otimes Q(G)) \cap (V \otimes Q(G/H))$ is semiprime and left Goldie. Conversely, assume that $J^\natural$ is semiprime and left Goldie. By Lemma 6.3(a) with $\Omega = \Delta(H)$, $\mu^{-1}(J \otimes Q(G)) \cong J^\natural \otimes_{Q(G/H)} Q(G)$. Hence $\mu^{-1}(J \otimes Q(G))$ is semiprime and left Goldie by Proposition 6.5(a) with $\Omega = \Delta(H)$. Thus $J \otimes Q(G)$ is semiprime and left Goldie, and so is J by Proposition 6.5(b) with $\Omega = \Delta(G)$. ∎

We have now the missing technical ingredient to reprove [V_1, Theorem 5.1] without assuming that every semiprime ideal of V is left Goldie.

6.7. *Proof of Theorem* 2.10(*b*). This follows as in the proof of [V_1, 5.1]. There the assumption that every semiprime ideal of V is Goldie is used only once, namely to show that the semiprime ideal J is left Goldie. This follows from Theorem 6.6(b) since the semiprime ideal $J^\natural$ was already shown to be left Goldie. ∎

We turn to the proof of Theorem 6.6(c). We need two auxiliary results.

6.8. LEMMA. *Let K be a field, and L a finitely generated separable field extension of K. Let J be a prime ideal of a K-algebra R. Then* $J \otimes_K L$ *is the intersection of finitely many prime ideals of* $R \otimes_K L$.

Proof. Let L_0 be a purely transcendental field extension of K such that L is a finite separable extension of L_0. Replacing L by the normal closure of L over L_0, we may assume that L is Galois over L_0 with finite

Galois group Ω. The ideal $J \otimes_K L_0$ is clearly prime. Let P be an ideal of $R \otimes_K L$ maximal with respect to $P \cap (R \otimes_K L_0) = J \otimes_K L_0$. Then P is a prime ideal. Since $(P : \Omega)$ is Ω-stable, it is generated by its intersection with $R \otimes_K L_0$, which is $J \otimes_K L_0$ (see Lemma 6.3(a)). Hence $J \otimes_K L = (J \otimes_K L_0) \otimes_{L_0} L = (P : \Omega)$. Since Ω is finite, $(P : \Omega)$ is a finite intersection of prime ideals. ∎

6.9. PROPOSITION. *Let H be a closed subgroup of G, and assume that both G and H are connected. Let J be a prime ideal of $V \otimes Q(G/H)$. Then $J \otimes_{Q(G/H)} Q(G)$ is a prime ideal of $V \otimes Q(G)$.*

This result extends Lemma 3.5(a), and in fact reproves that result (in case $H = G$) without using the unirationality of $Q(G^\circ)$ over k. We will see in Example 6.11 that $J \otimes_{Q(G/H)} Q(G)$ may not be prime if H is not connected.

Proof. Set $I = J \otimes_{Q(G/H)} Q(G)$, and $\Omega = (\mathrm{id} \otimes \Delta)(H)$. Since $Q(G/H)$ is the fixed field of $Q(G)$ under the action of a group, namely $\Delta(H)$, $Q(G)$ is separable over $Q(G/H)$, see [Ros, p. 4]. By Lemma 6.8, I is a finite intersection of prime ideals. Say $P_1, \ldots, P_n$ are the prime ideals of $V \otimes Q(G)$ which are minimal over I. Then their intersection is I, and they are permuted by Ω. Since J is prime, it follows from Lemma 6.3(a) that I is Ω-prime. Since I is the finite intersection of the Ω-stable ideals $(P_i : \Omega)$, it follows that $I = (P_i : \Omega)$ for some i, implying that Ω permutes the P_i transitively. Set $p_i = P_i \cap (V \otimes A(G))$, where $A(G)$ denotes the algebra of regular functions on G. Since $V \otimes Q(G)$ is a localization at regular (central) elements of $V \otimes A(G)$, $P_i = p_i Q(G)$. Note that Ω permutes $p_1, \ldots, p_n$ transitively. Since the action of the connected group Ω on $V \otimes A(G)$ is rational, the p_i are Ω-stable, so that $n = 1$. Consequently, $I = P_1$, so that I is prime. ∎

6.10. *Proof of Theorem* 6.6(c). Assume first that J is prime and G connected. Then $J \otimes Q(G)$ is prime by Lemma 3.5(a). Thus $\mu^{-1}(J \otimes Q(G))$ is prime, and so is $J^\natural = \mu^{-1}(J \otimes Q(G)) \cap (V \otimes Q(G/H))$. Conversely, assume that $J^\natural$ is prime, and that both G and H are connected. Set $I = \mu^{-1}(J \otimes Q(G)) = J^\natural \otimes_{Q(G/H)} Q(G)$. By Proposition 6.9, I is prime. Hence $J = \mu(I) \cap V$ is prime. ∎

6.11. EXAMPLE. The conclusions of Theorem 6.6(c) and Proposition 6.9 can be wrong if the connectedness assumptions are not satisfied. To be precise, let J be an ideal of V.

(a) Suppose J is prime. If G is not connected, then $J \otimes Q(G)$ is not prime, and $J^\natural$ may not be prime.

(b) Suppose $J^\natural$ is prime. If H is not connected, then both J and $J^\natural \otimes_{Q(G/H)} Q(G)$ may not be prime (even if G is connected).

(a) Suppose first that J is prime and G not connected. Then $Q(G)$ is not a field but a direct sum of fields, implying that $J \otimes Q(G)$ is not prime. If $H = 1$, then $J^{\natural} = \mu^{-1}(J \otimes Q(G))$, so is also not prime.

(b) Let $G = \mathbb{G}_m$ be the multiplicative group of k, and let $V = k[x]$ be the commutative polynomial ring in one variable. The action β of G on V is given as follows: $a \in G$ sends x to $a^{-1}x$. The algebra of regular functions on G can be represented as $A(G) = k[t, t^{-1}]$, where t is a central indeterminate over k such that $\Gamma(a)t = \Delta(a)t = at$ for all $a \in G$. Then $Q(G) = k(t)$. Note that $V \otimes Q(G) = k[x] \otimes k(t) \cong k(t)[x]$ is a polynomial ring in one variable over a field. Let P be the prime ideal of $V \otimes Q(G)$ defined by $P = (xt + 1) = (x + t^{-1})$. It is $(\beta \otimes \Gamma)(G)$-stable. Let H be any finite, non-trivial subgroup of G. Set $M = P \cap (V \otimes Q(G/H))$. Then M is a $(\beta \otimes \Gamma)(G)$-stable prime ideal of $V \otimes Q(G/H)$. Set $J = M^{\#}$. Then $M = J^{\natural}$. We will show that $M \otimes_{Q(G/H)} Q(G) = \mu(J \otimes Q(G))$ is not prime. It then follows by Lemma 3.5(a) that J is not prime, either.

Let $\Omega = (\mathrm{id} \otimes \Delta)(H)$. Let $1 \neq a \in H$. Then $(\mathrm{id} \otimes \Delta)(a)(xt + 1) = axt + 1$. If P were Ω-stable, then P would contain the non-zero scalar $1 - a = (axt + 1) - a(xt + 1)$, a contradiction. Consequently, $(P : \Omega)$ is strictly contained in P, and therefore not prime (since Ω is finite). Since it is Ω-stable, it is by Lemma 6.3(a) generated by its intersection with $V \otimes Q(G/H)$, which is M. So $M \otimes_{Q(G/H)} Q(G) = (P : \Omega)$ is not prime.

7. THE MAPS ν_J

In this section, the technical heart of this paper, we prove several of the main results: Theorem 2.2, which asserts that G acts transitively on each stratum in $\mathrm{Rat}(V)$; Theorem 2.3, which states that rational ideals are maximal in their strata in $\mathrm{Spec}(V)$; and the more technical Theorem 2.12. At the end of the section, in Subsection 7.12 and Diagram 7.13, we summarize much of the information obtained, showing how V/J and $V/(J : G)$ are related for a rational ideal J of V. This is the starting point for our investigations in the later sections.

Our approach is based on the work of Moeglin and Rentschler. In constructing the maps ν_J below, we follow to some extent the strategy in the proof of "(ii) ⇒ (i)" in [MR$_4$, Théorème 2], but working in arbitrary characteristic, and not assuming that all semiprime ideals of V are Goldie. We then use some of the ideas in [MR$_1$, Sect. 2]. We also heavily use the results we obtained in the earlier sections.

7.1. *Throughout this section*, we assume that J is a prime left Goldie ideal with $(J : G) = 0$. Then V is G-prime, and left Goldie by Theorem 2.1. Let S be the set of regular elements of V. By assumption, S is a left Ore set, and $Q(V) = S^{-1}V$. We denote by C the center of $Q(V)$, and by $\mathrm{Hom}_G(C, Q(G))$ the k-algebra homomorphisms from C to $Q(G)$ intertwining the actions β and Γ of G. *From Subsection* 7.8 *on*, we will additionally assume that J is rational. Then V will be G-rational by Theorem 2.10(a).

7.2. *The Map* φ_J. We denote by φ_J the composition of the following algebra homomorphisms:

$$\varphi_J\colon V \otimes Q(G) \xrightarrow{\mu} V \otimes Q(G) \twoheadrightarrow V/J \otimes Q(G).$$

Here the second map is the canonical projection.

Since V/J is prime and left Goldie, $V/J \otimes Q(G)$ is semiprime left Goldie by Proposition 6.5(a), applied with $L = Q(G)$ and $\Omega = \Delta(G)$. Hence $Q(V/J \otimes Q(G))$ exists, and $V/J \otimes Q(G) \subseteq Q(V/J \otimes Q(G))$. We therefore sometimes treat φ_J as a map from $V \otimes Q(G)$ into $Q(V/J \otimes Q(G))$.

We summarize some of the properties of φ_J in the following lemma.

LEMMA. *Let H be a closed subgroup of G stabilizing J.*

(a) φ_J *intertwines the actions* $\beta \otimes \Gamma$ *and* id $\otimes\ \Gamma$ *of* G.

(b) φ_J *intertwines the actions* id $\otimes\ \Delta$ *and* $\beta \otimes \Delta$ *of* H.

(c) $\ker \varphi_J = \mu^{-1}(J \otimes Q(G))$

(d) $\ker(\varphi_J|_{V \otimes Q(G/H)}) = J^{\natural}$

(e) $\varphi_J|_V$ *is injective.*

Proof. Parts (a) and (b) follow from the first and second intertwining property, respectively, see [MR_3, 0.4]. Part (c) is clear and implies (d) by the definition of $J^{\natural}$. Part (e) follows from (d) since $J^{\natural} \cap V = (J : G) = 0$ by [MR_3, I.5], see Proposition 3.3. ∎

7.3. *The Map* $\tilde{\varphi}_J$. The left Ore set S of regular elements of V is also a left Ore set of regular elements of $V \otimes Q(G)$, and $S^{-1}(V \otimes Q(G)) = Q(V) \otimes Q(G)$; cf. the proof of Proposition 6.5(a).

LEMMA. *The set* $\varphi_J(S)$ *consists of regular elements of* $V/J \otimes Q(G)$. *Hence* φ_J *extends to a map*

$$\tilde{\varphi}_J\colon Q(V) \otimes Q(G) \to Q(V/J \otimes Q(G)),$$

which intertwines the actions $\beta \otimes \Gamma$ *and* id $\otimes\ \Gamma$ *of* G. *Moreover, the restriction of* $\tilde{\varphi}_J$ *to* $Q(V)$ *is injective.*

Proof. This is shown in the proof of [MR_4, Théorème 2]; we reproduce the argument for the reader's convenience. Set $T = \varphi_J(S)$, and

$$X = \{x \in V/J \otimes Q(G) \mid tx = 0 \text{ for some } t \in T\}.$$

We first show that X is an ideal of $V/J \otimes Q(G)$. Because $T = \varphi_J(S)$ satisfies the left Ore condition in $\varphi_J(V)$, X is closed under addition (so that it is a right ideal in $V/J \otimes Q(G)$), and also closed under left multiplication by elements in $\varphi_J(V)$. Since $V/J \otimes Q(G) = \varphi_J(V)Q(G)$, X is thus also a left ideal and hence a two-sided ideal.

Suppose X is non-zero. From the fact that φ_J intertwines the actions $\beta \otimes \Gamma$ and $\mathrm{id} \otimes \Gamma$, it follows that T and hence X are $(\mathrm{id} \otimes \Gamma)(G)$-stable. By Lemma 6.3(a) with $\Omega = \Gamma(G)$, X is generated by its intersection with V/J. Since V/J is a prime left Goldie ring, its non-zero ideal $X \cap (V/J)$ contains a regular element x. Since regular elements of V/J remain regular in $V/J \otimes Q(G)$, and since the elements of T are non-zero (as $\varphi_J|_V$ is injective), this is a contradiction. Hence $X = 0$, and the elements of T are right regular in $V/J \otimes Q(G)$ (see [McR, 2.1.2]). Since S is left Ore in $V \otimes Q(G)$, and since φ_J maps $V \otimes Q(G)$ onto $V/J \otimes Q(G)$, $T = \varphi_J(S)$ is left Ore in $V/J \otimes Q(G)$. Using this, one sees that the elements of T are right regular in the ring of left fractions $Q(V/J \otimes Q(G))$. Since the latter algebra is semisimple Artinian, the elements of T are regular in $V/J \otimes Q(G)$. This proves the existence of $\tilde{\varphi}_J$.

It is clear that $\tilde{\varphi}_J$ intertwines the actions $\beta \otimes \Gamma$ and $\mathrm{id} \otimes \Gamma$ of G. Finally, since V is G-prime, the prime ideals of the semisimple Artinian algebra $Q(V)$ are permuted transitively by the action β of G, so their intersection is zero. Hence any proper $\beta(G)$-stable ideal of $Q(V)$ is zero. This applies in particular to the kernel of $\tilde{\varphi}_J|_{Q(V)}$. ∎

7.4. LEMMA. *If J is rational, then $\tilde{\varphi}_J(C)$ is contained in $Q(G)$, which we identify with its natural image in $Q(V/J \otimes Q(G))$.*

Proof. Suppose that J is rational. Then $Q(V/J) \otimes Q(G)$ is a finite direct sum of simple rings, and has center $Q(G)$. Consequently, $Q(V/J \otimes Q(G)) = Q(Q(V/J) \otimes Q(G))$ has center $Q(G)$ (see [McR, 2.1.16]). Since $V/J \otimes Q(G) = \varphi_J(V)Q(G)$, $\tilde{\varphi}_J(C)$ is central in $Q(V/J \otimes Q(G))$ and thus contained in $Q(G)$. ∎

7.5. *The Map ν_J.* Suppose that $\tilde{\varphi}_J(C) \subseteq Q(G)$. (By Lemma 7.4, this is in particular satisfied if J is rational.) Then restricting $\tilde{\varphi}_J$ to C gives rise to a map $\nu_J\colon C \to Q(G)$. By Subsection 7.3, ν_J is injective and intertwines the actions β and Γ of G. So $\nu_J \in \mathrm{Hom}_G(C, Q(G))$.

7.6. LEMMA. *Let J_1 and J_2 be two distinct prime Goldie ideals of V with $(J_i : G) = 0$. Suppose that both $\tilde{\varphi}_1(C)$ and $\tilde{\varphi}_2(C)$ are contained in $Q(G)$. (This is, for example, satisfied if J_1 and J_2 are both rational.) Then $\nu_{J_1} \neq \nu_{J_2}$.*

Proof. Let H be the trivial subgroup of G. Then $\ker(\varphi_{J_1}) = J_1^\natural \neq J_2^\natural = \ker(\varphi_{J_2})$. Denote the kernel of $\tilde{\varphi}_{J_i}$ by M_i. So the M_i are ideals of $Q(V) \otimes Q(G)$. Since φ_{J_i} is the restriction of $\tilde{\varphi}_{J_i}$ to $V \otimes Q(G)$, it follows that $M_1 \neq M_2$. Since $Q(V) \otimes Q(G) \cong Q(V) \otimes_C (C \otimes Q(G))$, Lemma 3.9 implies that $M_1 \cap (C \otimes Q(G)) \neq M_2 \cap (C \otimes Q(G))$. Hence the restrictions of $\tilde{\varphi}_{J_i}$ to $C \otimes Q(G)$ have distinct kernels. But $\tilde{\varphi}_{J_i}|_{C \otimes Q(G)} = \nu_{J_i} \otimes \mathrm{id}$. Hence $\nu_{J_1} \neq \nu_{J_2}$. ∎

We are now able to prove Theorem 2.3.

7.7. *Proof of Theorem* 2.3. Factoring out by $(J : G)$, we may assume that $(J : G) = 0$. So now the hypotheses in Subsection 7.1 are satisfied. By Lemma 3.4(a), we may assume that P is maximal with respect to $(P : G) = 0$. By Corollary 5.5, P is prime left Goldie. So we can construct φ_P and $\tilde{\varphi}_P$. We denote by C the center of $Q(V)$. By Lemma 7.4, $\tilde{\varphi}_J(C) \subseteq Q(G)$, so that ν_J exists. We will show that $\tilde{\varphi}_P(C) \subseteq Q(G)$, so that ν_P exists, and that $\nu_J = \nu_P$. Then $J = P$ by Lemma 7.6.

Let $x \in C$. Then $x = s^{-1}v$ for some $v \in V$ and s in S. Denote by π the natural $Q(G)$-linear projection from $V/J \otimes Q(G)$ onto $V/P \otimes Q(G)$. Then $\varphi_P = \pi \circ \varphi_J$. From $v = sx$, we obtain $\varphi_J(v) = \varphi_J(s)\tilde{\varphi}_J(x)$. Using the fact that π is $Q(G)$-linear, it follows that $\varphi_P(v) = \pi(\varphi_J(v)) = \pi(\varphi_J(s)\tilde{\varphi}_J(x)) = \pi(\varphi_J(s))\tilde{\varphi}_J(x) = \varphi_P(s)\tilde{\varphi}_J(x)$. Hence $\tilde{\varphi}_P(x) = \varphi_P(s)^{-1}\varphi_P(v) = \tilde{\varphi}_J(x) = \nu_J(x)$. This shows firstly that $\tilde{\varphi}_P(C) = \nu_J(C) \subseteq Q(G)$, and then that $\nu_P = (\tilde{\varphi}_P)|_C = \nu_J$. ∎

7.8. *Until the end of this section*, we assume in addition to the hypotheses in Subsection 7.1 that V is G-rational, and that J is rational (cf. Theorem 2.10). It follows in particular that $\tilde{\varphi}_J$ induces the map $\nu_J : C \to Q(G)$.

7.9. Lemma. *For any* $\gamma \in G$, $\nu_{\beta(\gamma)J} = \Delta(\gamma) \circ \nu_J$.

Proof. The second intertwining property states that $\mu \circ (\mathrm{id} \otimes \Delta)(\gamma) = (\beta \otimes \Delta)(\gamma) \circ \mu$. Moreover, $(\beta \otimes \Delta)(\gamma)(J \otimes Q(G)) = (\beta(\gamma)J) \otimes Q(G)$. Hence $\varphi_{\beta(\gamma)J} \circ (\mathrm{id} \otimes \Delta)(\gamma) = (\beta \otimes \Delta)(\gamma) \circ \varphi_J$. Consequently, also

$$\tilde{\varphi}_{\beta(\gamma)J} \circ (\mathrm{id} \otimes \Delta)(\gamma) = (\beta \otimes \Delta)(\gamma) \circ \tilde{\varphi}_J. \qquad (*)$$

Note that the restriction to C of $(\mathrm{id} \otimes \Delta)(\gamma)$ is the identity, and that the restriction of $(\beta \otimes \Delta)(\gamma)$ to $\tilde{\varphi}_J(C) = \nu_J(C)$ is $\Delta(\gamma)$. So restricting the maps in $(*)$ to C, we obtain $\nu_{\beta(\gamma)J} = \Delta(\gamma) \circ \nu_J$. ∎

7.10. Theorem. *Under the hypotheses in Subsections* 7.1 *and* 7.8, *the map*

$$\nu : (G\text{-stratum of } J \text{ in } \mathrm{Rat}\, V) \to \mathrm{Hom}_G(C, Q(G))$$

given by $P \mapsto \nu_P$ *is a* G*-equivariant bijection. In particular, the action of* G *on the* G*-stratum of* J *in* Rat V *is transitive*.

Here $\gamma \in G$ acts on the G-stratum of J via $\beta(\gamma)$, and it acts on $\mathrm{Hom}_G(C, Q(G))$ by composition with $\Delta(\gamma)$. Note that Theorem 2.2 is an immediate consequence of this result. Theorem 2.12, which we will prove shortly, gives an explicit inverse for the bijection $P \mapsto \nu_P$. For Noetherian rings in characteristic zero, Theorem 7.10 is due to Moeglin and Rentschler [MR_1, 2.9].

Proof. By Lemma 7.6, ν is injective. By Lemma 7.9, ν is G-equivariant. By Theorem 4.7, the action on $\text{Hom}_G(C, Q(G))$ is transitive. Hence ν is onto, and also the action on the G-stratum of J in $\text{Rat}\, V$ is transitive. ∎

7.11. *Proof of Theorem* 2.12. Factoring out by $I = (J : G)$, we may assume that $I = (J : G) = 0$. So now the hypotheses in Subsections 7.1 and 7.8 are satisfied. It follows from Theorem 7.10 that $\gamma \in G$ belongs to the stabilizer of J iff $\nu_J = \Delta(\gamma) \circ \nu_J$ iff $\Delta(\gamma) f = f$ for all $f \in \nu_J(C)$. Hence the stabilizer of J is

$$H = \{\gamma \in G | \Delta(\gamma) f = f \text{ for all } f \in \nu_J(C)\}.$$

By Proposition 4.5, H is the unique closed subgroup of G such that $Q(G/H)$ is a finite purely inseparable extension of $\nu_J(C)$ (cf. Definition 4.1). It remains to show that $J = \pi^{-1}(\pi(V) m_{(H)})$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} V \otimes Q(G/H) & \xrightarrow{\pi'} & B & \overset{\text{def}}{=\!=} & Q(V) \otimes_C Q(G/H) \\ & \searrow^{\varphi_J|_{V \otimes Q(G/H)}} & & & \Big\downarrow \alpha \overset{\text{def}}{=\!=} (\tilde{\varphi}_J|_{Q(V)}) \otimes_C \text{id} \\ & & & & Q(V/J \otimes Q(G)) \end{array}$$

Here $Q(G/H)$ is a C-module via the map ν_J, and π' is the $Q(G/H)$-linear extension of the natural map $\pi \colon V \to B = Q(V) \otimes_C Q(G/H)$. We show first that α is injective. Write $Q(V) = \oplus R_i$ where the R_i are simple rings with centers C_i. Write $Q(G/H) = \oplus K_i$, where each K_i is a (purely inseparable) field extension of $\nu_J(C_i)$. Recall from Subsection 6.1 that $Q(V) \otimes_C Q(G/H) \cong \oplus_i (R_i \otimes_{C_i} K_i)$. Each $R_i \otimes_{C_i} K_i$ is a simple ring and thus not contained in the kernel of α, since the restriction of $\tilde{\varphi}_J$ to $R_i \subseteq Q(V)$ is injective. Since every ideal of the semisimple Artinian algebra $Q(V) \otimes_C Q(G/H)$ contains one of the $R_i \otimes_{C_i} K_i$, it follows that α is injective. This fact together with Lemma 7.2(d) implies that $\ker(\pi') = \ker(\varphi_j|_{V \otimes Q(G/H)}) = J^{\natural}$.

Consider the map $\pi' : V \otimes Q(G/H) \to B$. Set $\mathscr{M} \overset{\text{def}}{=} \ker(\pi') = J^{\natural}$, so that $J = \mathscr{M}^{\#}$. Then by [MR$_3$, I.5(ii)],

$$J = \mathscr{M}^{\#} = V \cap \left(\left[\mathscr{M} \cap \left(V \otimes A_{(H)} \right) \right] + \left(V \otimes m_{(H)} \right) \right),$$

where $A_{(H)} \overset{\text{def}}{=} \mathscr{O}_{(H/H), G/H}$ (which is contained in $Q(G/H)$, see Subsection 3.1). It follows easily that $J = \pi^{-1}(\pi(V) m_{(H)})$: write $\pi' = g \circ f$ where f: $V \to V \otimes A_{(H)}$ and g: $V \otimes A_{(H)} \to Q(V) \otimes_C Q(G/H)$ are the obvious maps. Then $\ker(g) = \mathscr{M} \cap (V \otimes A_{(H)})$, and $g(V \otimes m_{(H)}) = \pi(V) m_{(H)}$. Consequently, $[\mathscr{M} \cap (V \otimes A_{(H)})] + (V \otimes m_{(H)}) = g^{-1}(\pi(V) m_{(H)})$, implying that $J = f^{-1}(g^{-1}(\pi(V) m_{(H)})) = \pi^{-1}(\pi(V) m_{(H)})$, as desired. ∎

7.12. *Diagram* 7.13 *summarizes the relationship between the algebras* V/J *and* $V/(J:G)$ *via a series of central ring extensions*. This relationship was discovered and heavily exploited by Moeglin and Rentschler (though extension (3) did not occur in their work in characteristic zero). We will refer to this diagram repeatedly. For convenience, we summarize the current hypotheses: J is a rational ideal of V, and H is the stabilizer of J in G. We assume that $(J:G) = 0$ (so that $V = V/(J:G)$), and denote by C the center of $Q(V) = Q(V/(J:G))$. In the last two rows, it is indicated how G and H act on each column of extensions, and how μ^{-1} and f intertwine these actions.

Diagram 7.13. Relating V/J and $V/(J:G)$ via central ring extensions:

$$\begin{array}{ccccc}
(V/J) \otimes Q(G) & \xrightarrow[\sim]{\mu^{-1}} & T \otimes_{Q(G/H)} Q(G) = (V \otimes Q(G))/\mu^{-1}(J \otimes Q(G)) & & \\
\Big| & & \Big| (2) & & \\
\Big| (1) & & T \overset{\text{def}}{=\!=\!=} (V \otimes Q(G/H))/J^{\natural} & \xrightarrow[\sim]{f} & VC \otimes_C Q(G/H) \\
\Big| & & & & \Big| (3) \\
V/J & & & & VC \\
 & & & & \Big| (4) \\
 & & & & V = V/(J:G) \\
 & & & & \\
(\mathrm{id} \otimes \Gamma)(G) & \leftrightsquigarrow & (\beta \otimes \Gamma)(G) & \leftrightsquigarrow & (\beta \otimes \Gamma)(G) \\
(\beta \otimes \Delta)(H) & \leftrightsquigarrow & (\mathrm{id} \otimes \Delta)(H) & &
\end{array}$$

In the diagram, all (vertical) ring extensions are central (i.e., the larger algebra is generated over the smaller by its center), and all but extension (4) are given by tensoring.

Extensions (1) and (2) are given by tensoring with $Q(G)$ over k and over $Q(G/H)$, respectively. Note that if G is connected, then $Q(G)$ is a field; moreover, $Q(G)$ is in this case unirational over k (though not necessarily over the intermediate field $Q(G/H)$).

Extension (3) is a finite central extension. Here $Q(G/H)$ is a C-module via ν_J. By Theorem 2.12, $Q(G/H)$ is a finite purely inseparable extension of $\nu_J(C)$ (cf. Definition 4.1). This extension is of course trivial in characteristic zero, but can complicate the situation in prime characteristic; see Example 9.1 and Remark 9.2.

Extension (4) behaves similar to a central localization: VC is the subalgebra of $Q(V)$ generated by V and by the center C of $Q(V)$.

The First Row. The isomorphism in the first row is induced by μ^{-1}. Since $J \otimes Q(G)$ is $(\beta \otimes \Delta)(H)$-stable, the second intertwining property implies that $\mu^{-1}(J \otimes Q(G))$ is $(\mathrm{id} \otimes \Delta)(H)$-stable. So by Lemma 6.3(a), $\mu^{-1}(J \otimes Q(G)) = J^{\natural} \otimes_{Q(G/H)} Q(G)$. Hence

$$(V \otimes Q(G))/\mu^{-1}(J \otimes Q(G)) = T \otimes_{Q(G/H)} Q(G).$$

The Second Row. The isomorphism f sends the class of a tensor in the algebra $T = (V \otimes Q(G/H))/J^{\natural}$ to the corresponding tensor in $VC \otimes_C Q(G/H)$. To be precise, given $v \in V$ and $a \in Q(G/H)$, the map f sends $(v \otimes_k a) + J^{\natural}$ to $v \otimes_C a$. We now show that this map is well-defined and an isomorphism.

Denote by $\overline{\varphi}_J$ the embedding of $T = (V \otimes Q(G/H))/J^{\natural}$ into $Q(V/J \otimes Q(G))$ induced by φ_J (see Lemma 7.2(d)). We saw in the proof of Theorem 2.12 (see 7.11), that the map $\alpha = (\tilde{\varphi}_J|_{Q(V)}) \otimes_C \mathrm{id}$ from $Q(V) \otimes_C Q(G/H)$ to $Q(V/J \otimes Q(G))$ is injective. Hence so is its restriction α_0 to $VC \otimes_C Q(G/H)$. So we have the following $Q(G/H)$-linear embeddings:

$$T = (V \otimes Q(G/H))/J^{\natural} \xrightarrow{\overline{\varphi}_J} Q(V/J \otimes Q(G)) \xleftarrow{\alpha_0} VC \otimes_C Q(G/H).$$

Both $\overline{\varphi}_J$ and $\tilde{\varphi}_J$ are induced by μ. Hence for $v \in V$, $\overline{\varphi}_J(v \otimes 1) = \tilde{\varphi}_J(v \otimes 1) = \alpha_0(v \otimes 1)$. Since $\overline{\varphi}_J$ and α_0 are both $Q(G/H)$-linear, they have the same image. Hence T is isomorphic to $VC \otimes_C Q(G/H)$ via the map $g = \alpha_0^{-1} \circ \overline{\varphi}_J$. This map is clearly $Q(G/H)$-linear, and for $v \in V$, $g(v \otimes 1) = v \otimes 1$. Hence $g = f$, and f is a well-defined isomorphism.

The Last Two Rows. It follows from the intertwining properties of μ that μ^{-1} intertwines the indicated actions. Note that $(\mathrm{id} \otimes \Delta)(H)$ is trivial on T. Since the actions of G induced by Γ and Δ commute, T is $(\beta \otimes \Gamma)(G)$-stable. Since ν_J intertwines the actions β and Γ, G acts on $VC \otimes_C Q(G/H)$ via $\beta \otimes \Gamma$. It is clear from the description of f above that f is $(\beta \otimes \Gamma)(G)$-equivariant. ∎

8. LOCALLY CLOSED RATIONAL IDEALS

In this section we study locally closed rational ideals. Note that some results are proved under special assumptions; see Subsections 8.5 and 8.10. The main result is Theorem 2.6, which says that a rational ideal is locally closed if and only if its orb is G-locally closed. One direction is an easy consequence of the fact that rational ideals are maximal in their strata in $\operatorname{Spec}(V)$ (Theorem 2.3). We present the proof of this direction first. For the proof of the other direction, we follow the strategy in [MR_1, Sect. 3]. More care is needed because of the greater generality in which we are working, especially because we do not assume that V is Noetherian. After we complete the proof of Theorem 2.6 (see 8.13), we deduce some consequences. In particular, we prove Theorem 2.5, which is a special case of Theorem 8.15. At the end of the section, we investigate how the correspondence $\natural$ behaves with respect to locally closed ideals.

8.1. *Proof of "(b) ⇒ (a)" in Theorem* 2.6. We assume that J is a rational ideal such that $(J:G)$ is G-locally closed. We show that J is locally closed. Suppose to the contrary that $J = \bigcap_{P \in \mathscr{M}} P$, where $\mathscr{M}$ denotes the set of all prime ideals strictly containing J. Then the G-stable ideal $\bigcap_{P \in \mathscr{M}} (P:G)$ is contained in J and contains $(J:G)$, so is equal to $(J:G)$. By Theorem 2.3, J is maximal in its stratum in $\operatorname{Spec}(V)$. That is, $(J:G) \subsetneq (P:G)$ for all $P \in \mathscr{M}$. Hence $(J:G)$ is the intersection of certain G-stable semiprime ideals all of which strictly contain $(J:G)$, a contradiction to the assumption that $(J:G)$ is G-locally closed. ∎

We now turn to the proof of the other implication in Theorem 2.6.

8.2. LEMMA. *Let J be a semiprime ideal of V, and let H be a closed subgroup of G stabilizing J. Then J is a $\beta(H)$-locally closed ideal of V if and only if $J^\natural$ is a $(\beta \otimes \Gamma)(G)$-locally closed ideal of $V \otimes Q(G/H)$.*

Proof. The bijection $\natural$ between the $\beta(H)$-stable ideals of V and the $(\beta \otimes \Gamma)(G)$-stable ideals of $V \otimes Q(G/H)$ is inclusion preserving, and maps semiprime ideals to semiprime ideals by Theorem 6.6(a). So N is a $\beta(H)$-stable semiprime ideal of V strictly containing J and contained in all $\beta(H)$-stable semiprime ideals of V which strictly contain J, if and only if $N^\natural$ is a $(\beta \otimes \Gamma)(G)$-stable semiprime ideal of $V \otimes Q(G/H)$ strictly containing $J^\natural$ and contained in all $(\beta \otimes \Gamma)(G)$-stable semiprime ideals of $V \otimes Q(G/H)$ which strictly contain $J^\natural$. This proves the lemma. ∎

8.3. LEMMA. *Let J be a locally closed rational ideal of V, and let H be a closed subgroup of G stabilizing J. Then $J^\natural$ is a $(\beta \otimes \Gamma)(G)$-locally closed ideal of $V \otimes Q(G/H)$.*

Proof. We first prove that if Ω is a group acting on a ring R, and if the zero ideal of R is locally closed, then it is also Ω-locally closed. To see this, let Q be a non-zero Ω-stable semiprime ideal of R. Then Q is the intersection of non-zero prime ideals of R. Hence Q contains the non-zero intersection of all non-zero prime ideals of R. Thus the zero ideal of R is

Ω-locally closed. It follows from this fact that J is $\beta(H)$-locally closed. Hence by Lemma 8.2, $J^\natural$ is a $(\beta \otimes \Gamma)(G)$-locally closed ideal of $V \otimes Q(G/H)$. ▮

8.4. LEMMA. *For the proof of the implication* (a) $\Rightarrow$ (b) *in Theorem* 2.6, *we may assume that* $(J : G) = 0$ *and that* G *is connected.*

Proof. We may clearly assume that $(J : G) = 0$. Since V is G-rational, the finitely many minimal prime ideals of the semiprime left Goldie ring V are permuted transitively by G, so they are G°-stable as G acts rationally. One of them is $P = (J : G^\circ)$. It suffices then to show that $0 = (J : G)$ is G-locally closed if $P = (J : G^\circ)$ is G°-locally closed.

Let I be a G°-stable ideal of V strictly containing P. We show first that then $(I : G) \neq 0$. Note that $(I : G)$ is the intersection of the finitely many G-conjugates of I. If $(I : G)$ were zero, the prime ideal P would contain $\gamma(I)$ for some $\gamma \in G$. Thus I would strictly contain $\gamma^n(I)$ for all positive integers n. This is impossible as G/G° is a finite group and I is G°-stable. Thus $(I : G) \neq 0$.

Denote by Q the intersection of all G°-stable semiprime ideals of V containing P strictly. We are assuming that Q contains P strictly. As we have seen, this implies that $(Q : G) \neq 0$. Now let I be a non-zero G-stable semiprime ideal of V. In order to show that the zero ideal is G-locally closed, it suffices to show that I contains $(Q : G)$. As every G-stable semiprime ideal is an intersection of G-prime ideals, we may assume that I is G-prime. So there is a prime ideal M such that $I = (M : G)$. As M contains some minimal prime ideal of V, and as the minimal prime ideals of V are permuted transitively by G, we may assume that M contains P. Then $(M : G^\circ)$ contains P. Suppose $(M : G^\circ) = P$. Then $I = (M : G) = (P : G) = 0$, a contradiction. So $(M : G^\circ)$ contains P strictly and contains hence Q. Therefore $I = (M : G)$ contains $(Q : G)$. ▮

Lemma 8.4 allows us to make the following assumptions.

8.5. Until 8.13, the end of the proof of the implication "(a) $\Rightarrow$ (b)" in Theorem 2.6, we assume that the linear algebraic group G is *connected*. Moreover, we assume that V is a G-rational algebra, and that J is a *locally closed* rational ideal with $(J : G) = 0$. We denote by H the stabilizer of J in G. It is a closed subgroup of G. Since G is connected, $(J : G) = 0$ is a prime ideal, so that V is a prime ring. We denote by S the set of regular elements of V, and by C the center of $Q(V)$. We will make additional assumptions in Subsection 8.10.

8.6. LEMMA. *Let* $A \subseteq B$ *be a finite centralizing extension of prime rings. Let* Ω *be a group acting on* B *such that* A *is* Ω*-stable. Then the zero ideal of* A *is* Ω*-locally closed if and only if the zero ideal of* B *is* Ω*-locally closed.*

Proof. Denote by A_0 and B_0 the intersections of all non-zero Ω-stable semiprime ideals of A and B, respectively. We have to show that $A_0 \neq 0$ iff $B_0 \neq 0$.

Suppose first that $A_0 \neq 0$. Let Q be any non-zero Ω-stable semiprime ideal of B, and set $q = Q \cap A$. Then q is clearly an Ω-stable semiprime ideal of A. By Bergman's incomparability theorem for finite centralizing extensions of prime rings (see [RS, 4.5]), every non-zero ideal of B has non-zero intersection with A. Hence $q \neq 0$, so it contains A_0. Thus Q contains A_0. Consequently also B_0 contains A_0 and is therefore non-zero.

Suppose now that $B_0 \neq 0$. Let q be a non-zero, Ω-stable semiprime ideal of A. Let p be a prime ideal of A containing q. By Bergman's lying over theorem for finite centralizing extensions (see [RS, 4.1]), there is a prime ideal P of B lying over p. That is, $P \cap A = p$. Since $(p : \Omega)$ contains $q \neq 0$, $(P : \Omega)$ is a non-zero Ω-stable semiprime ideal of B. Hence P contains B_0. It follows from Bergman's incomparability theorem that $B_0 \cap A \neq 0$. Hence p contains the non-zero ideal $B_0 \cap A$ of A. As this is true for all prime ideals containing the semiprime ideal q, also q contains $B_0 \cap A$. Consequently, A_0 contains $B_0 \cap A$ and is therefore non-zero. (Note that we proved that $B_0 \supseteq A_0 \supseteq B_0 \cap A$; consequently, $A_0 = B_0 \cap A$.) ∎

8.7. PROPOSITION. *The zero ideal of the algebra VC is G-locally closed.*

Proof. Recall Diagram 7.13. We are assuming that J is locally closed. By Lemma 8.3, the zero ideal of $T = (V \otimes Q(G/H))/J^{\natural}$ is $(\beta \otimes \Gamma)(G)$-locally closed. Since $T \cong VC \otimes_C Q(G/H)$, the proposition follows now from Lemma 8.6, applied to the ring extension $VC \subseteq VC \otimes_C Q(G/H)$ and the action of the group $\Omega = (\beta \otimes \Gamma)(G)$. ∎

Having established that the zero ideal of VC is G-locally closed, we now show that the zero ideal of V is also G-locally closed. We follow the strategy in [MR_1, Sect. 3]; Lemmas 8.8 and 8.9, Proposition 8.12, and Subsection 8.13 correspond more or less to [MR_1, 3.4, 3.5, 3.6, and 3.8], respectively. More care is needed in our context because of the greater generality in which we are working, especially because we do not assume that V is Noetherian.

8.8. LEMMA. *Let B be a k-subalgebra of C which is G-stable. Then the zero ideal of B is G-locally closed.*

Proof. Using the G-equivariant embedding ν_J of C into $Q(G)$ (see Subsection 7.5), it suffices to prove the following: if B is a $\Gamma(G)$-stable subalgebra of $Q(G)$, then the zero ideal of B is $\Gamma(G)$-locally closed.

Consider the finitely generated B-algebra $B \cdot A(G)$. It is a domain since $Q(G)$ is a field. By generic flatness (see [D, 2.6.3]), there is some non-zero $t \in B$ such that $B_t A(G)$ is a free B_t-module; here $B_t = B[t^{-1}]$. We will show that every non-zero $\Gamma(G)$-stable semiprime ideal I of B contains t.

Suppose that I is a $\Gamma(G)$-stable semiprime ideal of B which does not contain t. Then there is some prime ideal of B containing I which does not contain t. Consequently, $I_t \neq B_t$. Then $I_t A(G)$ is a proper ideal of $B_t A(G)$, and so $IA(G) \cap A(G)$ is a proper ideal of $A(G)$. Since $IA(G)$ is $\Gamma(G)$-stable, and since $\Gamma(G)$ permutes the maximal ideals of $A(G)$ transitively, it follows that $IA(G) \cap A(G) = 0$. Hence $I = 0$: suppose not. Then there is a non-zero $x \in I \subseteq Q(G)$. Hence for some non-zero $f \in A(G)$, $xf \in A(G)$, so $xf \in IA(G) \cap A(G) = 0$, a contradiction since $Q(G)$ is a field. ∎

8.9. Lemma. *There is a finite subset F of C such that the subalgebra B of C generated by $\bigcup_{\gamma \in G} \gamma(F)$ has the following properties*: $Q(B) = C$, *and the only proper G-stable ideal of B is the zero ideal.*

Proof. Let F' be a finite set of generators of C as a field extension of k, and let $B' = k[\gamma(F') | \gamma \in G]$. Applying Lemma 8.8 to B', there is a non-zero element $t \in B'$ such that t is contained in every non-zero G-stable semiprime ideal of B'. Let $F = F' \cup \{t^{-1}\}$, and let $B = k[\gamma(F) | \gamma \in G] = B'[\gamma(t)^{-1} | \gamma \in G]$. Then $Q(B) = C$. Let I be a proper G-stable semiprime ideal of B. Then $I \cap B'$ is a proper, G-stable semiprime ideal of B' not containing t. Hence $I \cap B' = 0$, so that $I = 0$. Finally, if I is any proper G-stable ideal of B, then the radical of I is a G-stable semiprime ideal and thus zero. Hence I is zero. ∎

8.10. Until 8.13, the end of the proof of the implication "(a) ⇒ (b)" in Theorem 2.6, we make the following assumptions in addition to the hypotheses in Subsection 8.5. We choose F and B as in Lemma 8.9, and set

$$d = \{x \in V \mid xF \subseteq V\} \qquad \text{and} \qquad p_C = \sum_{\gamma \in G} \gamma(d).$$

Since F is a subset of the center C of $Q(V)$, d is an ideal of V. Since F is a finite subset of $Q(V)$, d is non-zero. Hence p_C is a non-zero G-stable ideal of V.

8.11. Lemma. *Let p be a prime ideal of V which is G-stable and does not contain p_C. Let $w_1, \ldots, w_n$ be finitely many elements in VB. Then there is a non-zero ideal I of V which is not contained in p with the property that $w_j I \subseteq V$ for all j. Moreover, if all w_j belong to pB, then we can find such an I such that $w_j I \subseteq p$ for all j.*

Proof. There is a finite subset G' of G such that every w_j is a finite sum of terms of the form $vb_1 \cdots b_r$ with $v \in V$ and $b_j \in G'(F) = \{\gamma(f) | \gamma \in G', f \in F\}$. If all $w_j \in pB$, we can assume here that $v \in p$. Set $D = \bigcap_{\gamma \in G'} \gamma(d)$. Then $D \neq 0$ since V is a prime ring and the intersection is finite. Suppose D is contained in p. Then $\gamma(d) \subseteq p$ for some $\gamma \in G'$. But then $p_C = \sum_{\delta \in G} \delta\gamma(d)$ is contained in the G-stable ideal p, a contradiction. Thus D is not contained in p. Since p is a prime ideal, p also does not contain any power of D. Since the b_j are central in VB, it follows that

$$(vb_1 \cdots b_r)D^r = v(b_1 D) \cdots (b_r D)$$

is a subset of V (or even of p if $v \in p$). So for some large m and all j, $w_j D^m$ is a subset of V (or even of p if $w_j \in pB$). Set $I = D^m$. ∎

8.12. PROPOSITION. *Let p be a prime ideal of V which is G-stable and does not contain p_C. Then there is a G-stable prime ideal of VC lying over p.*

Proof. We show first that $p = pB \cap V$. Let $w \in pB \cap V$. By Lemma 8.11, there is a non-zero ideal I_0 of V not contained in p such that $wI_0 \subseteq p$. Since p is prime and I_0 is not contained in p, it follows that $w \in p$. Hence $p = pB \cap V$.

Now let P' be an ideal of VB maximal with respect to the property that $P' \cap V = p$. Then P' is a prime ideal of VB. Let $P = (P' : G)$. Then P is a G-stable ideal of VB. Since p is G-stable and contained in P', it follows that $P \cap V = p$. We show now that P is a prime ideal of VB. Note that this does not immediately follow from the fact that G is connected, since G does not necessarily act rationally on VB.

Let $w_1, w_2 \in VB$ such that $w_1 VB w_2 \subseteq P$. By Lemma 8.11, there is an ideal I of V not contained in p such that $w_1 I, w_2 I \subseteq V$. Then $w_1 I V w_2 I$ is contained both in V and in P, and thus in $P \cap V = p$. As p is prime, one of the $w_j I$ is a subset of p. Say $w_1 I \subseteq p$. Let Q be any prime ideal of VB with $Q \cap V = p$. Then Q contains $w_1 I$ and also $w_1(VB)I = w_1 IB$. The last equality holds since I is an ideal of V and B is central in VB. Hence Q contains w_1 or I. If Q contained the ideal I of V, then $I \subseteq Q \cap V = p$, a contradiction. Hence $w_1 \in Q$. Now for every $\gamma \in G$, $\gamma(P')$ is a prime ideal of VB with $\gamma(P') \cap V = p$. Hence w_1 belongs to every $\gamma(P')$. Consequently $w_1 \in P$, and P is a prime ideal of VB.

As $p = P \cap V$, the G-stable ideal $P \cap B$ of B is proper and thus zero. Since $C = Q(B)$, VC is a central localization of VB. So since P is prime, PC is a prime ideal of VC, and $P = PC \cap VB$. Consequently,

$$p \subseteq PC \cap V \subseteq (PC \cap VB) \cap V = P \cap V = p.$$

Hence PC is a G-stable prime ideal of VC satisfying $p = PC \cap V$. ∎

8.13. *Proof of* "(a) ⇒ (b)" *in Theorem* 2.6. By Lemma 8.4, we can make the assumptions in Subsection 8.5. By Proposition 8.7, VC is G-locally closed, i.e., the intersection N of all non-zero G-stable semiprime ideals of VC is non-zero. Since $VC \subseteq Q(V)$, $N \cap V \neq 0$. Let p be a non-zero prime ideal of V which does not contain p_C (see Subsection 8.10). By Proposition 8.12, there is a G-stable prime ideal P of VC with $P \cap V = p$. Then P contains N, so that p contains the non-zero ideal $N \cap V$. Thus every non-zero G-stable prime ideal of V contains $I = p_C \cap (N \cap V)$, which is non-zero as V is a prime ring. Now let Q be a non-zero G-stable semiprime ideal of V, and let P be any prime ideal containing Q. Then $(P : G)$ is G-stable, and it is prime since G is connected and acts rationally on V. Hence $(P : G)$ contains I. As P was an arbitrary prime containing Q, this shows that Q contains I. Consequently $0 = (J : G)$ is G-locally closed. This concludes the proof of Theorem 2.6. ∎

We now derive some consequences of Theorem 2.6. First, we prove that if J is a rational ideal of V which is locally closed, then the orbit of J in $\mathrm{Rat}(V)$ is open in its closure:

8.14. *Proof of Corollary* 2.7. We may assume that the orbit of J is not equal to its closure, which consists of all rational ideals containing $(J : G)$. Denote by I the intersection of all G-stable semiprime ideals strictly containing $(J : G)$. By Theorem 2.6, $(J : G)$ is G-locally closed, so strictly contained in I. Let J' be a rational ideal containing $(J : G)$ which is not in the orbit of J. Then Theorem 2.2 implies that J' is not in the G-stratum of J, i.e., $(J' : G) \neq 0$. Hence $(J' : G)$ contains I. Consequently, the complement of the orbit of J in its closure consists of all rational ideals containing I, so is a closed set. ∎

8.15. THEOREM. *Let J be a locally closed rational ideal of V. Assume that every prime ideal of $V/(J : G)$ is an intersection of rational ideals. If P is an ideal of V maximal with respect to $(P : G) = (J : G)$, then P is rational (and thus contained in the G-orbit of J).*

Note that the hypotheses of this theorem are in particular satisfied in the following two interesting special cases: if either V is a finitely generated k-algebra and J is a rational ideal such that V/J is finite-dimensional over k, see Corollary 8.16 below. Or, more generally, if J is locally closed, $V/(J : G)$ is a Jacobson ring, and every primitive ideal of $V/(J : G)$ is rational; this result was already stated as Theorem 2.5.

Proof. By Lemma 3.4(a), P is prime. Factoring out by $(J : G)$, we may assume that $(J : G) = 0$. Since J is locally closed, it follows by Corollary 2.7 that there is a non-zero G-stable ideal N which is contained in every rational ideal not in the orbit of J. Since $(P : G) = 0$, the ideal N cannot be contained in P. Since P is an intersection of rational ideals, P must be contained in some rational ideal J' in the G-orbit of J. Then $(J' : G) = (J : G) = 0$, so that $P = J'$ by the maximality of P. ∎

8.16. COROLLARY. *Assume that V is a finitely generated k-algebra. Let J be a rational ideal of V such that V/J is a vector space of finite dimension over k. If P is an ideal of V maximal with respect to $(P : G) = (J : G)$, then P is rational (and belongs thus to the G-orbit of J).*

Proof. Factoring out by $(J : G)$, we may assume that $(J : G) = 0$. Then V is a PI-algebra. Hence the primitive ideals coincide with the maximal ideals by Kaplansky's theorem [Row, 1.5.16]. Kaplansky's theorem implies also easily that the maximal ideals coincide with the rational ideals, see, e.g., [V_1, 2.6]. In particular, J is a maximal ideal and hence locally closed. Finally, being an affine PI-algebra, V is Jacobson by a theorem of Amitsur and Procesi [Row, 4.4.6]. Hence the hypotheses of Theorem 8.15 (and of Theorem 2.5) are satisfied. ∎

Since the relationship between the $\beta(H)$-stable ideals of V and the $(\beta \otimes \Gamma)(G)$-stable ideals of $V \otimes Q(G/H)$ given by $\natural$ plays such an important role in the theory, we include the following result, which is not needed in the sequel. Part of this result was already established in Lemmas 8.2 and 8.3.

8.17. THEOREM. *Let J be a rational ideal of V, and let H be a closed subgroup of G stabilizing J. The following are equivalent*:

(a) *J is a locally closed ideal of V.*

(b) *J is a $\beta(H)$-locally closed ideal of V.*

(c) *$J^\natural$ is a $(\beta \otimes \Gamma)(G)$-locally closed ideal of $V \otimes Q(G/H)$.*

If G is connected, these statements are also equivalent to:

(d) *$J^\natural$ is a locally closed ideal of $V \otimes Q(G/H)$.*

It is worthwhile to remark that the proof of (b) ⇒ (a) (and hence of (c) ⇒ (a)) depends, via Corollary 2.4, on Theorem 2.3.

We will see in the proof that (d) always implies the other three statements, even if G is not connected. But (d) can fail if G is not connected: in that case, $J \otimes Q(G)$ is not a prime ideal since $Q(G)$ is not a field (only a direct sum of fields). So if $H = 1$, then $J^\natural = \mu^{-1}(J \otimes Q(G))$ and is therefore not a prime ideal. Thus $J^\natural$ is not locally closed, as locally closed ideals are always prime.

Proof. (a) ⇒ (b). This was seen in the proof of Lemma 8.3.

(b) ⇒ (a). Let P be a prime ideal of V strictly containing J. Then by Corollary 2.4, $(P : H) \neq J$. As $(P : H)$ is a semiprime $\beta(H)$-stable ideal of

V strictly containing J, P contains the non-zero intersection of all semiprime $\beta(H)$-stable ideals which strictly contain J. Hence J is locally closed.

(b) $\Leftrightarrow$ (c). This is a special case of Lemma 8.2.

(d) $\Rightarrow$ (c). This holds even if G is not connected, and follows from the fact established in the proof of Lemma 8.3.

(a) $\Rightarrow$ (d). We now assume that G is connected. We show first that $J \otimes Q(G)$ is a locally closed prime ideal of $V \otimes Q(G)$. Factoring out by J, we may for this purpose temporarily assume that $J = 0$. By Lemma 3.5, $V \otimes Q(G)$ is a prime ring. Denote by S the left Ore set of regular elements of V. Let P be a prime ideal of $V \otimes Q(G)$ such that $P \cap V = 0$. As $Q(V) \otimes Q(G) = S^{-1}(V \otimes Q(G))$ is Noetherian, $S^{-1}P$ is a two-sided ideal of $Q(V) \otimes Q(G)$ which is clearly proper (see [McR, 2.1.16]). But the latter algebra is simple. Hence $S^{-1}P = 0$, implying $P = 0$. So if Q is any non-zero prime ideal of $V \otimes Q(G)$, then $Q \cap V \neq 0$. Since $V \subseteq V \otimes Q(G)$ is a centralizing extension, $Q \cap V$ is a prime ideal of V. Hence Q contains the non-zero intersection of all non-zero prime ideals of V. Consequently, the zero ideal of $V \otimes Q(G)$ is locally closed.

We proved that $J \otimes Q(G)$ is a locally closed prime ideal of $V \otimes Q(G)$. Thus also $\mu^{-1}(J \otimes Q(G))$ is a locally closed prime ideal of $V \otimes Q(G)$. To simplify the argument, we introduce some additional notation. Set $K = Q(G/H)$, $L = Q(G)$, $R = V \otimes K$, $P = \mu^{-1}(J \otimes Q(G))$, and $p = J^{\natural}$. Then $R \otimes_K L = V \otimes Q(G)$, and $p = P \cap R$. Moreover, since $J \otimes Q(G)$ is $(\beta \otimes \Delta)(H)$-stable, P is $(\mathrm{id} \otimes \Delta)(H)$-stable by the second intertwining property. Thus $P = p \otimes_K L$ by Lemma 6.3(a). Denote by A the intersection of all prime ideals of $R \otimes_K L$ which contain P strictly. Then A contains P strictly. Moreover, A is $(\mathrm{id} \otimes \Delta)(H)$-stable (since P is). Hence $A = (A \cap R) \otimes_K L$, and $A \cap R$ contains p strictly. Now if q is a prime ideal of R strictly containing p, then $q \otimes_K L$ is an ideal of $R \otimes_K L$ whose intersection with R is q. By a Zorn's lemma argument, there is an ideal Q of $R \otimes_K L$ maximal with respect to $Q \cap R = q$. Then Q is prime, and contains $P = p \otimes_K L$ strictly, so contains A. Hence $q = Q \cap R$ contains $A \cap R$. Consequently, $p = J^{\natural}$ is locally closed. ∎

9. EXAMPLES AND FURTHER APPLICATIONS

Recall Diagram 7.13. Example 9.1 shows that extension (3) can indeed be quite non-trivial in prime characteristic, even for a faithful action of a connected, abelian linear algebraic group G. In the remainder of the section, we relate the uniform dimensions and the Gelfand–Kirillov dimensions of V/J and $V/(J : G)$, proving in particular Proposition 2.9 and Theorem 2.8, see 9.3 and 9.5, respectively.

9.1. EXAMPLE. Assume that the characteristic of k is $p \neq 0$. There is a k-algebra V with a rational action of a connected, abelian linear algebraic group G, having the following properties.

(a) V is a finitely generated Noetherian k-algebra which is a domain. Moreover, V is an Azumaya algebra, and V has PI-degree p (the latter means in this case that $Q(V)$ is a division algebra of dimension p^2 over its center).

(b) There is a maximal (hence rational) ideal J of V with $(J : G) = 0$. The stabilizer H of J in G is a finite normal subgroup of order p.

(c) Extension (3) in Diagram 7.13 is non-trivial. In fact, VC is a division algebra with center C, but $T \cong VC \otimes_C Q(G/H)$ is isomorphic to $p \times p$-matrices over the field $Q(G/H)$.

(d) Here $\operatorname{udim} V = \operatorname{udim} VC = 1 < p = \operatorname{udim} VC \otimes_C Q(G/H) = \operatorname{udim} V/J$. So the inequality in Proposition 2.9 can be strict because the uniform dimension is not constant in extension (3).

(e) The variety G/H is affine, and the action of H on $V/J \cong M_p(k)$ factors through $\mathrm{GL}_p(k)$. So [MR_4, Proposition 4(ii)] does not extend to prime characteristic.

We will discuss parts (d) and (e) below after 9.3.

Construction of Example 9.1. Let u be an indeterminate over k, and let σ be the k-algebra automorphism of $k[u]$ given by $\sigma(u) = u + 1$; its order is p. Let V be the skew-Laurent polynomial ring $V = k[u]\{X, X^{-1}; \sigma\}$; that is, for $f(u) \in k[u]$, $f(u)X = X\sigma(f(u)) = Xf(u+1)$. So the defining relations of V are $uX = X(u+1)$, $uX^{-1} = X^{-1}(u-1)$, and $XX^{-1} = X^{-1}X = 1$. Then V is an affine, Noetherian domain. Its center is $Z = k[u]^{\langle\sigma\rangle}[X^p, X^{-p}]$. Denote by C the total ring of fractions of Z. Then $C = k(u)^{\langle\sigma\rangle}(X^P)$, and VC is a domain of dimension p^2 over C. It follows that VC is a division algebra, so equal to $Q(V)$, and that $C = Z(Q(V))$.

Let $G = \mathbb{G}_a \times \mathbb{G}_m$, where $\mathbb{G}_a$ and $\mathbb{G}_m$ denote the additive and multiplicative group of k, respectively. The element $\gamma = (a, b) \in G$ acts on V by $\gamma u = u + a$ and $\gamma X = bX$. This action is well-defined since the action of γ on $k[u]$ commutes with σ; for example, given $f(u) \in k[u]$, $(\gamma f(u))(\gamma X) = f(u+a)(bX) = (bX)f(u+a+1) = (\gamma X)\sigma(\gamma f(u))$. We can identify σ with the element $(1, 1)$ of G. Note that this action on the Laurent polynomial ring $k[u, X^p, X^{-p}]$ permutes the maximal ideals transitively, and that the stabilizer of every maximal ideal is trivial. Since this algebra is integral over Z, the action of G permutes also the maximal ideals of Z transitively. Consequently, the ideal of Z generated by the evaluations of the central polynomials of V is not contained in any maximal ideal, so it is equal to Z. It follows by the Artin–Procesi theorem that V is an Azumaya algebra, see [Co_3, 10.7.8]. This proves the assertions in (a).

Let M be any maximal ideal of Z. Since V is an Azumaya algebra, there is a unique maximal ideal of V lying over M, namely $J = MV$. Since G permutes the maximal ideals of Z transitively, $(M : G) = 0$. Consequently also $(J : G) = 0$ (this follows, for example, by incomparability for the finite centralizing extension $Z \subseteq V$). Denote the stabilizer of J in G by H. Since G is abelian, H is a normal subgroup. Since J is the unique maximal ideal of V lying over M, H is also the stabilizer of M. In particular, H contains $\langle \sigma \rangle$. We will show that $H = \langle \sigma \rangle$. Let I be a maximal ideal of $k[u, X^p, X^{-p}]$ lying over M. Since $Z = k[u, X^p, X^{-p}]^{\langle \sigma \rangle}$, σ permutes the prime ideals lying over M transitively, see [AM, Chap. 5, Exercise 13]. Since H permutes the prime ideals of Z lying over M, and since the stabilizer in G of I is trivial, it follows that $H = \langle \sigma \rangle$. In particular, H is a normal subgroup of G with p elements, proving (b).

We now turn to part (c). We already saw that $VC = Q(V)$ is a division algebra with center C. We have to show that $Q(G/H)$ splits VC. Since $VC \otimes_C Q(G/H)$ is a central simple algebra of degree p with center $Q(G/H)$, it is either a division algebra, or isomorphic to $M_p(Q(G/H))$. Hence it suffices to show that it is not a domain. Set $Z_1 = k[u]^{\langle \sigma \rangle}[X, X^{-1}]$. The map $Z_1 \to Z$ induced by $X \mapsto X^p$ is a G-equivariant k-algebra isomorphism. Hence Z_1 has a maximal ideal M_1 with $(M_1 : G) = 0$; moreover, the stabilizer of M_1 is H. So there is a G-equivariant embedding ν of $Q(Z_1) = C(X)$ into $Q(G/H)$. Restriction of ν to C yields a G-equivariant embedding of C into $Q(G/H)$. Replacing ν by $\Delta(\gamma) \circ \nu$ for some $\gamma \in G$, we may by Theorem 4.7 assume that $\nu|_C = \nu_J$. Hence $VC \otimes_C Q(G/H)$ contains $C(X) \otimes_C \nu(C(X)) \cong C(X) \otimes_C C(X)$. The latter algebra is not a domain, since X is algebraic over C but does not belong to C. Consequently, the simple Artinian algebra $VC \otimes_C Q(G/H)$ with center $Q(G/H)$ is not a domain and hence isomorphic to $M_p(Q(G/H))$. This proves (c), and (d) follows immediately.

Finally, we prove the assertions in (e). Since H is a subgroup of an abelian group, it is normal. Hence the quotient group G/H is affine. Since $(J : G) = 0$, V/J is a simple, finitely generated algebra over k of PI-degree p and hence isomorphic to $M_p(k)$. Denote the image of X in V/J by $\overline{X}$. Since M was an arbitrary maximal ideal of Z, we may assume that $X^p - 1 \in M \subseteq J$. Hence $\overline{X}^p = 1$, and $H \cong \langle \overline{X} \rangle$. Note that $\langle \overline{X} \rangle$ is a subgroup of the group of units of V/J, which is isomorphic to $\text{GL}_p(k)$. Since $fX = X\sigma(f)$ for $f \in k[u]$, conjugation by $\overline{X}$ gives the action of the generator $\sigma = (1, 1)$ of H on V/J. Hence the action of H on $V/J \cong M_p(k)$ factors through $\langle \overline{X} \rangle$.

9.2. *Remark*. Using the Frobenius map, it is easy to construct examples where extension (3) in Diagram 7.13 is non-trivial: one takes a "good" action β' of a group G on an algebra V (for which extension (3) is trivial), and forms a new, "bad" action β by preceding β' with a Frobenius morphism. Reference [V_1, Example 3.2] is of this form; there β' is essentially the action Γ of G on $A(G)$. It is worth noting that Example 9.1 is not of this type: in fact, in this example G acts faithfully and V contains non-zero homogeneous elements whose degree with respect to the action of the torus contained in G is one (e.g., the element $X \in V$).

We now turn to uniform dimension (Goldie rank).

9.3. *Proof of Proposition* 2.9. Since $Q(G^\circ)$ is unirational over k, V/J and $V/J \otimes Q(G^\circ)$ have the same uniform dimension. Since $Q(G)$ is the direct sum of $|G/G^\circ|$ copies of $Q(G^\circ)$, it follows that the uniform dimension of $V/J \otimes Q(G)$ is equal to $|G/G^\circ| \cdot \operatorname{udim}(V/J)$. By Lemma 7.3, $Q(V/(J:G))$ embeds into $Q(V/J \otimes Q(G))$. Hence $\operatorname{udim}(V/(J:G)) \leq \operatorname{udim}(V/J \otimes Q(G)) = |G/G^\circ| \cdot \operatorname{udim}(V/J)$. ∎

Moeglin and Rentschler describe in [MR_4, Proposition 4] two cases where equality in Proposition 2.9 holds in characteristic zero for *connected* G. Here is a quick outline of some of their arguments. Recall Diagram 7.13. In extensions (2) and (3), the uniform dimension of the larger algebra is always greater than or equal to the uniform dimension of the smaller algebra. This holds for the finite centralizing extension (3) by a result of Lanski [L, Theorem 4]. Since the regular elements of T remain regular in $T \otimes_{Q(G/H)} Q(G)$, $Q(T)$ embeds into $Q(T \otimes_{Q(G/H)} Q(G))$, so that also in extension (2) the uniform dimension of the larger algebra bounds the uniform dimension of the smaller. Since $VC \subseteq Q(V)$, $\operatorname{udim} V = \operatorname{udim} VC$. As seen above, the uniform dimension on the top level of the diagram is $\operatorname{udim}(V/J)$ (since $G = G^\circ$). If this number is to be the uniform dimension of $V = V/(J:G)$, then uniform dimension cannot change in extensions (2) and (3).

As extension (3) is trivial in characteristic zero, it sufficed for Moeglin and Rentschler in [MR_4] to find conditions ensuring that T and $T \otimes_{Q(G/H)} Q(G)$ have the same uniform dimension. One of the two situations studied by Moeglin and Rentschler is the case that there is a rational section from G/H to G, where H is the stabilizer of J in G [MR_4, Proposition 4(i)]. Using this section, one shows easily that $Q(G)$ is unirational over $Q(G/H)$, ensuring that T and $T \otimes_{Q(G/H)} Q(G)$ have indeed the same uniform dimension. The same argument would work in prime characteristic, if one could show that uniform dimension does not go up in extension (3) provided there is such a rational section. Example 9.1(d) showed that uniform dimension may not be constant in extension (3); but in that example there is certainly no such section. The second case studied by Moeglin and Rentschler [MR_4, Proposition 4(ii)] cannot be directly generalized to prime characteristic, as Example 9.1(e) showed.

We conclude our discussion of uniform dimension with an example which shows that also in characteristic zero, the inequality in Proposition 2.9 can be strict.

9.4. EXAMPLE. The inequality in Proposition 2.9 can be strict, even for connected groups, both in characteristic zero and in prime characteristic.

Let n be an integer ≥ 2, and suppose that k contains a primitive nth root of unity ζ; that is, we assume that the characteristic of k is either zero or else does not divide n. Let V be the skew-polynomial ring $V = k\{x, y\}$ with relation $xy = \zeta yx$. Then V is a Noetherian domain. The center of V is $Z = k[x^n, y^n]$, a commutative polynomial ring in two variables, and V is a free Z-module of rank n^2. Let $G = k^* \times k^*$ be a two-dimensional torus. The element $\gamma = (a, b) \in G$ acts on V by $\gamma x = ax$ and $\gamma y = by$. Let M be the maximal ideal of Z generated by $x^n - 1$ and $y^n - 1$. Then the stabilizer H of M in G is finite (it is in fact isomorphic to $\mathbb{Z}/(n) \times \mathbb{Z}/(n)$). Hence the G-orbit of M has dimension two, implying that $(M : G) = 0$. (Alternatively, one could have seen this using Theorem 2.8.) Let J be a prime ideal of V lying over M. By incomparability for the finite centralizing extension $Z \subseteq V$, J is a maximal ideal of V. Since $(J : G)$ is a prime ideal of V, and $(J : G) \cap Z = (M : G) = 0$, incomparability for the finite centralizing extension $Z \subseteq V$ implies that $(J : G) = 0$. Hence $V/(J : G) = V$ has uniform dimension one. Since V has PI-degree n and $(J : G) = 0$, also V/J has PI-degree n. Since V/J is a finitely generated simple algebra of PI-degree n over the algebraically closed field k, it is isomorphic to $n \times n$ matrices over k. Hence V/J has uniform dimension n. Thus, $\mathrm{udim}(V/(J : G)) = 1 < n = |G/G^\circ| \cdot \mathrm{udim}(V/J)$. ∎

We now prove Theorem 2.8, which relates the Gelfand–Kirillov dimensions of V/J and $V/(J : G)$ for a rational ideal J of V. The basic references for Gelfand–Kirillov dimension are [BK, KL]. We use GK-dimension always with respect to the fixed base field k.

9.5. *Proof of Theorem* 2.8. Note that $(J : G)$ is left Goldie by Theorem 2.1. By Corollary 3.7(b), $(J : G)$ is the intersection of the finitely many G-conjugates of $(J : G^\circ)$. Hence $V/(J : G)$ and $V/(J : G^\circ)$ have the same GK-dimension [KL, 3.3]. Moreover, $\dim G/H = \dim G^\circ/(H \cap G^\circ)$. Hence it suffices to prove the theorem in case that G is connected, which we assume from now on. We may also assume that $(J : G) = 0$. Recall Diagram 7.13.

By [MR_1, 3.13], GK-dimension does not change in extension (4), and by [KL, 5.5], it does not change in the finite extension (3). So $\operatorname{GKdim} T = \operatorname{GKdim} V$.

Writing $Q(G)$ as a finite extension of a purely transcendental field extension of k, one sees that

$$\operatorname{GKdim}(V/J \otimes Q(G)) = \operatorname{GKdim}(V/J) + \dim G,$$

see [KL, 3.6, 4.2, and 5.5].

The transcendence degree of $Q(G)$ over $Q(G/H)$ is $\dim H$. Writing $Q(G)$ now as a finite extension of a purely transcendental field extension of $Q(G/H)$, one sees as above that

$$\operatorname{GKdim}\big(T \otimes_{Q(G/H)} Q(G)\big) = \operatorname{GKdim} T + \dim H = \operatorname{GKdim} V + \dim H.$$

Hence $\operatorname{GKdim} V + \dim H = \operatorname{GKdim}(V/J) + \dim G$. Consequently, $\operatorname{GKdim} V = \operatorname{GKdim}(V/J) + \dim(G/H)$. ∎

ACKNOWLEDGMENTS

I am grateful to Ken Goodearl, Ed Letzter, and Zinovy Reichstein for their interest in this work, and for detailed comments; and to Miguel Ferrero for showing me his part of Proposition 5.2, and for kindly allowing me to include it in this paper.

REFERENCES

[AK] E. Abe and T. Kanno, Some remarks on algebraic groups, *Tôhoku Math. J. (2)* **11** (1959), 376–384.

[AM] M. F. Atiyah and I. G. Macdonald, "Introduction to Commutative Algebra," Addison–Wesley, Reading, MA, 1969.

[Be] A. D. Bell, Goldie dimensions of prime factors of polynomial and skew-polynomial rings, *J. London Math. Soc.* (2) **29** (1984), 418–424.

[Bo] A. Borel, "Linear Algebraic Groups," 2nd ed., Springer-Verlag, New York, 1991.

[BGR] W. Borho, P. Gabriel, and R. Rentschler, "Primideale in Einhüllenden auflösbarer Lie-Algebren," Lecture Notes in Mathematics, Vol. 357, Springer-Verlag, New York/Berlin, 1973.

[BK] W. Borho and H. Kraft, Über die Gelfand–Kirillov-Dimension, *Math. Ann.* **220** (1976), 1–24.

[Ch] W. Chin, Actions of solvable algebraic groups on noncommutative rings, *in* "Azumaya Algebras, Actions, and Modules, Proceedings of a Conference in Honor of Goro Azumaya's Seventieth Birthday," Contemp. Math., Vol. 124, pp. 29–38, Birkhäuser, Basel, 1992.

[Co_2] P. M. Cohn, "Algebra," 2nd ed., Vol. 2, Wiley, Chichester, 1989.

[Co_3] P. M. Cohn, "Algebra," 2nd ed., Vol. 3, Wiley, Chichester, 1991.

[D] J. Dixmier, "Enveloping Algebras," English translation, Amer. Math. Soc., Providence, 1996.

[Fa] D. R. Farkas, Groups acting on affine algebras, *Trans. Amer. Math. Soc.* **310** (1988), 485–497; Remarks and corrections, *Trans. Amer. Math. Soc.* **319** (1990), 821–823.
[Fe] M. Ferrero, Prime and principal closed ideals in polynomial rings, *J. Algebra* **134** (1990), 45–59.
[FP] M. Ferrero and M. M. Parmenter, A note on Jacobson rings and polynomial rings, *Proc. Amer. Math. Soc.* **105** (1989), 281–286.
[GL] K. R. Goodearl and E. S. Letzter, The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras, *Trans. Amer. Math. Soc.*, to appear.
[HLT] T. J. Hodges, T. Levasseur, and M. Toro, Algebraic structure of multi-parameter quantum groups, *Adv. in Math.* **126** (1997), 52–92.
[J_1] A. Joseph, Idéaux premiers et primitifs de l'algèbre des fonctions sur un groupe quantique, *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), 1139–1142.
[J_2] A. Joseph, On the prime and primitive spectra of the algebra of functions on a quantum group, *J. Algebra* **169** (1994), 441–511.
[J_3] A. Joseph, "Quantum Groups and Their Primitive Ideals," Ergeb. Math. Grenzgeb. (3), Vol. 29, Springer-Verlag, Berlin, 1995.
[KL] G. R. Krause and T. H. Lenagan, "Growth of Algebras and Gelfand–Kirillov Dimension," Pitman, Boston, 1985.
[L] C. Lanski, Goldie conditions in finite normalizing extensions, *Proc. Amer. Math. Soc.* **79** (1980), 515–519.
[McR] J. C. McConnell and J. C. Robson, "Noncommutative Noetherian Rings," Wiley, Chichester, 1987.
[MR_1] C. Moeglin and R. Rentschler, Orbites d'un groupe algébrique dans l'espace des idéaux rationnels d'une algèbre enveloppante, *Bull. Soc. Math. France* **109** (1981), 403–426.
[MR_2] C. Moeglin and R. Rentschler, Sur la classification des idéaux primitifs des algèbres enveloppantes, *Bull. Soc. Math. France* **112** (1984), 3–40.
[MR_3] C. Moeglin and R. Rentschler, Sous-corps commutatifs ad-stables des anneaux de fractions des quotients des algèbres enveloppantes; espaces homogènes et induction de Mackey, *J. Funct. Anal.* **69** (1986), 307–396.
[MR_4] C. Moeglin and R. Rentschler, Idéaux G-rationnels, rang de Goldie, preprint, 1986.
[M] S. Montgomery, "Hopf Algebras and Their Actions on Rings," CBMS Regional Conference Series in Mathematics, Vol. 82, American Mathematical Society, Providence, RI, 1993.
[P] D. S. Passman, "Infinite Crossed Products," Academic Press, Boston, 1989.
[Re] R. Rentschler, Primitive ideals in enveloping algebras (general case), *in* "Noetherian Rings and Their Applications" (L. W. Small, Ed.), Math. Surveys and Monographs, Vol. 24, pp. 37–57, Amer. Math. Soc., Providence, 1987.
[RS] J. C. Robson and L. W. Small, Liberal extensions, *Proc. London Math. Soc.* **42** (1981), 87–103.
[Ros] M. Rosenlicht, Automorphisms of function fields, *Trans. Amer. Math. Soc.* **79** (1955), 1–11.
[Row] L. H. Rowen, "Polynomial Identities in Ring Theory," Academic Press, New York, 1980.
[V_1] N. Vonessen, Actions of algebraic groups on the spectrum of rational ideals, *J. Algebra* **182** (1996), 383–400.
[V_2] N. Vonessen, Actions of linearly reductive groups on affine PI-algebras, *Mem. Amer. Math. Soc.* **414** (1989).
[W] W. C. Waterhouse, "Introduction to Affine Group Schemes," Graduate Texts in Mathematics, Vol. 66, Springer-Verlag, New York, 1979.