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Current algebras, highest weight categories and quivers ☆

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Abstract

We study the category of graded finite-dimensional representations of the polynomial current algebra associated to a simple Lie algebra. We prove that the category has enough injectives and compute the graded character of the injective envelopes of the simple objects as well as extensions between simple objects. The simple objects in the category are parametrized by the affine weight lattice. We show that with respect to a suitable refinement of the standard ordering on the affine weight lattice the category is highest weight. We compute the Ext quiver of the algebra of endomorphisms of the injective cogenerator of the subcategory associated to an interval closed finite subset of the weight lattice. Finally, we prove that there is a large number of interesting quivers of finite, affine and star-shaped type, as well as tame quasi-hereditary algebras, that arise from our study.

Keywords: Current algebra; Highest weight category; Kirillov–Reshetikhin modules; Path algebra

0. Introduction

In this paper we study the category $\mathcal{G}$ of graded finite-dimensional representations of the polynomial current algebra $\mathfrak{g}[t]$ associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$. There are numerous interesting and related families of examples of such representations: the Demazure modules arising from the positive level representations of the affine algebra, the fusion products

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of finite-dimensional representations of $\mathfrak{g}[t]$ defined in [8], the Kirillov–Reshetikhin modules studied in [3,4] and the Weyl modules introduced in [5] and studied in [2,9]. All these representations are in general reducible but always indecomposable.

The isomorphism classes of simple objects in $\mathcal{G}$ are indexed by the set $\Lambda = P^+ \times \mathbf{Z}_+$ where P^+ is the set of dominant integral weights of $\mathfrak{g}$. The set Λ can be identified in a natural way with a subset of the lattice of integral weights $\widehat{P}$ of the untwisted affine Lie algebra associated to $\mathfrak{g}$. We define an interval finite partial order $\preccurlyeq$ on Λ which is a refinement of the usual order on $\widehat{P}$ and show (Theorem 1) that $\mathcal{G}$ is a highest weight category, in the sense of [6], with the poset of weights $(\Lambda, \preccurlyeq)$. To do this, we study first the category $\widehat{\mathcal{G}}$ of graded $\mathfrak{g}[t]$-modules with finite-dimensional graded pieces. This category has enough projectives and the graded character of the projective modules can be described explicitly. Then, using a certain duality, we are able to show that the category $\mathcal{G}$ has enough injectives and we compute the graded character of the injective envelope of any simple object. We then prove that $\mathcal{G}$ is a directed highest weight category by computing the extensions between simple objects.

In Section 3 we study algebraic structures associated with Serre subcategories of $\mathcal{G}$. For an interval closed subset Γ of Λ, let $\mathcal{G}[\Gamma]$ be the full subcategory of $\mathcal{G}$ consisting of objects whose simple constituents are parametrized by elements of Γ and let $I(\Gamma)_\Gamma$ be the injective cogenerator of $\mathcal{G}[\Gamma]$. It is well known that there is an equivalence of categories between $\mathcal{G}[\Gamma]$ and the category of finite-dimensional right $\mathfrak{A}(\Gamma) = \operatorname{End}_{\mathcal{G}[\Gamma]} I(\Gamma)_\Gamma$-modules. Moreover $\mathfrak{A}(\Gamma)$ is a quotient of the path algebra of its Ext quiver $Q(\Gamma)$ and has a compatible grading. By using the character formula for the injective envelopes, we show that $Q(\Gamma)$ can be computed quite explicitly in terms of finite-dimensional representations of $\mathfrak{g}$.

In Sections 4 and 5 we show that there are many interesting quivers arising from our study. Thus, in Section 4 we see that for all $\mathfrak{g}$ (in some cases one has to exclude $\mathfrak{sl}_2$ or $\mathfrak{g}$ of type C_ℓ), there exists interval closed finite subsets Γ such that the corresponding algebra $\mathfrak{A}(\Gamma)$ is hereditary and $Q(\Gamma)$ is (a) a generalized Kronecker quiver; (b) a quiver of type $\mathbb{A}_\ell$, $\mathbb{D}_\ell$; (c) an affine quiver of type $\tilde{\mathbb{D}}_\ell$; (d) any star shaped quiver with three branches. In Section 5 we study an example which arises from the theory of Kirillov–Reshetikhin modules for $\mathfrak{g}[t]$ where $\mathfrak{g}$ is of type D_n. In this case the algebra $\mathfrak{A}(\Gamma)$ is not hereditary, but is still of tame representation type.

1. The category $\mathcal{G}$

1.1. The simple Lie algebras and the associated current algebras

Throughout the paper $\mathfrak{g}$ denotes a finite-dimensional complex simple Lie algebra and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Set $I = \{1, \ldots, \dim \mathfrak{h}\}$ and let $\{\alpha_i\colon i \in I\} \subset \mathfrak{h}^*$ be a set of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $R \subset \mathfrak{h}^*$ (respectively R^+, P^+, Q^+) be the corresponding set of roots (respectively positive roots, dominant integral weights, the $\mathbf{Z}_+$-span of R^+) and let $\theta \in R^+$ be the highest root. Let $W \subset \operatorname{Aut}(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$ and $w_\circ$ be the longest element of W. For $\alpha \in R$ denote by $\mathfrak{g}_\alpha$ the corresponding root space. The subspaces $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$ are Lie subalgebras of $\mathfrak{g}$. Fix a Chevalley basis $x_\alpha^\pm$, $\alpha \in R^+$, h_i, $i \in I$ of $\mathfrak{g}$ and for $\alpha \in R^+$, set $h_\alpha = [x_\alpha, x_{-\alpha}]$. Note that $h_{\alpha_i} = h_i$, $i \in I$. For $i \in I$, let $\omega_i \in P^+$ be defined by $\omega_i(h_j) = \delta_{ij}$ for all $j \in I$.

Let $\mathcal{F}(\mathfrak{g})$ be the category of finite-dimensional $\mathfrak{g}$-modules with the morphisms being maps of $\mathfrak{g}$-modules. In particular, we write $\operatorname{Hom}_\mathfrak{g}$ for $\operatorname{Hom}_{\mathcal{F}(\mathfrak{g})}$. The set P^+ parametrizes the isomorphism classes of simple objects in $\mathcal{F}(\mathfrak{g})$. For $\lambda \in P^+$, let $V(\lambda)$ be the simple module in the corresponding isomorphism class which is generated by an element $v_\lambda \in V(\lambda)$ satisfying the defining relations:

$$\mathfrak{n}^+ v_\lambda = 0, \qquad h v_\lambda = \lambda(h) v_\lambda, \qquad \left(x_{\alpha_i}^-\right)^{\lambda(h_i)+1} v_\lambda = 0,$$

for all $h \in \mathfrak{h}$, $i \in I$. The module $V(-w_\circ\lambda)$ is the $\mathfrak{g}$-dual of $V(\lambda)$. If $V \in \operatorname{Ob}\mathcal{F}(\mathfrak{g})$, write

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where $V_\lambda = \{v \in V\colon\ hv = \lambda(h)v,\ \forall h \in \mathfrak{h}\}$. Set $\operatorname{wt}(V) = \{\lambda \in \mathfrak{h}^*\colon\ V_\lambda \neq 0\}$. Finally, recall also that the category $\mathcal{F}(\mathfrak{g})$ is semi-simple, i.e. any object in $\mathcal{F}(\mathfrak{g})$ is isomorphic to a direct sum of the modules $V(\lambda)$, $\lambda \in P^+$. We shall use the following standard results in the course of the paper (cf. [12] for (iv)).

Lemma.

(i) *Let* $\lambda \in P^+$. *Then* $\operatorname{wt}(V(\lambda)) \subset \lambda - Q^+$.
(ii) *Let* $V \in \mathcal{F}(\mathfrak{g})$.*Then* $w \operatorname{wt}(V) \subset \operatorname{wt}(V)$ *for all* $w \in W$ *and* $\dim V_\lambda = V_{w\lambda}$.
(iii) *Let* $V \in \mathcal{F}(\mathfrak{g})$. *Then*

$$\dim \operatorname{Hom}_{\mathfrak{g}}\bigl(V(\lambda), V\bigr) = \dim\bigl\{v \in V_\lambda\colon\ \mathfrak{n}^+ v = 0\bigr\}.$$

(iv) *Let* $\lambda, \mu \in P^+$. *Then the module* $V(-w_\circ\lambda) \otimes V(\mu)$ *is generated as a* $\mathbf{U}(\mathfrak{g})$*-module by the element* $v = v_{-\lambda} \otimes v_\mu$ *with defining relations*:

$$\left(x_{\alpha_i}^+\right)^{\lambda(h_i)+1} v = \left(x_{\alpha_i}^-\right)^{\mu(h_i)+1} v = 0, \qquad hv = (\mu - \lambda)(h)v,$$

for all $i \in I$ *and* $h \in \mathfrak{h}$. $\square$

Given any Lie algebra $\mathfrak{a}$ let $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbf{C}[t]$ be the polynomial current algebra of $\mathfrak{a}$. Let $\mathfrak{a}[t]_+$ be the Lie ideal $\mathfrak{a} \otimes t\mathbf{C}[t]$. Both $\mathfrak{a}[t]$ and $\mathfrak{a}[t]_+$ are $\mathbf{Z}_+$-graded Lie algebras with the grading given by powers of t. Let $\mathbf{U}(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$. Then $\mathbf{U}(\mathfrak{a}[t])$ has a natural $\mathbf{Z}_+$-grading as an associative algebra and we let $\mathbf{U}(\mathfrak{a})[k]$ be the kth-graded piece. The algebra $U(\mathfrak{a})$ is a Hopf algebra, the comultiplication being given by extending the assignment $x \to x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{a}$ to an algebra homomorphism of $U(\mathfrak{a})$. In the case of $\mathbf{U}(\mathfrak{a}[t])$ the comultiplication is a map of graded algebras.

In the course of the paper, we shall repeatedly use the fact that $\mathbf{U}(\mathfrak{a}[t])$ *is generated as a graded algebra by* $\mathfrak{a}$ *and* $\mathfrak{a} \otimes t$ *without a further comment.*

1.2. The category $\widehat{\mathcal{G}}$

Let $\widehat{\mathcal{G}}$ be the category whose objects are graded $\mathfrak{g}[t]$-modules with finite-dimensional graded pieces and where the morphisms are graded maps of $\mathfrak{g}[t]$-modules. More precisely, if $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ then

$$V = \bigoplus_{r \in \mathbf{Z}_+} V[r],$$

where $V[r]$ is a finite-dimensional subspace of V such that $(xt^k)V[r] \subset V[r+k]$ for all $x \in \mathfrak{g}$ and $r, k \in \mathbf{Z}_+$. In particular, $V[r] \in \operatorname{Ob}\mathcal{F}(\mathfrak{g})$. Also, if $V, W \in \operatorname{Ob}\widehat{\mathcal{G}}$, then

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W) = \bigl\{f \in \operatorname{Hom}_{\mathfrak{g}[t]}(V, W)\colon\ f\bigl(V[r]\bigr) \subset W[r],\ r \in \mathbf{Z}_+\bigr\}.$$

For $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W)$ let $f[r]$ be the restriction of f to $V[r]$. Clearly, $f[r] \in \operatorname{Hom}_{\mathfrak{g}}(V[r], W[r])$.

Define a covariant functor $\operatorname{ev} : \mathcal{F}(\mathfrak{g}) \to \widehat{\mathcal{G}}$ by the requirements:

$$\operatorname{ev}(V)[0] = V, \qquad \operatorname{ev}(V)[r] = 0, \quad r > 0,$$

and with $\mathfrak{g}[t]$-action given by

$$\bigl(xt^k\bigr)v = \delta_{k,0}xv, \quad \forall x \in \mathfrak{g},\ k \in \mathbf{Z}_+,\ v \in V,$$

and

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}\bigl(\operatorname{ev}(V), \operatorname{ev}(W)\bigr) = \operatorname{Hom}_{\mathfrak{g}}(V, W). \tag{1.1}$$

For $s \in \mathbf{Z}_+$ let τ_s be the grading shift given by

$$(\tau_s V)[k] = V[k-s], \quad k \in \mathbf{Z}_+,\ V \in \operatorname{Ob}\widehat{\mathcal{G}}.$$

Clearly $\tau_s(V) \in \operatorname{Ob}\widehat{\mathcal{G}}$.

1.3. Simple objects in $\widehat{\mathcal{G}}$

Let $\Lambda = P^+ \times \mathbf{Z}_+$. For $(\lambda, r) \in \Lambda$, set

$$V(\lambda, r) = \tau_r\bigl(\operatorname{ev}\bigl(V(\lambda)\bigr)\bigr). \tag{1.2}$$

Proposition. *A simple object in $\widehat{\mathcal{G}}$ is isomorphic to $V(\lambda, r)$ for some $(\lambda, r) \in \Lambda$ and we have*

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}\bigl(V(\lambda, r), V(\mu, s)\bigr) = 0, \qquad (\lambda, r) \neq (\mu, s),$$
$$\operatorname{Hom}_{\widehat{\mathcal{G}}}\bigl(V(\lambda, r), V(\lambda, r)\bigr) \cong \mathbf{C}.$$

In particular, the isomorphism classes of simple objects in $\widehat{\mathcal{G}}$ are parametrized by elements of Λ. Moreover if $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ is such that $V = V[n]$ for some $n \in \mathbf{Z}_+$, then V is semi-simple.

Proof. The modules $V(\lambda, r)$, $(\lambda, r) \in \Lambda$ are obviously simple and non-isomorphic. Moreover, if $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ is such that $V \neq V[r]$ for some r, there exists $m, m' \in \mathbf{Z}_+$ with $m' > m$ such that $V[j] \neq 0$, for $j \in \{m, m'\}$. Hence the subspace $\bigoplus_{k>m} V[k]$ is a nontrivial proper graded $\mathfrak{g}[t]$-submodule of V and it follows that V is not simple. Assume now that V is simple so that $V = V[r]$ for some r. This implies that V is finite-dimensional and also that $\mathfrak{g}[t]_+V = 0$. It follows that V must be isomorphic to $V(\lambda)$ for some $\lambda \in P^+$ as a $\mathfrak{g}$-module and hence $V \cong V(\lambda, r)$ as $\mathfrak{g}[t]$-modules. The other statements are now obvious. $\square$

1.4. Tensor structure of the category $\widehat{\mathcal{G}}$

Let $V, W \in \operatorname{Ob}\widehat{\mathcal{G}}$. Then $V \otimes W$ is a $\mathfrak{g}[t]$-module with the action being given by the comultiplication. Given $k \in \mathbf{Z}_+$, set

$$(V \otimes W)[k] = \bigoplus_{i \in \mathbf{Z}_+} V[i] \otimes W[k-i],$$

with the usual convention that $W[j] = 0$ if $j < 0$. The following is trivially checked.

Lemma.

(i) $V \otimes W = \bigoplus_{k \in \mathbf{Z}_+} (V \otimes W)[k]$ *and for all* $r \in \mathbf{Z}_+$, *we have*

$$\big(xt^r\big)\big((V \otimes W)[k]\big) \subset (V \otimes W)[k+r].$$

In particular, $\widehat{\mathcal{G}}$ is a tensor category.

(ii) *For all* $r, s \in \mathbf{Z}_+$

$$\tau_s V \cong V \otimes V(0,s), \qquad \tau_{r+s}(V \otimes W) \cong (\tau_r V) \otimes (\tau_s W). \tag{1.3}$$

1.5. The subcategories $\mathcal{G}$ and $\mathcal{G}_{\leqslant s}$

Let $\mathcal{G}_{\leqslant s}$ be the full subcategory of $\widehat{\mathcal{G}}$ whose objects V satisfy

$$V[r] = 0, \quad \forall r > s,$$

and let $\mathcal{G}$ be the full subcategory of $\mathcal{G}$ consisting of $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ such that $V \in \operatorname{Ob}\mathcal{G}_{\leqslant s}$ for some $s \in \mathbf{Z}_+$. It follows from the definition that $\mathcal{G}_{\leqslant s}$ is a full subcategory of $\mathcal{G}_{\leqslant r}$ for all $s < r \in \mathbf{Z}_+$. Given $V \in \operatorname{Ob}\mathcal{G}$, let $\operatorname{soc}(V) \in \operatorname{Ob}\mathcal{G}$ be the maximal semi-simple subobject of V. Similarly, given $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, let $\operatorname{head}(V)$ be the maximal semi-simple quotient of V.

Given $s \in \mathbf{Z}_+$ and $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, define

$$V_{>s} = \bigoplus_{r>s} V[r], \qquad V_{\leqslant s} = V/V_{>s}.$$

Then $V_{\leqslant s} \in \operatorname{Ob}\mathcal{G}_{\leqslant s}$. Furthermore, if $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W)$, then $V_{>s}$ is contained in the kernel of the canonical morphism $\bar{f} : V \to W_{\leqslant s}$ and hence we have a natural morphism $f_{\leqslant s} \in \operatorname{Hom}_{\mathcal{G}_{\leqslant s}}(V_{\leqslant s}, W_{\leqslant s})$.

Lemma.

(i) *For all* $r, s \in \mathbf{Z}_+$, *and* $V \in \operatorname{Ob}\mathcal{G}_{\leqslant r}$, $W \in \operatorname{Ob}\mathcal{G}_{\leqslant s}$ *we have*

$$V \otimes W \in \operatorname{Ob}\mathcal{G}_{\leqslant r+s}.$$

In particular $\mathcal{G}$ is a tensor subcategory of $\widehat{\mathcal{G}}$.

(ii) *The assignments* $V \mapsto V_{\leqslant r}$ *for all* $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ *and* $f \mapsto f_{\leqslant r}$ *for all* $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W)$, $V, W \in \operatorname{Ob}\widehat{\mathcal{G}}$ *define a full, exact and essentially surjective functor from* $\widehat{\mathcal{G}}$ *to* $\mathcal{G}_{\leqslant r}$.
(iii) *For any* $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, $\lambda \in P^+$ *and* $r, s \in \mathbf{Z}_+$ *with* $s \geqslant r$, *we have*

$$\big(V \otimes V(\lambda, r)\big)_{\leqslant s} \cong V_{\leqslant s-r} \otimes V(\lambda, r).$$

Proof. Parts (i) and (ii) are obvious. For the last part, consider the natural map of graded $\mathfrak{g}[t]$-modules $V \otimes V(\lambda, r) \to V_{\leqslant s-r} \otimes V(\lambda, r)$. The assertion follows by noting that $(V \otimes V(\lambda, r))[k] = V[k-r] \otimes V(\lambda, r)$ for all $k \in \mathbf{Z}_+$. $\square$

From now on, given $V \in \operatorname{Ob}\mathcal{G}$ we denote by $[V : V(\lambda, r)]$ the multiplicity of $V(\lambda, r)$ in a composition series for V. Furthermore, given $W \in \operatorname{Ob}\widehat{\mathcal{G}}$, we set $[W : V(\lambda, r)] := [W_{\leqslant r} : V(\lambda, r)]$. Observe that $[V : V(\lambda, r)]$ equals the $\mathfrak{g}$-module multiplicity of $V(\lambda)$ in $V[r]$. For any $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, define

$$\Lambda(V) = \big\{(\lambda, r) \in \Lambda\colon \big[V : V(\lambda, r)\big] \neq 0\big\}.$$

1.6. We recall the following definition (which motivated much of this paper) of a directed category following [6,11], in the context of interest to us. Thus let $\mathcal{C}$ be an abelian category over $\mathbf{C}$ whose objects are complex vector spaces, have finite length and such that $\operatorname{Hom}_{\mathcal{C}}(M, N)$ is finite-dimensional for all $M, N \in \operatorname{Ob}\mathcal{C}$.

Definition. We say that $\mathcal{C}$ is a directed category if

1°. The simple objects in $\mathcal{C}$ are parametrized by a poset $(\Pi, \leqslant)$ (*the poset of weights*) such that for all $\tau \in \Pi$, the set $\{\xi \in \Pi\colon \xi < \tau\}$ is finite.
2°. Given $\sigma \in \Pi$, let $S(\sigma)$ be a simple object in the corresponding isomorphism class. Then

$$\operatorname{Ext}^1_{\mathcal{C}}\big(S(\sigma), S(\tau)\big) \neq 0 \quad\Rightarrow\quad \sigma < \tau.$$

It is immediate from [7,14,15] that a directed category has enough injectives. Given $\Xi \subset \Pi$, let $\mathcal{C}[\Xi]$ be the full subcategory of $\mathcal{C}$ whose objects satisfy

$$M \in \operatorname{Ob}\mathcal{C}[\Xi], \quad \big[M : S(\tau)\big] \neq 0 \quad\Rightarrow\quad \tau \in \Xi.$$

It is clear that if $\mathcal{C}$ is a directed category with the poset of weights Π, then for any subset $\Xi \subset \Pi$, the category $\mathcal{C}[\Xi]$ is also directed with the poset of weights Ξ. Given $\sigma, \tau \in \Xi$ with $\sigma < \tau$, we denote by $[\sigma, \tau]$ the interval $\{\xi\colon \sigma \leqslant \xi \leqslant \tau\}$. A subset Ξ of Π is said to be *interval closed* if $\sigma < \tau \in \Xi$ implies that $[\sigma, \tau] \subset \Xi$.

1.7. We can now state the main result of this section. Set

$$\Lambda_{\leqslant r} = \big\{(\nu, l) \in \Lambda\colon l \leqslant r\big\}, \quad r \in \mathbf{Z}_+.$$

Define a strict partial order on Λ in the following way. Given $(\lambda, r), (\mu, s) \in \Lambda$, say that (μ, s) covers (λ, r) if and only if $s = r + 1$ and $\mu - \lambda \in R \sqcup \{0\}$. It follows immediately that for any $(\mu, s) \in \Lambda$ the set of $(\lambda, r) \in \Lambda$ such that (μ, s) covers (λ, r) is finite. Let $\preccurlyeq$ be the unique partial order on Λ generated by this cover relation. Then $\{(\mu, s)\colon (\mu, s) \preccurlyeq (\lambda, r)\}$ is finite for all $(\lambda, r) \in \Lambda$. Note that if $(\lambda, r) \preccurlyeq (\mu, s) \in \Lambda_{\leqslant k}$ then $(-w_\circ\mu, k - s) \preccurlyeq (-w_\circ\lambda, k - r)$. If $(\mu, s) \preccurlyeq (\lambda, r)$ and $(\mu, s) \neq (\lambda, r)$, we write $(\mu, s) \prec (\lambda, r)$.

Theorem 1. *For all $\Gamma \subset \Lambda$, the category $\mathcal{G}[\Gamma]$ is a directed category with poset the of weights $(\Gamma, \preccurlyeq)$.*

We prove this theorem in the next section (Proposition 2.5).

2. Injective and projective objects

2.1. Projectives in $\widehat{\mathcal{G}}$ and $\mathcal{G}_{\leqslant r}$

Given $(\lambda, r) \in \Lambda$, observe that $V(\lambda, r)$ is a $\mathfrak{g}$-module by restriction and set

$$P(\lambda, r) = \mathbf{U}\big(\mathfrak{g}[t]\big) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda, r).$$

Clearly, $P(\lambda, r)$ is an infinite-dimensional graded $\mathfrak{g}[t]$-module. Using the PBW theorem we have an isomorphism of graded vector spaces

$$\mathbf{U}\big(\mathfrak{g}[t]\big) \cong \mathbf{U}\big(\mathfrak{g}[t]_+\big) \otimes \mathbf{U}(\mathfrak{g}),$$

and hence we get

$$P(\lambda, r)[k] = \mathbf{U}\big(\mathfrak{g}[t]_+\big)[k - r] \otimes V(\lambda, r), \tag{2.1}$$

where we understand that $\mathbf{U}(\mathfrak{g}[t]_+)[k - r] = 0$ if $k < r$. This shows that $P(\lambda, r) \in \operatorname{Ob} \widehat{\mathcal{G}}$ and also that

$$P(\lambda, r)[r] = 1 \otimes V(\lambda, r).$$

Set $p_{\lambda,r} = 1 \otimes v_{\lambda,r}$.

Proposition. *Let $(\lambda, r) \in \Lambda$, $s \in \mathbf{Z}_+$ and $s \geqslant r$.*

(i) *$P(\lambda, r)$ is generated as a $\mathfrak{g}[t]$-module by $p_{\lambda,r}$ with defining relations*:

$$\big(\mathfrak{n}^+\big) p_{\lambda,r} = 0, \qquad h p_{\lambda,r} = \lambda(h) p_{\lambda,r}, \qquad \big(x_{\alpha_i}^-\big)^{\lambda(h_i)+1} p_{\lambda,r} = 0,$$

for all $h \in \mathfrak{h}$, $i \in I$. Hence, $P(\lambda, r)$ is the projective cover in the category $\widehat{\mathcal{G}}$ of its unique simple quotient $V(\lambda, r)$. Moreover the kernel $K(\lambda, r)$ of the canonical projection $P(\lambda, r) \twoheadrightarrow V(\lambda, r)$ is generated as a $\mathfrak{g}[t]$-module by $P(\lambda, r)[r + 1] = \mathfrak{g} \otimes V(\lambda, r)$.

(ii) *$P(\lambda, r) \cong P(0, 0) \otimes V(\lambda, r)$ as objects in $\widehat{\mathcal{G}}$.*

(iii) *The modules $P(\lambda, r)_{\leqslant s}$ are projective in $\mathcal{G}_{\leqslant s}$ and*

$$P(\lambda, r)_{\leqslant s} \cong P(0, 0)_{\leqslant s-r} \otimes V(\lambda, r).$$

(iv) *As* $\mathfrak{g}$*-modules, we have*

$$P(0,0)[k] \cong \mathbf{U}\big(\mathfrak{g}[t]_+\big)[k] \cong \bigoplus_{(r_1,\dots,r_k)\in\mathbf{Z}_+^k:\ \sum_{j=1}^k jr_j=k} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}),$$

where $S^p(\mathfrak{g})$ *denotes the pth symmetric power of* $\mathfrak{g}$.

(v) *Let* $(\mu,s)\in\Lambda$. *Then* $[K(\lambda,r):V(\mu,s)]\neq 0$ *only if* $(\lambda,r)\prec(\mu,s)$.

(vi) *Let* $V\in\operatorname{Ob}\widehat{\mathcal{G}}$. *Then* $\dim\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda,r),V)=[V:V(\lambda,r)]$.

Proof. The fact that $P(\lambda,r)$ is projective in the category $\widehat{\mathcal{G}}$ is standard in the relative homological algebra (cf. [10]). The other statements in (i) are immediate from the discussion preceding the proposition. For part (ii), note that the element $p_{0,0}\otimes v_{\lambda,r}$ satisfies the defining relations of $P(\lambda,r)$. Moreover it is easily seen that

$$\mathbf{U}\big(\mathfrak{g}[t]\big)(p_{0,0}\otimes v_{\lambda,r}) = P(0,0)\otimes V(\lambda,r).$$

Hence we have a surjective morphism $P(\lambda,r)\to P(0,0)\otimes V(\lambda,r)$ in $\widehat{\mathcal{G}}$. On the other hand, (2.1) implies that $P(\lambda,r)\cong P(0,0)\otimes V(\lambda,r)$ as vector spaces and (ii) is proved. Part (iii) is immediate from Lemma 1.5(ii), (iii). The first isomorphism in (iv) is obvious. To prove the second, we may assume that $k>0$ since $\mathbf{U}(\mathfrak{g}[t]_+)[0]=\mathbf{C}$. For $s\geqslant 0$, let $\mathbf{U}(\mathfrak{g}[t]_+)_{\leqslant s}$, $s\geqslant 0$ be the subspace of $\mathbf{U}(\mathfrak{g}[t]_+)$ spanned by the set

$$\big\{\big(y_1t^{r_1}\big)\cdots\big(y_kt^{r_k}\big):\ y_jt^{r_j}\in\mathfrak{g}[t]_+,\ 1\leqslant j\leqslant k\leqslant s\big\},$$

and set

$$\mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]_{\leqslant s} = \mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]\cap\mathbf{U}\big(\mathfrak{g}[t]_+\big)_{\leqslant s}.$$

Then $\mathbf{U}(\mathfrak{g}[t]_+)[k]=\mathbf{U}(\mathfrak{g}[t]_+)[k]_{\leqslant k}$ and for $0\leqslant s\leqslant k$ the subspaces $\mathbf{U}(\mathfrak{g}[t]_+)[k]_{\leqslant s}$ define an increasing filtration on $\mathbf{U}(\mathfrak{g}[t]_+)[k]$. Moreover, regarding $\mathfrak{g}[t]_+$ as a $\mathfrak{g}$-module via the adjoint action on $\mathfrak{g}[t]$, we see that this filtration is in fact $\mathfrak{g}$-equivariant. Since $\mathbf{U}(\mathfrak{g}[t]_+)[k]_{\leqslant r}$ is finite-dimensional we get an isomorphism of $\mathfrak{g}$-modules,

$$\begin{aligned}\mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]_{\leqslant r} &\cong_{\mathfrak{g}} \mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]_{\leqslant r-1}\oplus\big(\mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]_{\leqslant r}/\mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]_{\leqslant r-1}\big)\\ &\cong_{\mathfrak{g}} \mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]_{\leqslant r-1}\oplus S^r\big(\mathfrak{g}[t]_+\big)[k],\end{aligned}$$

the second isomorphism being a consequence of the PBW theorem. It follows by a downward induction on r that

$$\mathbf{U}\big(\mathfrak{g}[t]_+\big)[k]\cong_{\mathfrak{g}}\bigoplus_{r=1}^{k} S^r\big(\mathfrak{g}[t]\big)_+[k] = S\big(\mathfrak{g}[t]_+\big)[k].$$

Given a partition $\mathbf{n}=(n_s\geqslant n_{s-1}\geqslant\cdots\geqslant n_1>0)$ of k, let $V(\mathbf{n})$ be the subspace of $S(\mathfrak{g}[t]_+)[k]$ spanned by the elements

$$\big\{\big(x_1t^{n_1}\big)\cdots\big(x_st^{n_s}\big):\ x_j\in\mathfrak{g},\ 1\leqslant j\leqslant s\big\}.$$

Clearly $V(\mathbf{n})$ is a $\mathfrak{g}$-submodule of $S(\mathfrak{g}[t]_+)[k]$ and we have

$$S\big(\mathfrak{g}[t]_+\big)[k] = \bigoplus_{\mathbf{n}\text{ a partition of }k} V(\mathbf{n}).$$

The result follows since

$$V(\mathbf{n}) \cong_{\mathfrak{g}} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}),$$

where $r_j = |\{1 \leqslant r \leqslant s\colon\ n_r = j\}|$.

Part (v) is now obvious. To establish (vi), it is enough to observe that by (i) the natural map

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}\big(P(\lambda, r), V\big) \to \mathrm{Hom}_{\mathcal{G}_{\leqslant r}}\big(P(\lambda, r)_{\leqslant r}, V_{\leqslant r}\big)$$

is injective and hence is an isomorphism (Lemma 1.5(ii)). The statement follows since $P(\lambda, r)_{\leqslant r} \cong V(\lambda, r)$ is projective in $\mathcal{G}_{\leqslant r}$ and every object in $\mathcal{G}_{\leqslant r}$ has a finite length. $\square$

In what follows, we shall write

$$S^{(k)}(\mathfrak{g}) = \bigoplus_{(r_1,\dots,r_k)\in\mathbf{Z}_+^k\colon\ \sum_{j=1}^k jr_j = k} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}).$$

Observe that $S^{(k)}(\mathfrak{g})$ is a $\mathfrak{g}$-module quotient of $\mathfrak{g}^{\otimes k}$. Indeed, the map $\mathfrak{g}^{\otimes k} \to \mathbf{U}(\mathfrak{g}[t]_+)[k] \cong_{\mathfrak{g}} S^{(k)}(\mathfrak{g})$ given by extending $x_1 \otimes \cdots \otimes x_k \mapsto (x_1 t)\cdots(x_k t)$, $x_j \in \mathfrak{g}$, $1 \leqslant j \leqslant k$ is a surjective $\mathfrak{g}$-module homomorphism.

2.2. *Morphisms between projectives*

The next proposition is trivial, but we include it here explicitly since it is used repeatedly in the later sections where we discuss examples of interesting subcategories of $\mathcal{G}$.

Let $s \leqslant r \leqslant \ell \in \mathbf{Z}_+$. It is immediate from the definitions and Proposition 2.1 that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{G}}\big(P(\lambda, r)_{\leqslant \ell}, P(\mu, s)_{\leqslant \ell}\big) &= \mathrm{Hom}_{\widehat{\mathcal{G}}}\big(P(\lambda, r), P(\mu, s)\big)\\ &\cong \mathrm{Hom}_{\mathfrak{g}}\big(V(\lambda), \mathbf{U}\big(\mathfrak{g}[t]_+\big)[r-s] \otimes V(\mu)\big).\end{aligned} \tag{2.2}$$

In this section we make the last isomorphism explicit. Let $s \leqslant r \in \mathbf{Z}_+$ and $\lambda, \mu \in P^+$. Given $f \in \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \mathbf{U}(\mathfrak{g}[t]_+)[r-s] \otimes V(\mu))$, define $\mathbf{f} \in \mathrm{Hom}_{\mathbf{C}}(P(\lambda, r), P(\mu, s))$ by

$$\mathbf{f}(u \otimes v) = \sum_p u u_p \otimes v_p,$$

where $u \in \mathbf{U}(\mathfrak{g}[t]_+)$, $v \in V(\lambda, r)$ and $\sum_p u_p \otimes v_p = f(v)$. Let $m : \mathbf{U}(\mathfrak{g}[t]) \otimes \mathbf{U}(\mathfrak{g}[t]) \to \mathbf{U}(\mathfrak{g}[t])$ be the multiplication map. The next proposition is a straightforward consequence of Proposition 2.1 and we omit the details of the calculations.

Proposition. *Let* $(\lambda, r), (\mu, s) \in \Lambda$.

(i) *The assignment* $\mathbf{f} \mapsto \mathbf{f}[r]$ *defines an isomorphism of vector spaces*

$$\phi_{(\mu,s)}^{(\lambda,r)} : \mathrm{Hom}_{\widehat{\mathcal{G}}}\big(P(\lambda,r), P(\mu,s)\big) \to \mathrm{Hom}_{\mathfrak{g}}\big(V(\lambda), \mathbf{U}\big(\mathfrak{g}[t]_+\big)[r-s] \otimes V(\mu)\big).$$

Moreover if $\mathbf{f} \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda,r), P(\mu,s))$, $\mathbf{g} \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\mu,s), P(\nu,k))$, *then*

$$(\mathbf{g} \circ \mathbf{f})[r] = (m \otimes 1) \circ \big(1 \otimes \mathbf{g}[s]\big) \circ \mathbf{f}[r].$$

(ii) *The assignment* $f \mapsto \mathbf{f}$ *is an isomorphism*

$$\psi_{(\mu,s)}^{(\lambda,r)} : \mathrm{Hom}_{\mathfrak{g}}\big(V(\lambda), \mathbf{U}\big(\mathfrak{g}[t]_+\big)[r-s] \otimes V(\mu)\big) \to \mathrm{Hom}_{\widehat{\mathcal{G}}}\big(P(\lambda,r), P(\mu,s)\big)$$

which is the inverse of $\phi_{(\mu,s)}^{(\lambda,r)}$. *Moreover, if* $f \in \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \mathbf{U}(\mathfrak{g}[t]_+)[r-s] \otimes V(\mu))$, $g \in \mathrm{Hom}_{\mathfrak{g}}(V(\mu), \mathbf{U}(\mathfrak{g}[t]_+)[s-k] \otimes V(\nu))$ *then*

$$\psi_{(\nu,k)}^{(\lambda,r)}\big((m \otimes 1) \circ (1 \otimes g) \circ f\big) = \psi_{(\nu,k)}^{(\mu,s)}(g) \circ \psi_{(\mu,s)}^{(\lambda,r)}(f).$$

2.3. Duality in $\mathcal{G}_{\leqslant s}$

Given $V, W \in \mathrm{Ob}\,\mathcal{G}$, the vector space $\mathrm{Hom}_{\mathbf{C}}(V, W)$ is a $\mathfrak{g}[t]$-module with respect to the usual action

$$\big(\big(xt^r\big) \cdot f\big)(v) = \big(xt^r\big) f(v) - f\big(\big(xt^r\big)v\big),$$

for all $x \in \mathfrak{g}$, $r \in \mathbf{Z}_+$, $f \in \mathrm{Hom}_{\mathbf{C}}(V, W)$ and $v \in V$. For $k \in \mathbf{Z}_+$, set

$$\mathrm{Hom}_{\mathbf{C}}(V, W)[k] = \bigoplus_{i \in \mathbf{Z}_+} \mathrm{Hom}_{\mathbf{C}}\big(V[i], W[i+k]\big) \tag{2.3}$$

and define

$$\mathrm{Hom}^+_{\mathbf{C}}(V, W) = \bigoplus_{k \in \mathbf{Z}_+} \mathrm{Hom}_{\mathbf{C}}(V, W)[k].$$

Since $V[i] = 0$ for all but finitely many $i \in \mathbf{Z}_+$, $\dim \mathrm{Hom}_{\mathbf{C}}(V, W)[k] < \infty$. Notice that $\mathrm{Hom}^+_{\mathbf{C}}(V, W) = \mathrm{Hom}_{\mathbf{C}}(V, W)$ provided that $V \in \mathrm{Ob}\,\mathcal{G}_{\leqslant r}$ and $W[i] = 0$ for all $i < r$. The proof of the following proposition is quite standard and is omitted.

Proposition. *Let* $V, W \in \mathrm{Ob}\,\mathcal{G}$, $r, s \in \mathbf{Z}_+$.

(i) *For all* $x \in \mathfrak{g}$, $i, r, k \in \mathbf{Z}_+$ *and* $f \in \mathrm{Hom}_{\mathbf{C}}(V[i], W[i+k])$,

$$\big(xt^r\big) \cdot f \in \mathrm{Hom}_{\mathbf{C}}\big(V[i], W[i+k+r]\big) \oplus \mathrm{Hom}_{\mathbf{C}}\big(V[i-r], W[i+k]\big).$$

In particular, $\mathrm{Hom}^+_{\mathbf{C}}(V, W) \in \mathrm{Ob}\,\mathcal{G}$ *and*

$$W \in \mathrm{Ob}\,\mathcal{G}_{\leqslant s} \quad \Rightarrow \quad \mathrm{Hom}^+_{\mathbf{C}}(V, W) \in \mathrm{Ob}\,\mathcal{G}_{\leqslant s}.$$

(ii) *Let* $^{\#_s} : \mathcal{G}_{\leqslant s} \to \mathcal{G}_{\leqslant s}$ *be the contravariant functor defined by* $\mathrm{Hom}_{\mathbf{C}}(-, V(0,s))$. *Then* $^{\#_s}$ *is exact and for all* $\lambda \in P^+$, $r \leqslant s$,

$$\left(V^{\#_r}\right)^{\#_s} \cong \tau_{s-r} V, \qquad V(\lambda, r)^{\#_s} \cong V(-w_\circ\lambda, s-r).$$

In particular, $^{\#_s}$ *defines an involutive auto-duality on the category* $\mathcal{G}_{\leqslant s}$.

(iii) *Suppose that* $V \in \mathrm{Ob}\,\mathcal{G}_{\leqslant r}$, $W \in \mathrm{Ob}\,\mathcal{G}_{\leqslant s}$. *As objects in* $\mathcal{G}$,

$$(V \otimes W)^{\#_{r+s}} \cong V^{\#_r} \otimes W^{\#_s}, \qquad V \otimes W^{\#_s} \cong \mathrm{Hom}_{\mathbf{C}}\big(W, V(0,s) \otimes V\big).$$

(iv) *For all* $V, W' \in \mathrm{Ob}\,\mathcal{G}$, $W \in \mathrm{Ob}\,\mathcal{G}_{\leqslant s}$, *we have an isomorphism of vector spaces,*

$$\mathrm{Hom}_{\mathcal{G}}(V, W \otimes W') \cong \mathrm{Hom}_{\mathcal{G}}\big(V \otimes W^{\#_s}, V(0,s) \otimes W'\big).$$

2.4. *Injective objects in* $\mathcal{G}$ *and* $\widehat{\mathcal{G}}$

We begin with the following remark: any injective object of $\mathcal{G}$ is also injective in $\widehat{\mathcal{G}}$. To prove this, let $I \in \mathrm{Ob}\,\mathcal{G}$ be injective and assume that $I[s] = 0$ for all $s \geqslant r$. Suppose that $\iota \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(V, W)$ is injective and let $f \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(V, I)$. Since $f_{\leqslant r} \in \mathrm{Hom}_{\mathcal{G}}(V_{\leqslant r}, I)$ and $\iota_{\leqslant r} \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(V_{\leqslant r}, W_{\leqslant r})$ is injective there exists $\tilde{f} \in \mathrm{Hom}_{\mathcal{G}}(W_{\leqslant r}, I)$ such that $\tilde{f} \circ \iota_{\leqslant r} = f_{\leqslant r}$. Let $\bar{f} = \tilde{f} \circ p_r(W)$ where $p_r(W) : W \to W_{\leqslant r}$ be the canonical projection. It is now easily checked that $\bar{f} \circ \iota = f$.

For $(\lambda, r) \in \Lambda$, set

$$I(\lambda, r) \cong P(-w_\circ\lambda, 0)_{\leqslant r}{}^{\#_r}.$$

It follows from Propositions 2.5 and 2.3 that $I(\lambda, r) \cong I(\lambda, 0) \otimes V(0, r)$.

Proposition. *Let* $(\lambda, r) \in \Lambda$.

(i) *The object* $I(\lambda, r)$ *is the injective envelope of* $V(\lambda, r)$ *in* $\mathcal{G}$.
(ii) *For* $k \in \mathbf{Z}_+$ *we have*

$$I(\lambda, r)[r-k] \cong_{\mathfrak{g}} S^{(k)}(\mathfrak{g}) \otimes V(\lambda).$$

(iii) *Let* $(\mu, s) \in \Lambda$. *Then* $[I(\lambda, r)/V(\lambda, r) : V(\mu, s)] \neq 0$ *only if* $(\mu, s) \prec (\lambda, r)$.

Proof. It is immediate from Proposition 2.3 that $I(\lambda, r)$ is injective in $\mathcal{G}_{\leqslant r}$. To prove that $I(\lambda, r)$ is injective in $\mathcal{G}$, it suffices now to show that $\mathrm{Ext}^1_{\mathcal{G}}(V(\mu, s), I(\lambda, r)) = 0$ if $s > r$, in other words that every short exact sequence of the form

$$0 \to I(\lambda, r) \to V \to V(\mu, s) \to 0$$

splits if $s > r$. Writing $V = \bigoplus_{k \in \mathbf{Z}_+} V[k]$, we see that $V[k] = 0$ if $k > s$. Hence $\mathfrak{g}[t]V[s] \subset V[s]$. Moreover, since $\bigoplus_{k \leqslant r} V[k] \cong_{\mathcal{G}} I(\lambda, r)$ it follows now that we have a decomposition of $\mathfrak{g}[t]$-modules

$$V \cong I(\lambda, r) \oplus V(\mu, s)$$

and hence the short exact sequence splits. To prove that it is the injective envelope of $V(\lambda, r)$ it suffices to use Proposition 2.1 and Lemma 1.1 to notice that $V(\lambda, r)$ is the unique irreducible subobject of $I(\lambda, r)$.

The proof of (ii) and (iii) is immediate from Lemma 1.1 and Proposition 2.1. □

Corollary. *Let $V, W \in \operatorname{Ob}\mathcal{G}$.*

(i) *For all $j \geqslant 0$, we have*

$$\operatorname{Ext}^j_{\mathcal{G}}(V, W) \cong \operatorname{Ext}^j_{\widehat{\mathcal{G}}}(V, W).$$

(ii) *Let I be the injective envelope of V. If $(\lambda, r) \in \Lambda(I)$ then $(\lambda, r) \preccurlyeq (\mu, s)$ for some $(\mu, s) \in \Lambda(\operatorname{soc}(V))$.*

2.5. Extensions between simple objects

The following proposition proves Theorem 1.

Proposition. *For $(\lambda, r), (\mu, s) \in \Lambda$, we have*

$$\operatorname{Ext}^1_{\mathcal{G}}\big(V(\lambda, r), V(\mu, s)\big) = 0, \quad s \neq r+1,$$
$$\operatorname{Ext}^1_{\mathcal{G}}\big(V(\lambda, r), V(\mu, r+1)\big) \cong \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda), \mathfrak{g} \otimes V(\mu)\big).$$

In other words, $\operatorname{Ext}^1_{\mathcal{G}}(V(\lambda, r), V(\mu, s)) = 0$ unless (μ, s) covers (λ, r).

Proof. Applying $\operatorname{Hom}_{\mathcal{G}}(V(\lambda, r), -)$ to the short exact sequence

$$0 \to V(\mu, s) \to I(\mu, s) \to J(\mu, s) \to 0$$

gives

$$\operatorname{Hom}_{\mathcal{G}}\big(V(\lambda, r), J(\mu, s)\big) \cong \operatorname{Ext}^1_{\mathcal{G}}\big(V(\lambda, r), V(\mu, s)\big).$$

The proposition obviously follows if we prove that

$$\operatorname{Hom}_{\mathcal{G}}\big(V(\lambda, r), J(\mu, s)\big) \cong \begin{cases} \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda), V(\mu)), & \text{if } s = r+1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi : V(\lambda, r) \to J(\mu, s)$ be a non-zero element of $\operatorname{Hom}_{\mathcal{G}}(V(\lambda, r), J(\mu, s))$. It follows from Proposition 2.4 that $(\lambda, r) \prec (\mu, s)$ and hence in particular that $r < s$. Suppose that $r < s-1$. Since $V(\mu, s)$ is essential in $I(\mu, s)$, there exists $V \subset I(\mu, s)$ such that $V/V(\mu, s) \cong \psi(V(\lambda, r))$. Then $V = V[s] \oplus V[r]$ and since $r < s-1$, it follows that $\mathfrak{g}[t]V[r] \subset V$. Hence $V[r]$ is in $\operatorname{soc}(I(\mu, s))$ which is impossible. Thus, $s = r-1$.

The following isomorphisms which are consequences of Proposition 2.4 establish the proposition.

$$\operatorname{Hom}_{\mathfrak g}\big(V(\lambda),\mathfrak g\otimes V(\mu)\big)\cong\operatorname{Hom}_{\mathcal G}\big(V(\lambda,r),\tau_r\operatorname{ev}\big(\mathfrak g\otimes V(\mu)\big)\big),$$
$$\tau_r\operatorname{ev}\big(J(\mu,r+1)[r]\big)\cong_{\mathcal G}\tau_r\operatorname{ev}\big(\mathfrak g\otimes V(\mu)\big),$$
$$\operatorname{Hom}_{\mathcal G}\big(V(\lambda,r),J(\mu,r+1)\big)\cong\operatorname{Hom}_{\mathcal G}\big(V(\lambda,r),\tau_r\operatorname{ev}\big(J(\mu,r+1)[r]\big)\big).\qquad\square$$

2.6. Given $\Gamma\subset\Lambda$, set

$$V_\Gamma^+=\big\{v\in V[s]_\mu\colon\ \mathfrak n^+v=0,\ (\mu,s)\in\Gamma\big\},$$
$$V_\Gamma=\mathbf U(\mathfrak g)V_\Gamma^+,\qquad V^\Gamma=V/V_{\Lambda\setminus\Gamma}.$$

Furthermore, given $f\in\operatorname{Hom}_{\mathcal G}(V,W)$, let $f_\Gamma:=f|_{V_\Gamma}$ and let f^Γ be the induced map $V^\Gamma\to W^\Gamma$. It follows from the definitions that f_Γ and f^Γ are morphisms of graded vector spaces and $\mathfrak g$-modules.

Proposition. *Let $V\in\operatorname{Ob}\widehat{\mathcal G}$ and let Γ be an interval closed subset of Λ.*

(i) *Suppose that for each $(\lambda,r)\in\Lambda(V)\setminus\Gamma$ there exists $(\mu,s)\in\Gamma$ with $(\lambda,r)\prec(\mu,s)$. Then $V_\Gamma\in\operatorname{Ob}\widehat{\mathcal G}[\Gamma]$ and $V/V_\Gamma\in\operatorname{Ob}\widehat{\mathcal G}[\Lambda\setminus\Gamma]$.*

(ii) *Suppose that for each $(\lambda,r)\in\Lambda(V)\setminus\Gamma$ there exists $(\mu,s)\in\Gamma$ with $(\mu,s)\prec(\lambda,r)$. Then $V_{\Lambda\setminus\Gamma}\in\operatorname{Ob}\widehat{\mathcal G}[\Lambda\setminus\Gamma]$ and $V^\Gamma\in\operatorname{Ob}\widehat{\mathcal G}[\Gamma]$.*

(iii) *Let*

$$0\to V\xrightarrow{f}U\xrightarrow{g}W\to0$$

be a short exact sequence in $\mathcal G$. Then U satisfies (i) (*respectively* (ii)) *if and only if V and W satisfy* (i) (*respectively* (ii)) *and*

$$0\to V_\Gamma\xrightarrow{f_\Gamma}U_\Gamma\xrightarrow{g_\Gamma}W_\Gamma\to0\tag{2.4}$$

(*respectively*

$$0\to V^\Gamma\xrightarrow{f^\Gamma}U^\Gamma\xrightarrow{g^\Gamma}W^\Gamma\to0)\tag{2.5}$$

is an exact sequence of objects in $\mathcal G[\Gamma]$.

Proof. Consider the map $\mathfrak g\otimes V_\Gamma\to V$ given by $x\otimes v\mapsto(xt)v$. This is clearly a map of $\mathfrak g$-modules. Let U be a $\mathfrak g$-module complement to V_Γ in V. Suppose that V_Γ is not a subobject of V in $\widehat{\mathcal G}$, i.e. there exists $x\in\mathfrak g$ and $v\in V_\Gamma$ such that $(xt)v\notin V_\Gamma$. Since $(\mathfrak g\otimes t)\mathbf U(\mathfrak g)\subset\mathbf U(\mathfrak g)(\mathfrak g\otimes t)$, we may assume without loss of generality that $v\in V_\Gamma^+\cap V[r]_\lambda$ for some $(\lambda,r)\in\Lambda$ and hence $\mathbf U(\mathfrak g)v\cong_{\mathfrak g}V(\lambda,r)$. In other words, the induced map of $\mathfrak g$-modules $\mathfrak g\otimes V(\lambda,r)\to U$ is non-zero and so there exists $(\nu,r+1)\notin\Gamma$ such that the composite map $\mathfrak g\otimes V(\lambda,r)\to U\to V(\nu,r+1)$ is non-zero. This implies immediately that $(\lambda,r)\prec(\nu,r+1)$. Choose $(\mu,s)\in\Gamma$ such that $(\nu,r+1)\prec(\mu,s)$ for some $(\mu,s)\in\Gamma$. This gives $(\nu,r+1)\in[(\lambda,r),(\mu,s)]$ which is impossible since Γ is interval closed. Hence

$$(xt)v\in V_\Gamma,\quad\forall x\in\mathfrak g,\ v\in V_\Gamma^+,$$

and (i) is proved.

In order to prove the second part, assume that $V_{\Lambda\setminus\Gamma}$ is not a $\widehat{\mathcal{G}}$-subobject of V. Then $(xt)v \notin V_{\Lambda\setminus\Gamma}$ for some $x \in \mathfrak{g}$, $v \in V_{\Lambda\setminus\Gamma}$ and as before we may assume, without loss of generality that $v \in {V_{\Lambda\setminus\Gamma}}^+ \cap V[r]_\lambda$ for some $(\lambda, r) \in \Lambda \setminus \Gamma$. Let U' be a $\mathfrak{g}$-module complement of $V_{\Lambda\setminus\Gamma}$. Then we have a non-zero $\mathfrak{g}$-module map $\mathfrak{g} \otimes V(\lambda, r) \to U'$, given by extending $x \otimes v \mapsto (xt)v$, and hence a non-zero $\mathfrak{g}$-module map $\mathfrak{g} \otimes V(\lambda, r) \to V(\nu, r+1)$ for some $(\nu, r+1) \in \Gamma$. By assumption, there exists $(\mu, s) \in \Gamma$ such that $(\mu, s) \prec (\lambda, r)$. Thus, $(\lambda, r) \in [(\mu, s), (\nu, r+1)] \subset \Gamma$ since Γ is interval closed, which is a contradiction.

The first statement of (iii) is obvious since $\Lambda(U) = \Lambda(V) \cup \Lambda(W)$. For the second, note that we have

$$U \cong V \oplus W,$$

as graded $\mathfrak{g}$-modules and hence as $\mathfrak{g}$-module we have

$$U_\Gamma \cong V_\Gamma \oplus W_\Gamma.$$

Since V_Γ, W_Γ and U_Γ are objects in $\mathcal{G}[\Gamma]$ the result follows. $\square$

Remark. Part (i) of the preceding proposition holds for all $V \in \mathrm{Ob}\,\mathcal{G}$ and $\Gamma \subset \Lambda$ interval closed such that $\Lambda(\mathrm{soc}(V)) \subset \Gamma$ while part (ii) holds if $\Lambda(\mathrm{head}(V)) \subset \Gamma$. In fact, for any $(\lambda, r) \in \Lambda(V)$, there exists $(\mu, s) \in \Lambda(\mathrm{soc}(V))$ and $(\nu, p) \in \Lambda(\mathrm{head}(V))$ such that

$$(\nu, p) \preccurlyeq (\lambda, r) \preccurlyeq (\mu, s).$$

This is an immediate consequence of the fact that V embeds in the injective envelope of $\mathrm{soc}(V)$ and is a quotient of the projective cover of $\mathrm{head}(V)$ together with Propositions 2.1(iii) and 2.4(ii). Moreover, in this case if we let $\overline{V_\Gamma}$ be the maximal subobject of V that is in $\mathcal{G}[\Gamma]$, then Proposition implies that

$$\overline{V_\Gamma} \cong V_\Gamma$$

if for each $(\lambda, r) \in \Lambda(V) \setminus \Gamma$ there exists $(\mu, s) \in \Gamma$ with $(\lambda, r) \preccurlyeq (\mu, s)$ and similarly for $\overline{V}^\Gamma$.

2.7. We isolate some consequences of the preceding proposition since we use them repeatedly in the following sections.

Proposition. *Let $\Gamma \subset \Lambda$ be finite and interval closed and assume that $(\lambda, r), (\mu, s) \in \Gamma$.*

(i) *The object $I(\lambda, r)_\Gamma$ is the injective envelope of $V(\lambda, r)$ in $\mathcal{G}[\Gamma]$ while $P(\lambda, r)^\Gamma$ is the projective cover of $V(\lambda, r)$ in $\mathcal{G}[\Gamma]$. In particular, $\mathcal{G}[\Gamma]$ has enough projectives.*

(ii) *We have*

$$\left[P(\lambda, r)^\Gamma : V(\mu, s)\right] = \left[P(\lambda, r) : V(\mu, s)\right] = \left[I(\mu, s) : V(\lambda, r)\right] = \left[I(\mu, s)_\Gamma : V(\lambda, r)\right].$$

(iii) *For all $j \geqslant 0$, we have*

$$\mathrm{Ext}^j_{\mathcal{G}}\big(V(\lambda, r), V(\mu, s)\big) \cong \mathrm{Ext}^j_{\mathcal{G}[\Gamma]}\big(V(\lambda, r), V(\mu, s)\big).$$

(iv) *Let* $\boldsymbol{p}^\Gamma(\lambda, r) : P(\lambda, r) \twoheadrightarrow P(\lambda, r)^\Gamma$ (*respectively* $\boldsymbol{\iota}_\Gamma(\lambda, r) : I(\lambda, r)_\Gamma \hookrightarrow I(\lambda, r)$) *be the canonical projection* (*respectively the canonical embedding*). *There exists an isomorphism* $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, s)) \to \mathrm{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma)$ *given by* $f \to f^\Gamma$ *such that*

$$\boldsymbol{p}^\Gamma(\mu, s) \circ f = f^\Gamma \circ \boldsymbol{p}^\Gamma(\lambda, r),$$

and similarly an isomorphism $\mathrm{Hom}_{\mathcal{G}}(I(\mu, s), I(\lambda, r)) \to \mathrm{Hom}_{\mathcal{G}[\Gamma]}(I(\mu, s)_\Gamma, I(\lambda, r)_\Gamma)$ *given by* $g \to g_\Gamma$ *such that*

$$g \circ \boldsymbol{\iota}_\Gamma(\mu, s) = \boldsymbol{\iota}_\Gamma(\lambda, r) \circ g_\Gamma.$$

Proof. It follows from Proposition 2.4(iii) that

$$(\nu, k) \prec (\lambda, r), \quad \forall (\nu, k) \in \Lambda\big(I(\lambda, r)\big) \setminus \big\{(\lambda, r)\big\}.$$

Proposition 2.6(i) now gives,

$$I(\lambda, r)_\Gamma \in \mathrm{Ob}\,\mathcal{G}[\Gamma], \qquad \mathrm{soc}\big(I(\lambda, r)_\Gamma\big) = V(\lambda, r)$$

and

$$\mathrm{Hom}_{\mathcal{G}}\big(V(\mu, s), I(\lambda, r)/I(\lambda, r)_\Gamma\big) = 0, \quad \forall (\mu, s) \in \Gamma.$$

It follows immediately that $\mathrm{Ext}^1_{\mathcal{G}[\Gamma]}(V(\mu, s), I(\lambda, r)_\Gamma) = 0$ for all $(\mu, s) \in \Gamma$ which implies the first statement in (i). The proof of the second statement is similar. The first and the last equality in (ii) follow immediately from Proposition 2.6 while the second equality is an obvious consequence of Propositions 2.1(ii), (iv) and 2.4.

Part (iii) is obvious if $j = 0$. Set $Q_{-1}(\mu, s) = V(\mu, s)$. For $j \geqslant 0$ define inductively the objects $I_j(\mu, s)$ as the injective envelope of $Q_{j-1}(\mu, s)$ in $\mathcal{G}$ and $Q_j(\mu, s) = \mathrm{coker}(Q_{j-1}(\mu, s) \hookrightarrow I_j(\mu, s))$. Then

$$0 \to V(\mu, s) \to I_0(\mu, s) \to I_1(\mu, s) \to \cdots \to I_k(\mu, s) \to 0$$

is an injective resolution for $V(\mu, s)$ in $\mathcal{G}$ and

$$\mathrm{Ext}^j_{\mathcal{G}}\big(V(\lambda, r), V(\mu, s)\big) \cong \mathrm{Hom}_{\mathcal{G}}\big(V(\lambda, r), I_j(\mu, s)\big), \quad j > 0. \tag{2.6}$$

It follows from Corollary 2.4(ii) by a straightforward induction on j that

$$(\nu, k) \neq (\mu, s) \in \Lambda\big(I_j(\mu, s)\big) \cup \Lambda\big(Q_j(\mu, s)\big) \quad \Rightarrow \quad (\nu, k) \prec (\mu, s).$$

Hence by Proposition 2.6,

$$Q_j(\mu, s)_\Gamma, I_j(\mu, s)_\Gamma \in \mathrm{Ob}\,\mathcal{G}[\Gamma], \qquad I_j(\mu, s)/I_j(\mu, s)_\Gamma \in \mathcal{G}[\Lambda \setminus \Gamma]$$

and the sequence

$$0 \to V(\mu, s) \to I_0(\mu, s)_\Gamma \to I_1(\mu, s)_\Gamma \to \cdots \to I_k(\mu, s)_\Gamma \to 0 \tag{2.7}$$

is exact in $\mathcal{G}[\Gamma]$. It follows from part (i) that (2.7) is an injective resolution of $V(\mu,s)$ in $\mathcal{G}[\Gamma]$. Furthermore, for all $(\lambda,r)\in\Gamma$

$$\operatorname{Hom}_{\mathcal{G}}\big(V(\lambda,r), I_j(\mu,s)\big)\cong\operatorname{Hom}_{\mathcal{G}}\big(V(\lambda,r), I_j(\mu,s)_\Gamma\big) \tag{2.8}$$

and similarly

$$\operatorname{Hom}_{\mathcal{G}}\big(V(\lambda,r), Q_j(\mu,s)\big)\cong\operatorname{Hom}_{\mathcal{G}}\big(V(\lambda,r), Q_j(\mu,s)_\Gamma\big).$$

In particular, this implies that

$$\operatorname{soc}\big(I_j(\mu,s)_\Gamma\big)\cong\operatorname{soc}\big(Q_{j-1}(\mu,s)_\Gamma\big)$$

and so $I_j(\mu,s)_\Gamma$ is the injective envelope of $Q_{j-1}(\mu,s)_\Gamma$ in $\mathcal{G}[\Gamma]$. By Proposition 2.6(iii), $I_j(\mu,s)_\Gamma/Q_{j-1}(\mu,s)_\Gamma\cong Q_j(\mu,s)_\Gamma$. Then

$$\operatorname{Ext}^j_{\mathcal{G}[\Gamma]}\big(V(\lambda,r), V(\mu,s)\big)\cong\operatorname{Hom}_{\mathcal{G}[\Gamma]}\big(V(\lambda,r), I_j(\mu,s)_\Gamma\big)$$

and (iii) follows from (2.6) and (2.8).

To prove (iv), let $f\in\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda,r),P(\mu,s))$, $f\neq 0$. Then $f^\Gamma\neq 0$ since $f^\Gamma(1\otimes V(\lambda,r))=f(1\otimes V(\lambda,r)) \pmod{P(\lambda,r)_{\Lambda\setminus\Gamma}}\neq 0$. Thus, we have an injective map

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}\big(P(\lambda,r), P(\mu,s)\big)\to\operatorname{Hom}_{\mathcal{G}[\Gamma]}\big(P(\lambda,r)^\Gamma, P(\mu,s)^\Gamma\big).$$

Since both spaces have the same dimension by (ii) and Proposition 2.2, the isomorphism follows. To prove the statement for injectives, observe that the natural map

$$\operatorname{Hom}_{\mathcal{G}}\big(I(\mu,s), I(\lambda,r)\big)\to\operatorname{Hom}_{\mathcal{G}}\big(I(\mu,s)_\Gamma, I(\lambda,r)\big)$$

is surjective, while $\operatorname{Hom}_{\mathcal{G}}(I(\mu,s)_\Gamma, I(\lambda,r)_\Gamma)\cong\operatorname{Hom}_{\mathcal{G}}(I(\mu,s)_\Gamma, I(\lambda,r))$. The assertion follows from (ii). $\square$

3. Algebras associated with the category $\mathcal{G}$

In this section, we let Γ be a finite interval closed subset of Λ. Given a finite-dimensional algebra A, let $A-\operatorname{mod}_f$ (respectively mod_f-A) be the category of finite-dimensional left (respectively right) A-modules.

3.1. The algebra $\mathfrak{A}(\Gamma)$ and an equivalence of categories

Set

$$I(\Gamma)=\bigoplus_{(\lambda,r)\in\Gamma} I(\lambda,r),\qquad \mathfrak{A}(\Gamma)=\operatorname{End}_{\mathcal{G}} I(\Gamma).$$

Then $\mathfrak{A}(\Gamma)$ is an associative algebra. Moreover, it is immediate from Proposition 2.7 that

$$\mathfrak{A}(\Gamma)\cong\mathfrak{A}_\Gamma(\Gamma):=\operatorname{End}_{\mathcal{G}} I(\Gamma)_\Gamma. \tag{3.1}$$

In particular, $\mathfrak{A}(\Gamma) - \mathrm{mod}_f$ is equivalent to $\mathfrak{A}_\Gamma(\Gamma) - \mathrm{mod}_f$ and similarly for the categories of right modules. Since $I(\Gamma)_\Gamma$ is the injective cogenerator of $\mathcal{G}[\Gamma]$, a standard argument now shows that the contravariant functor $\mathrm{Hom}_{\mathcal{G}}(-, I(\Gamma)_\Gamma)$ from $\mathcal{G}[\Gamma]$ to the category $\mathfrak{A}_\Gamma(\Gamma) - \mathrm{mod}_f$ is exact and provides a duality of categories. Similarly, the functor $\mathrm{Hom}_{\mathcal{G}}(-, I(\Gamma)_\Gamma)^*$ from $\mathcal{G}[\Gamma]$ to the category $\mathrm{mod}_f - \mathfrak{A}_\Gamma(\Gamma)$ is exact and provides an equivalence of categories. Thus, $\mathcal{G}[\Gamma]$ is equivalent to $\mathrm{mod}_f - \mathfrak{A}(\Gamma)$ and is dual to $\mathfrak{A}(\Gamma) - \mathrm{mod}_f$.

It is clear from the definition that the simple objects in $\mathfrak{A}(\Gamma) - \mathrm{mod}_f$ are one-dimensional, that is to say $\mathfrak{A}(\Gamma)$ is basic, and their isomorphism classes are parametrized by elements of Γ. Given $(\lambda, r) \in \Gamma$, let $S_{\lambda,r}$ be the corresponding simple left $\mathfrak{A}(\Gamma)$-module.

Proposition. *Let $\Gamma \subset \Lambda$ be finite and interval closed. For all $(\lambda, r), (\mu, s) \in \Gamma$, we have*

$$\dim \mathrm{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,s}) = \delta_{r,s+1} \dim \mathrm{Hom}_{\mathfrak{g}}\big(V(\mu), \mathfrak{g} \otimes V(\lambda)\big). \tag{3.2}$$

In particular, the algebra $\mathfrak{A}(\Gamma)$ is quasi-hereditary.

Proof. Since Γ is interval closed, it follows from Corollary 2.7 that if $(\mu, s), (\lambda, r) \in \Gamma$, then

$$\mathrm{Hom}_{\mathcal{G}[\Gamma]}\big(V(\mu, s), I(\lambda, r)_\Gamma / V(\lambda, r)\big) \cong \mathrm{Hom}_{\mathcal{G}}\big(V(\mu, s), I(\lambda, r)/V(\lambda, r)\big).$$

Hence, we have,

$$\mathrm{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,s}) \cong \mathrm{Ext}^1_{\mathcal{G}[\Gamma]}\big(V(\mu, s), V(\lambda, r)\big) \cong \mathrm{Ext}^1_{\mathcal{G}}\big(V(\mu, s), V(\lambda, r)\big).$$

Equation (3.2) follows from Proposition 2.5 which also proves that $\mathfrak{A}(\Gamma) - \mathrm{mod}_f$ is a directed highest weight category with the poset of weights $(\Gamma, \preccurlyeq^{\mathrm{op}})$. Now [6, Theorem 3.6] implies that the algebra $\mathfrak{A}(\Gamma)$ is quasi-hereditary. □

3.2. Let $Q(\Gamma)$ be the Ext-quiver of $\mathfrak{A}(\Gamma)$, that is, the quiver whose set of vertices is Γ and the number of arrows from (λ, r) to (μ, s) in $Q(\Gamma)$ is $\dim \mathrm{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,s})$. Note that the number of paths from (λ, r) to (μ, s) is non-zero only if $(\mu, s) \preccurlyeq (\lambda, r)$. In particular, $Q(\Gamma)$ has no oriented loops. Let $\mathbf{C}Q(\Gamma)$ be the path algebra of $Q(\Gamma)$ and $\mathbf{C}Q(\Gamma)[k]$ be the subspace spanned by all paths of length k. Then

$$\mathbf{C}Q(\Gamma) = \bigoplus_{k \in \mathbf{Z}_+} \mathbf{C}Q(\Gamma)[k],$$

is a tightly graded associative algebra. Since $\mathfrak{A}(\Gamma)$ is basic, a classical result of Gabriel's (cf. for example [13, 2.1(2)]) proves that $\mathfrak{A}(\Gamma)$ is isomorphic to a quotient of the path algebra $\mathbf{C}Q(\Gamma)$ of $Q(\Gamma)$ by an ideal $R(\Gamma)$ which is contained in the ideal of paths of a length at least two. In particular this means that an arrow between (λ, r) and $(\mu, r-1)$ maps to a non-zero element of $\mathrm{Hom}_{\mathcal{G}}(I(\lambda, r), I(\mu, r-1))$. Given $(\lambda, r) \in \Gamma$, let $1_{\lambda,r}$ be the corresponding primitive idempotent in $\mathbf{C}Q(\Gamma)$. Note that $1_{\lambda,r}$ maps to the element $\mathrm{id}_{\lambda,r} \in \mathrm{End}_{\mathcal{G}} I(\Gamma)$ defined by $\mathrm{id}_{\lambda,r}(I(\mu, s)) = \delta_{(\lambda,r),(\mu,s)} \mathrm{id}$ for $(\mu, s) \in \Gamma$. In particular, $\mathrm{id}_{\mu,s} \mathfrak{A}(\Gamma) \mathrm{id}_{\lambda,r} \cong \mathrm{Hom}_{\mathcal{G}}(I(\lambda, r), I(\mu, s))$ as a vector space.

3.3. A grading on $\mathfrak{A}(\Gamma)$

Given $k \leqslant r \in \mathbf{Z}_+$ define

$$\mathfrak{A}(\Gamma)[k] = \bigoplus_{(\lambda,r),(\mu,r-k)\in\Gamma} \mathrm{Hom}_{\mathcal{G}}\big(I(\lambda,r), I(\mu,r-k)\big).$$

Since $[I(\lambda,r) : V(\mu,s)] = 0$ unless $(\mu,s) \preccurlyeq (\lambda,r)$ (cf. Proposition 2.4(iii)), it follows immediately that

$$\mathfrak{A}(\Gamma) = \bigoplus_{k\in\mathbf{Z}_+} \mathfrak{A}(\Gamma)[k], \qquad \mathfrak{A}(\Gamma)[j]\mathfrak{A}(\Gamma)[k] \subset \mathfrak{A}(\Gamma)[j+k], \quad \forall j,k \in \mathbf{Z}_+.$$

Thus, $\mathfrak{A}(\Gamma)$ is a graded associative algebra and $\mathfrak{A}(\Gamma)[0]$ is a commutative semi-simple subalgebra of $\mathfrak{A}(\Gamma)$. It is trivial to observe that with this grading the algebra $\mathfrak{A}(\Gamma)$ is in fact a graded quotient of $\mathbf{C}Q(\Gamma)$ and hence the ideal $R(\Gamma)$ is graded. In particular, $\mathfrak{A}(\Gamma)$ is tightly graded.

3.4. The dimension of $R(\Gamma)$

Proposition. *Let Γ be interval closed and finite and $(\lambda,r), (\mu,s) \in \Gamma$. Then the number of paths from (λ,r) to (μ,s) in $Q(\Gamma)$ is* $\dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), \mathfrak{g}^{\otimes(r-s)} \otimes V(\lambda))$.

Proof. Let $N((\lambda,r),(\mu,r-s))$ be the number of paths in $Q(\Gamma)$ from (λ,r) to $(\mu,r-s)$ and set $N'(\lambda,\mu,s) = \dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), \mathfrak{g}^{\otimes s} \otimes V(\lambda))$. It is easy to see that

$$N'(\lambda,\mu,0) = \delta_{\lambda,\mu} = N\big((\lambda,r),(\mu,r)\big),$$

while

$$\begin{aligned} N'(\lambda,\mu,1) &= \dim \mathrm{Hom}_{\mathfrak{g}}\big(V(\mu), \mathfrak{g}\otimes V(\lambda)\big) = \dim \mathrm{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,r-1}) \\ &= N\big((\lambda,r),(\mu,r-1)\big), \end{aligned}$$

where we used (3.2). We now prove that N and N' satisfy the same recurrence relation which establishes the proposition. It is clear that

$$N\big((\lambda,r),(\mu,r-s)\big) = \sum_{\nu\in P^+} N\big((\nu,r-s+1),(\mu,r-s)\big)N\big((\lambda,r),(\nu,r-s+1)\big).$$

On the other hand, note that we can write

$$\mathfrak{g}^{\otimes(s-1)} \otimes V(\lambda) \cong \bigoplus_{\nu\in P^+} V(\nu)^{N'(\lambda,\nu,s-1)}.$$

Tensoring with $\mathfrak{g}$ gives,

$$N'(\lambda,\mu,s) = \dim \bigoplus_{\nu\in P^+} \mathrm{Hom}_{\mathfrak{g}}\big(V(\mu), \mathfrak{g}\otimes V(\nu)\big)^{N'(\lambda,\nu,s-1)} = \sum_{\nu\in P^+} N'(\lambda,\nu,s-1)N'(\nu,\mu,1),$$

and the proof is complete. $\square$

Corollary. *Given $(\mu, r-s), (\lambda, r) \in \Gamma$, we have*

$$\dim 1_{\mu,r-s} R(\Gamma) 1_{\lambda,r} = \dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\mu), \mathfrak{g}^{\otimes s} \otimes V(\lambda)\big) - \dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\mu), S^{(s)}(\mathfrak{g}) \otimes V(\lambda)\big).$$

In particular, the algebra $\mathfrak{A}(\Gamma)$ is hereditary if and only if

$$\dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\mu), \mathfrak{g}^{\otimes s} \otimes V(\lambda)\big) = \dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\mu), S^{(s)}(\mathfrak{g}) \otimes V(\lambda)\big)$$

for all $(\mu, r-s), (\lambda, r) \in \Gamma$.

Proof. Observe that

$$\begin{aligned}\dim 1_{\mu,r-s} R(\Gamma) 1_{\lambda,r} &= N\big((\lambda, r), (\mu, r-s)\big) - \dim \mathrm{id}_{\mu,r-s} \mathfrak{A}(\Gamma)\, \mathrm{id}_{\lambda,r} \\ &= N\big((\lambda, r), (\mu, r-s)\big) - \dim \operatorname{Hom}_{\mathcal{G}}\big(I(\lambda, r), I(\mu, r-s)\big).\end{aligned}$$

The first assertion is now immediate from the above Proposition and Proposition 2.4(ii). For the second, it is enough to observe that $\mathfrak{A}(\Gamma)$ is hereditary if and only if $R(\Gamma) = 0$ and that $R(\Gamma) = \bigoplus_{(\lambda,r),(\mu,r-s)\in\Gamma} 1_{\mu,r-s} R(\Gamma) 1_{\lambda,r}$. □

3.5. Given $\Gamma \subset \Lambda_{\leqslant r}$, let $\Gamma^{\#_r} = \{(-w_\circ\mu, r-s)\colon (\mu, s) \in \Gamma\}$. It is easy to see that $\Gamma^{\#_r}$ is interval closed if and only if Γ is interval closed.

Proposition. *Suppose that $\Gamma \subset \Lambda_{\leqslant r}$ is finite and interval closed. Then $\mathfrak{A}(\Gamma^{\#_r}) \cong \mathfrak{A}(\Gamma)^{\mathrm{op}}$.*

Proof. Let $(\lambda, r) \in \Gamma$. Then $V(\lambda, r)^{\#_r}$ is an object in $\mathcal{G}[\Gamma^{\#_r}]$ and it follows that $(\mathcal{G}[\Gamma])^{\#_r} = \mathcal{G}[\Gamma^{\#_r}]$. Thus, $\mathcal{G}[\Gamma]$ is dual to $\mathcal{G}[\Gamma^{\#_r}]$. It follows from 3.1 that $\mathfrak{A}(\Gamma)^{\mathrm{op}}$ and $\mathfrak{A}(\Gamma^{\#_r})$ are Morita equivalent. Since they are both basic, the assertion follows. □

4. Examples of Γ with $\mathfrak{A}(\Gamma)$ hereditary

Throughout this section we use the notations of [13] for the various types of quivers. This should eliminate confusion with the notation for the types of simple Lie algebras. For instance, $\mathbb{T}_{n_1,\dots,n_r}$ denotes a quiver whose underlying graph is a star with r branches where the ith branch contains n_i vertices, while $\tilde{\mathbb{X}}_k$ denotes the quiver whose underlying graph is of affine Dynkin type $X_k^{(1)}$. Notice that if $\Gamma \subset \Lambda_{\leqslant r}$, then Proposition 3.5 implies that $Q(\Gamma^{\#_r})$ is the opposite quiver of $Q(\Gamma)$.

4.1. The generalized Kronecker quivers

Let $\lambda \in P^+$ be non-zero and let

$$k_\lambda = \big|\big\{i \in I\colon \lambda(h_i) > 0\big\}\big|.$$

It is easily checked, by using Lemma 1.1 that

$$\dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda), \mathfrak{g} \otimes V(\lambda)\big) = k_\lambda.$$

For $r \in \mathbf{Z}_+$, set $\Gamma_{\lambda,r} = \{(\lambda, r), (\lambda, r+1)\}$. It follows that $Q(\Gamma_{\lambda,r})$ is the quiver with k_λ arrows from $(\lambda, r+1)$ to (λ, r). If $k_\lambda = 1$, the quiver is of type $\mathbb{A}_2$, if $k_\lambda = 2$ it is the Kronecker quiver $\tilde{\mathbb{A}}_1$, while for $k_\lambda > 2$ we get the generalized Kronecker quiver. Since there are no paths of length two in these quivers, it follows that $\mathfrak{A}(\Gamma_{\lambda,r}) \cong \mathbf{C}Q(\Gamma_{\lambda,r})$.

4.2. *Quivers of type* $\tilde{\mathbb{D}}_4$

Suppose that $\mathfrak{g}$ is not of type $\mathfrak{sl}_2$. Let $I_\bullet = \{i \in I\colon\ \theta - \alpha_i \in R^+\}$. Note that $|I_\bullet| = 1$ if $\mathfrak{g}$ is not of type $\mathfrak{sl}_{n+1}$ and let $i_\bullet$ be the unique element of $I_\bullet$. If $\mathfrak{g} \cong \mathfrak{sl}_{n+1}$, $n > 1$, $I_\bullet = \{1, n\}$.

Let $r \in \mathbf{Z}_+$. If $\mathfrak{g}$ is not of type $\mathfrak{sl}_{n+1}$ set

$$\Gamma = \bigl\{(\theta, r), (0, r+1), (2\theta, r+1), (2\theta - \alpha_{i_\bullet}, r+1), (\theta, r+1)\bigr\}.$$

Otherwise, set

$$\Gamma = \bigl\{(\theta, r), (0, r+1), (2\theta, r+1), (2\theta - \alpha_1, r+1), (2\theta - \alpha_n, r+1)\bigr\}.$$

In the first case, we find that $Q(\Gamma)$ is

$$\begin{tikzcd}
 & (2\theta - \alpha_{i_\bullet}, r+1) \arrow[d] & \\
(0, r+1) \arrow[r] & (\theta, r) & (2\theta, r+1) \arrow[l] \\
 & (\theta, r+1) \arrow[u] &
\end{tikzcd}$$

while in the second case $Q(\Gamma)$ is

$$\begin{tikzcd}
 & (2\theta - \alpha_1, r+1) \arrow[d] & \\
(0, r+1) \arrow[r] & (\theta, r) & (2\theta, r+1) \arrow[l] \\
 & (2\theta - \alpha_n, r+1). \arrow[u] &
\end{tikzcd}$$

The algebra $\mathfrak{A}(\Gamma)$ is hereditary since any path in $Q(\Gamma)$ has a length at most one. Note that Γ can be shifted by any $\lambda \in P^+$ such that $\dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda + \theta), \mathfrak{g} \otimes V(\lambda + \theta)) = 1$.

4.3. *Quivers of type* $\mathbb{A}_\ell$

If $\mathfrak{g}$ is not of type $\mathfrak{sl}_2$ choose $i_\bullet \in I_\bullet$.

Proposition. *Fix $\lambda \in P^+$ with $\lambda(h_{i_\bullet}) \neq 0$, $\ell \in \mathbf{Z}_+$. Let $r_j \in \mathbf{Z}_+$, $0 \leqslant j \leqslant \ell$ be such that $|r_k - r_{k+1}| = 1$ for $0 \leqslant k \leqslant \ell - 1$. Let $\alpha \in \{\theta, \theta - \alpha_{i_\bullet}\}$. The set*

$$\Gamma = \big\{(\lambda + j\alpha, r_j)\colon\ 0 \leqslant j \leqslant \ell\big\}$$

is interval closed, the quiver $Q(\Gamma)$ is of type $\mathbb{A}_{\ell+1}$ and the algebra $\mathfrak{A}(\Gamma)$ is hereditary.

Proof. We prove the proposition in the case when $\alpha = \theta - \alpha_{i_\bullet}$, the proof in the other case being similar and in fact simpler. Suppose that for some $0 \leqslant j, j' \leqslant \ell$ and $(\mu, s) \in \Lambda$, we have,

$$(\lambda + j\alpha, r_j) \prec (\mu, s) \prec (\lambda + j'\alpha, r_{j'}).$$

Then $r_j < s < r_{j'}$ and

$$\mu = \lambda + j'\alpha - \sum_{p=1}^{r_{j'}-s} \beta_p = \lambda + j\alpha + \sum_{q=1}^{s-r_j} \gamma_q,$$

for some $\beta_p, \gamma_q \in R \sqcup \{0\}$, $1 \leqslant p \leqslant r_{j'} - s$, $1 \leqslant q \leqslant s - r_j$. Assuming without loss of generality that $j \leqslant j'$, we find in particular, $0 < r_{j'} - r_j \leqslant j' - j$ and

$$(j' - j)(\theta - \alpha_{i_\bullet}) = \sum_{p=1}^{r_{j'}-s} \beta_p + \sum_{q=1}^{s-r_j} \gamma_q.$$

Equating the coefficients of α_i, $i \neq i_\bullet$ on both sides of the above expression we conclude that

$$\beta_p, \gamma_q \in \{\theta, \theta - \alpha_{i_\bullet}\}, \quad r_{j'} - r_j = j' - j,$$

for $1 \leqslant p \leqslant r_{j'} - s$ and $1 \leqslant q \leqslant s - r_j$, which gives

$$r_p = r_j + (p - j), \quad j \leqslant p \leqslant j'.$$

Next, equating the coefficient of $\alpha_{i_\bullet}$ on both sides now gives $\beta_p = \gamma_q = \theta - \alpha_{i_\bullet}$ for all $1 \leqslant p \leqslant r_{j'} - s$, $1 \leqslant q \leqslant s - r_j$. This proves that

$$(\mu, s) = (\lambda + s\alpha, s) = (\lambda + s\alpha, r_s),$$

and hence $(\mu, s) \in \Gamma$.

It follows from Proposition 3.1 that

$$\dim \operatorname{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda+j\alpha,r_j}, S_{\lambda+k\alpha,r_k}) = \delta_{r_j-r_k,1} \dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda + k\alpha), \mathfrak{g} \otimes V(\lambda + j\alpha)\big),$$

and applying Lemma 1.1 now gives

$$\dim \operatorname{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda+j\alpha,r_j}, S_{\lambda+k\alpha,r_k}) = \delta_{r_j-r_k,1}\delta_{|j-k|,1}.$$

This shows that there is precisely one arrow between $(\lambda + j\alpha, r_j)$ and $(\lambda + (j \pm 1)\alpha, r_{j\pm1})$ and no other arrow which has $(\lambda + j\alpha, r_j)$ as its head or tail. Therefore, $Q(\Gamma)$ is of type $\mathbb{A}_{\ell+1}$.

To prove that the algebra $\mathfrak{A}(\Gamma)$ is hereditary, let $(\lambda + k\alpha, r_k)$, $(\lambda + k'\alpha, r_{k'}) \in \Gamma$. The number of paths in $Q(\Gamma)$ between these vertices is zero unless $(\lambda + k\alpha, r_k)$, $(\lambda + k'\alpha, r_{k'})$ are strictly comparable. Assume without loss of generality that $(\lambda + k\alpha, r_k) \prec (\lambda + k'\alpha, r_{k'})$ and also that $k \leqslant k'$. But in this case, we have proved that $r_k = r_{k'} - k' + k$ and that there is exactly one path from $(\lambda + k'\alpha, r_{k'})$ to $(\lambda + k\alpha, r_{k'} - k' + k)$. The result now follows from Corollary 3.4 if we prove that

$$\dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda + k\alpha), S^{(k'-k)}(\mathfrak{g}) \otimes V(\lambda + k'\alpha)\big) = 1.$$

Since $\operatorname{Hom}_{\mathfrak{g}}(V((k-k')\theta), S^{(k'-k)}(\mathfrak{g})) \neq 0$ it suffices to prove that

$$\dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda + k\alpha), V\big((k-k')\theta\big) \otimes V(\lambda + k'\alpha)\big) = 1.$$

But this again follows from Lemma 1.1. $\square$

Remark. The restriction $\lambda(h_{i_\bullet}) \neq 0$ is not necessary if $\alpha = \theta$.

4.4. Quivers of type $\tilde{\mathbb{D}}_{\ell+1}$, $\ell \geqslant 4$

The arguments given in the previous section can be used with obvious modifications to prove the following. Let $\alpha = \theta - \alpha_{i_\bullet}$, $\ell \in \mathbf{Z}_+$ with $\ell \geqslant 4$ and $\lambda \in P^+$. Let $r_j \in \mathbf{Z}_+$, $2 \leqslant j \leqslant \ell - 1$ be such that $|r_k - r_{k+1}| = 1$ for $2 \leqslant k \leqslant \ell - 2$ and let

$$\Gamma = \Gamma_1 \cup \Gamma_2$$

where $\Gamma_1 = \{(\lambda + j\theta, r_j)\colon 2 \leqslant j \leqslant \ell - 1\}$ and

$$\Gamma_2 = \big\{(\lambda + \theta, r_2 + 1), (\lambda + \theta + \alpha, r_2 + 1), (\lambda + \ell\theta, r_{\ell-1} - 1), \big(\lambda + (\ell - 1)\theta + \alpha, r_{\ell-1} - 1\big)\big\}.$$

Then Γ is interval closed and the quiver $Q(\Gamma)$ is of type $\tilde{\mathbb{D}}_{\ell+1}$, where the set Γ_2 consists of precisely those vertices which are either the head or the tail of precisely one arrow. Moreover, the algebra $\mathfrak{A}(\Gamma)$ is hereditary.

4.5. Star-shaped quivers

Suppose that $\mathfrak{g}$ is not of type C_n. Fix $\lambda \in P^+$ and $\ell_1 \geqslant \ell_2 \geqslant \ell_3 > 0 \in \mathbf{Z}$. For $1 \leqslant p \leqslant 3$ and $0 \leqslant j \leqslant \ell_p$, let $r_{p,j} \in \mathbf{Z}_+$ be such that $|r_{p,j} - r_{p,j+1}| = 1$. Let

$$\begin{aligned}\Gamma_1 &= \big\{\big(\lambda + (\ell_1 - j)\theta, r_{1,j}\big)\colon 0 \leqslant j \leqslant \ell_1\big\},\\ \Gamma_2 &= \big\{\big(\lambda + (\ell_1 + j)\theta, r_{2,j}\big)\colon 0 \leqslant j \leqslant \ell_2\big\},\\ \Gamma_3 &= \big\{\big(\lambda + (\ell_1 + j)\theta - j\alpha_{i_\bullet}, r_{3,j}\big)\colon 0 \leqslant j \leqslant \ell_3 - 1\big\}.\end{aligned}$$

Set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Proposition. *Suppose that for all* $1 \leqslant p \leqslant 3$ *and* $0 \leqslant j_p < \ell_p$ *we have*

$$|r_{3,j_3} - r_{2,j_2}| \leqslant |j_3 - j_2|, \qquad |r_{3,j_3} - r_{1,j_1}| \leqslant |j_3 - j_1|.$$

Then Γ *is interval closed,* $Q(\Gamma)$ *is of type* $\mathbb{T}_{\ell_1,\ell_2,\ell_3}$ *and* $\mathfrak{A}(\Gamma)$ *is hereditary.*

Proof. Note that the conditions imply that

$$r_{1,j} = r_{2,j} = r_{3,j}, \quad 0 \leqslant j \leqslant \ell_3, \qquad \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 = \big\{(\lambda + \ell_1\theta, r_{1,0})\big\}.$$

Using Proposition 4.3 we see that $\Gamma_1 \cup \Gamma_2$ and Γ_3 are interval closed and that $\mathfrak{A}(\Gamma_1 \cup \Gamma_2)$ and $\mathfrak{A}(\Gamma_3)$ are hereditary while $Q(\Gamma_1 \cup \Gamma_2)$ and $Q(\Gamma_3)$ are, respectively, of type $\mathbb{A}_{\ell_1+\ell_2-1}$ and $\mathbb{A}_{\ell_3}$. The proposition follows at once if we prove that an element in Γ_3 is not comparable in the partial order $\preccurlyeq$ to any element in $\Gamma_1 \cup \Gamma_2$ except possibly to $(\lambda + \ell_1\theta, r_{1,0})$.

Suppose first that $(\lambda + (\ell_1 + j_3)\theta - j_3\alpha_{i_\bullet}, r_{3,j_3})$ is strictly comparable with $(\lambda + (\ell_1 + j_2)\theta, r_{2,j_2})$. Then $0 < |r_{2,j_2} - r_{3,j_3}| \leqslant |j_3 - j_2|$ and

$$(j_2 - j_3)\theta + j_3\alpha_{i_\bullet} = \sum_{p=1}^{|r_{2,j_2} - r_{3,j_3}|} \beta_p, \qquad \beta_p \in R \sqcup \{0\}. \tag{4.1}$$

If $j_3 > j_2$, then we see by comparing the coefficients of α_i with $i \neq i_\bullet$ on both sides, that $\beta_p \in \{-\theta, -\theta + \alpha_{i_\bullet}\}$ and that $|r_{2,j_2} - r_{3,j_3}| = j_3 - j_2$. Suppose that $-\theta$ occurs s times in the set $\{\beta_p\colon 1 \leqslant p \leqslant j_3 - j_2\}$. Then $0 \leqslant s \leqslant j_3 - j_2$ and $-\theta + \alpha_{i_\bullet}$ occurs $j_3 - j_2 - s$ times. It follows that $-(j_3 - j_2)\theta + j_3\alpha_{i_\bullet} = \sum_p \beta_p = -s\theta + (j_3 - j_2 - s)(-\theta + \alpha_{i_\bullet}) = -(j_3 - j_2)\theta + (j_3 - j_2 - s)\alpha_{i_\bullet}$, which implies $j_2 = s = 0$.

Furthermore, suppose that $j_2 > j_3$. By comparing the coefficients of α_i with $i \neq i_\bullet$ in both sides of (4.1), we conclude that $\beta_p \in \{\theta, \alpha_{i_\bullet}\}$ and $|r_{2,j_2} - r_{3,j_3}| = j_2 - j_3$. Suppose that $\alpha_{i_\bullet}$ occurs s' times. Then we must have $s' = j_3$ and $j_2 - j_3 - s' = (j_2 - j_3)$ which is only possible if $s' = j_3 = 0$.

Suppose now that $(\lambda + (k + j_3)\theta - j_3\alpha_{i_\bullet}, r_{3,j_3})$ is strictly comparable with $(\lambda + (k - j_1)\theta, r_{1,j_1})$. Then we must have $0 \leqslant r = |r_{3,j_3} - r_{1,j_1}| \leqslant |j_3 - j_1|$ and

$$(j_1 + j_3)\theta - j_3\alpha_{i_\bullet} = \sum_{p=1}^{r} \gamma_p, \quad \gamma_p \in R \sqcup \{0\}.$$

Comparing the coefficients of α_i, $i \neq i_\bullet$ in both sides shows that $\beta_p \in \{\theta, \theta - \alpha_{i_\bullet}\}$. If θ appears s times, then we have

$$r\theta - (r - s)\alpha_{i_\bullet} = (j_1 + j_3)\theta - j_3\alpha_{i_\bullet},$$

which implies $r = j_1 + j_3$ and $r - s = j_3$. Since $r \leqslant |j_1 - j_3|$ the first equality implies that either $j_1 = 0$ or $j_3 = 0$. $\square$

5. Quivers with relations

In this section we give an example of Γ for which $\mathfrak{A}(\Gamma)$ is not hereditary. The example is motivated by a family of $\mathfrak{g}[t]$-modules called the Kirillov–Reshetikhin modules (cf. [4]). We assume in this section that $\mathfrak{g}$ is of type D_n, $n \geqslant 6$. Recall that for $i \in I$ with $i \neq n-1, n$ we have

$$\omega_i = \sum_{j=1}^{i} j\alpha_j + i \sum_{j=i+1}^{n-2} \alpha_j + \frac{i}{2}(\alpha_{n-1} + \alpha_n).$$

Let Γ be the interval $[(2\omega_4, 0), (0, 4)]$. It is easily checked that

$$\Gamma = \bigl\{(2\omega_4, 0), (\omega_2 + \omega_4, 1), (\omega_4, 2), (2\omega_2, 2), (\omega_1 + \omega_3, 2), (\omega_2, 3), (0, 4)\bigr\}.$$

Since $\mathfrak{g} \cong_{\mathfrak{g}} V(\omega_2)$, it is now not hard to see by using Proposition 3.2 that the quiver $Q(\Gamma)$ is as follows:

$$\begin{tikzcd}
& (0,4) \arrow[d, "a"'] & \\
& (\omega_2,3) \arrow[dl, "b_1"'] \arrow[d, "b_2"'] \arrow[dr, "b_3"] & \\
(\omega_4,2) \arrow[dr, "c_1"'] & (2\omega_2,2) \arrow[d, "c_2"'] & (\omega_1+\omega_3,2) \arrow[dl, "c_3"] \\
& (\omega_2+\omega_4,1) \arrow[d, "d"'] & \\
& (2\omega_4,0). &
\end{tikzcd}$$

The path algebra $\mathbf{C}Q(\Gamma)$ has a basis consisting of the paths of length at most four which we list below for the reader's convenience:

$$\bigl\{1_{\lambda,r}\colon (\lambda,r) \in \Gamma\bigr\}, \qquad \{a, b_i, c_i, d\colon 1 \leqslant i \leqslant 3\},$$
$$\{b_i a, dc_i, c_i b_i\colon 1 \leqslant i \leqslant 3\}, \qquad \{c_i b_i a, dc_i b_i\colon 1 \leqslant i \leqslant 3\}, \qquad \{dc_i b_i a\colon 1 \leqslant i \leqslant 3\}.$$

We now compute the dimension of $\dim 1_{\mu,s} R(\Gamma) 1_{\lambda,r}$ for $(\mu,s), (\lambda,r) \in \Gamma$ with $r - s \geqslant 2$. By Corollary 3.4 it suffices to calculate $\dim \operatorname{Hom}_{\mathcal{G}}(I(\lambda,r), I(\mu,s))$. Using Proposition 2.7(ii) and the graded characters of injective envelopes of simples in $\mathcal{G}[\Gamma]$ listed in Appendix A.1 we find that

$$\dim 1_{\mu,s} R(\Gamma) 1_{\lambda,r} = 1,$$

if

$$\big((\lambda,r),(\mu,s)\big)\in\big\{\big((0,4),(\omega_1+\omega_3,2)\big),\big((\omega_2,3),(\omega_2+\omega_4,1)\big),\big((\omega_1+\omega_3,2),(2\omega_4,0)\big)\big\},$$

and

$$\dim 1_{\mu,s}R(\Gamma)1_{\lambda,r}=2,$$

if

$$\big((\lambda,r),(\mu,s)\big)\in\big\{\big((0,4),(\omega_2+\omega_4,1)\big),\big((0,4),(2\omega_4,0)\big)\big\},$$

while $\dim 1_{\mu,s}R(\Gamma)1_{\lambda,r}=0$ otherwise. This implies that there exists a unique (up to multiplication by non-zero constants) choice of complex numbers x_i, $1\leqslant i\leqslant 3$ and ξ_j,ζ_j,η_j, $j=1,2$ such that $\mathfrak{A}(\Gamma)$ is the quotient of $\mathbf{C}Q(\Gamma)$ by the following relations

$$b_3a=0=dc_3,\qquad x_1c_1b_1+x_2c_2b_2+x_3c_3b_3=0,\tag{5.1}$$

$$c_3b_3a=0,\qquad \xi_1c_1b_1a+\xi_2c_2b_2a=0,\qquad \zeta_1dc_1b_1+\zeta_2dc_2b_2=0,\tag{5.2}$$

and

$$\eta_1dc_1b_1a+\eta_2dc_2b_2a=0,\qquad dc_3b_3a=0.\tag{5.3}$$

Proposition. *The algebra $\mathfrak{A}(\Gamma)$ is the quotient of $\mathbf{C}Q(\Gamma)$ by the relations*:

$$b_3a=0,\qquad dc_3=0,\qquad c_1b_1+c_2b_2+c_3b_3=0.\tag{5.4}$$

In particular $\mathfrak{A}(\Gamma)$ is quadratic, of global dimension 2 and of tame representation type.

Remark. It is not hard to see by using the results of [4] and the equivalence of categories between $\mathfrak{A}(\Gamma)$-modules and $\mathcal{G}[\Gamma]$, that the projective cover in $\mathfrak{A}(\Gamma)-\mathrm{mod}_f$ of $S_{0,4}$ or the injective envelope of $S_{2\omega_4,0}$ corresponds to the Kirillov–Reshetikhin module $KR(2\omega_4)$, which is thus injective and projective in $\mathcal{G}[\Gamma]$. This connection will be pursued elsewhere.

Proof. The relations in (5.4) are clearly independent of each other. To see that all relations in $\mathfrak{A}(\Gamma)$ are consequences of those in (5.4) it is enough to prove that the space spanned by b_ic_i, b_jc_j with $1\leqslant i<j\leqslant 3$ is always of dimension two. Using the equivalence of categories, Propositions 3.5, 2.4 and 2.3(ii) this can be reformulated into the following question on morphisms in $\mathcal{G}$. Thus, for $\mu\in\{2\omega_2,\omega_4,\omega_1+\omega_3\}$ fix non-zero elements $f_\mu\in\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\omega_2+\omega_4,2),P(\mu,1))$ and $g_\mu\in\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\mu,1),P(\omega_2,0))$. We have to prove that the elements $g_\mu f_\mu$ and $g_\lambda f_\lambda$ are linearly independent in $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\omega_2+\omega_4,2),P(\omega_2,0))$. In turn, using Proposition 2.2 this question translates into the following question in the category $\mathcal{F}(\mathfrak{g})$. Let $\bar{f}_\mu$, $\bar{g}_\mu$ be the restrictions of f_μ and g_μ to $V(\omega_2+\omega_4)$ and $V(\mu)$, respectively, and $\mathbf{p}:\mathfrak{g}^{\otimes 3}\to S^2(\mathfrak{g})\otimes\mathfrak{g}$ be the canonical projection. Then $\mathbf{p}\circ(1\otimes\bar{g}_\mu)\circ\bar{f}_\mu$ and $\mathbf{p}\circ(1\otimes\bar{g}_\lambda)\circ\bar{f}_\lambda$ are linearly independent elements of $\mathrm{Hom}_{\mathfrak{g}}(V(\omega_2+\omega_4),S^2(\mathfrak{g})\otimes\mathfrak{g})$. This is done by an explicit computation of the maps, and the details can be found in Appendix A.2.

Since $\mathfrak{A}(\Gamma)$ is quadratic, it follows from [1, Theorem 1.1] that $\operatorname{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,s}) = 0$ unless $r = s+2$. We have

$$\begin{aligned}\dim \operatorname{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{0,4}, S_{\omega_1+\omega_3,2}) &= \dim \operatorname{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\omega_2,3}, S_{\omega_2+\omega_4,1}) \\ &= \dim \operatorname{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\omega_1+\omega_3,2}, S_{2\omega_4,0}) = 1\end{aligned}$$

and $\operatorname{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,r-2}) = 0$ in all other cases. We claim that $\operatorname{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,s}) = 0$ for all $(\lambda, r), (\mu, s) \in \Gamma$. Indeed, by [1, Theorem 1.1]

$$\begin{aligned}&\dim \operatorname{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{\lambda,r}, S_{\mu,s}) \\ &\quad = \dim 1_{\mu,s}\big(\big(\mathbf{C}Q(\Gamma)_+R(\Gamma) \cap R(\Gamma)\mathbf{C}Q(\Gamma)_+\big)/\big(R(\Gamma)^2 + \mathbf{C}Q(\Gamma)_+R(\Gamma)\mathbf{C}Q(\Gamma)_+\big)\big)1_{\lambda,r},\end{aligned}$$

where $\mathbf{C}Q(\Gamma)_+$ is the radical of $\mathbf{C}Q(\Gamma)$. If $r - s < 4$, it is clear that

$$\dim 1_{\mu,s}\big(\mathbf{C}Q(\Gamma)_+R(\Gamma) \cap R(\Gamma)\mathbf{C}Q(\Gamma)_+\big)1_{\lambda,r} = 0.$$

For $r - s = 4$, we have a unique pair $(\lambda, r), (\mu, r-4) \in \Gamma$, namely $(0, 4)$ and $(2\omega_4, 0)$, and two linearly independent elements in $1_{2\omega_4,0}(\mathbf{C}Q(\Gamma)_+R(\Gamma) \cap R(\Gamma)\mathbf{C}Q(\Gamma)_+)1_{0,4}$, namely dc_3b_3a and $dc_2b_2a + dc_1b_1a$. The first is contained in $R(\Gamma)^2$, since it can be written as $(dc_3)(b_3a)$ and $dc_3, b_3a \in R(\Gamma)$, while the second is contained in $\mathbf{C}Q(\Gamma)_+R(\Gamma)\mathbf{C}Q(\Gamma)_+$ since it can be written as $d(c_1b_1 + c_2b_2 + c_3b_3)a$. Thus, $\dim \operatorname{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{0,4}, S_{2\omega_4,0}) = 0$.

It remains to prove that the algebra is tame. Let $\Gamma_0 = \Gamma \setminus \{(2\omega_4, 0), (0, 4)\}$. Note that Γ_0 is interval closed and consider the subalgebra $\mathfrak{A}(\Gamma_0)$. This algebra is canonical (cf. [13, 3.7]) and of type $(2, 2, 2)$, hence tame concealed [13, 4.3(5)]. Let K be the subspace of $\mathfrak{A}(\Gamma_0)$ spanned by $\{b_3, c_3b_3\}$. Clearly, K is a $\mathfrak{A}(\Gamma_0)$-submodule of $\mathfrak{A}(\Gamma_0)1_{\omega_2,3}$. Let M be the quotient of $\mathfrak{A}(\Gamma_0)1_{\omega_2,3}$ by K. This $\mathfrak{A}(\Gamma_0)$-module has dimension vector

$$\begin{tikzcd} & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ 1 \arrow[dr, dash] & 1 \arrow[d, dash] & 0 \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$

and hence belongs to the tubular family of type $(2, 2, 2)$. Then it is easy to check that $\mathfrak{A}(\Gamma)$ is obtained as the one-point extension and one-point coextension of $\mathfrak{A}(\Gamma_0)$ at M and hence is tame (even domestic). We refer the reader to [13, 4.7] for details. □

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Appendix A

A.1. The graded characters of injectives envelopes of simple objects in $\mathcal{G}[\Gamma]$ given by Propositions 2.4 and 2.7 can be calculated explicitly by using the LiE computer program [16] and we list them below for the reader's convenience.

$$I(0,4)_\Gamma = V(0,4) \oplus V(\omega_2,3) \oplus \begin{pmatrix} V(\omega_4,2) \\ \oplus \\ V(2\omega_2,2) \end{pmatrix} \oplus V(\omega_2+\omega_4,1) \oplus V(2\omega_4,0),$$

$$I(\omega_2,3)_\Gamma = V(\omega_2,3) \oplus \begin{pmatrix} V(\omega_4,2) \\ \oplus \\ V(2\omega_2,2) \\ \oplus \\ V(\omega_1+\omega_3) \end{pmatrix} \oplus 2V(\omega_2+\omega_4,1) \oplus V(2\omega_4,0),$$

$$I(\omega_4,2)_\Gamma = V(\omega_4,2) \oplus V(\omega_2+\omega_4,1) \oplus V(2\omega_4,0),$$

$$I(2\omega_2,2)_\Gamma = V(2\omega_2,2) \oplus V(\omega_2+\omega_4,1) \oplus V(2\omega_4,0),$$

$$I(\omega_1+\omega_3,2)_\Gamma = V(\omega_1+\omega_3,2) \oplus V(\omega_2+\omega_4,1),$$

$$I(\omega_2+\omega_4,1)_\Gamma = V(\omega_2+\omega_4,1) \oplus V(2\omega_4,0),$$

$$I(2\omega_4,0)_\Gamma = V(2\omega_4,0).$$

A.2. We now establish the following result which was used in the proof of Proposition in Section 5. We use the notation of the proof freely.

Lemma. *Let $\lambda \neq \mu \in \{2\omega_2, \omega_4, \omega_1+\omega_3\}$. Then $\mathbf{p} \circ (1 \otimes \bar{g}_\mu) \circ \bar{f}_\mu$ and $\mathbf{p} \circ (1 \otimes \bar{g}_\lambda) \circ \bar{f}_\lambda$ are linearly independent elements of* $\operatorname{Hom}_{\mathfrak{g}}(V(\omega_2+\omega_4), S^2(\mathfrak{g}) \otimes \mathfrak{g})$.

Proof. Write x_i^- for $x_{\alpha_i}^-$. Let $w_{\omega_4} = \bar{g}_{\omega_4}(v_{\omega_4})$. Since $\mathfrak{g} \cong_{\mathfrak{g}} V(\omega_2)$, we have

$$\begin{aligned} w_{\omega_4} = {} & v_{\omega_2} \otimes x_2^- x_1^- x_3^- x_2^- v_{\omega_2} + x_2^- x_1^- x_3^- x_2^- v_{\omega_2} \otimes v_{\omega_2} - x_2^- v_{\omega_2} \otimes x_1^- x_3^- x_2^- v_{\omega_2} \\ & - x_1^- x_3^- x_2^- v_{\omega_2} \otimes x_2^- v_{\omega_2} + x_1^- x_2^- v_{\omega_2} \otimes x_3^- x_2^- v_{\omega_2} + x_3^- x_2^- v_{\omega_2} \otimes x_1^- x_2^- v_{\omega_2} \in S^2(\mathfrak{g}). \end{aligned}$$

Write $w_{\omega_4} = w_{\omega_4}^{(1)} \otimes w_{\omega_4}^{(2)}$ in the Sweedler notation. Then

$$v_1 = w_{\omega_4} \otimes v_{\omega_2}, \qquad v_2 = v_{\omega_2} \otimes w_{\omega_4}, \qquad v_3 = w_{\omega_4}^{(1)} \otimes v_{\omega_2} \otimes w_{\omega_4}^{(2)}$$

is a basis of $U = \{v \in \mathfrak{g}^{\otimes 3}{}_{\omega_2+\omega_4}\colon \mathfrak{n}^+ v = 0\}$. Let $s_1 = v_1$, $s_2 = v_2 + v_3$, $s_3 = v_2 - v_3$. Then s_1, s_2 form a basis of $U \cap S^2(\mathfrak{g}) \otimes \mathfrak{g}$, while s_3 spans $U \cap \bigwedge^2 \mathfrak{g} \otimes \mathfrak{g}$.

We have

$$\begin{aligned}\bar{f}_{2\omega_2}(v_{\omega_2+\omega_4}) = & -v_{\omega_2}\otimes x_1^- x_2^- x_3^- x_2^- v_{2\omega_2} + 2v_{\omega_2}\otimes x_2^- x_1^- x_3^- x_2^- v_{2\omega_2} \\ & - 3x_2^- v_{\omega_2}\otimes x_1^- x_3^- x_2^- v_{2\omega_2} + 3x_1^- x_2^- v_{\omega_2}\otimes x_3^- x_2^- v_{2\omega_2} \\ & + 3x_3^- x_2^- v_{\omega_2}\otimes x_1^- x_2^- v_{2\omega_2} - 3x_1^- x_3^- x_2^- v_{\omega_2}\otimes x_2^- v_{2\omega_2} \\ & + 6x_2^- x_1^- x_3^- x_2^- v_{\omega_2}\otimes v_{2\omega_2}\end{aligned}$$

and

$$\begin{aligned}\bar{f}_{\omega_1+\omega_3}(v_{\omega_2+\omega_4}) = & \, v_{\omega_2}\otimes x_1^- x_2^- x_3^- v_{\omega_1+\omega_3} - 2v_{\omega_2}\otimes x_2^- x_1^- x_3^- v_{\omega_1+\omega_3} \\ & + v_{\omega_2}\otimes x_3^- x_2^- x_1^- v_{\omega_1+\omega_3} + 2x_2^- v_{\omega_2}\otimes x_1^- x_3^- v_{\omega_1+\omega_3} \\ & - 2x_1^- x_2^- v_{\omega_2}\otimes x_3^- v_{\omega_1+\omega_3} - 2x_3^- x_2^- v_{\omega_2}\otimes x_1^- v_{\omega_1+\omega_3} \\ & + 2x_1^- x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_1+\omega_3}.\end{aligned}$$

Furthermore,

$$\bar{g}_{\omega_2}(v_{\omega_1+\omega_3}) = v_{\omega_2}\otimes x_2^- v_{\omega_2} - x_2^- v_{\omega_2}\otimes v_{\omega_2}.$$

Finally, $\bar{g}_{2\omega_2}(v_{2\omega_2}) = v_{\omega_2}\otimes v_{\omega_2}$.

The composite map

$$V(\omega_2+\omega_4)\xrightarrow{\bar{f}_{2\omega_2}}\mathfrak{g}\otimes V(2\omega_2)\xrightarrow{1\otimes\bar{g}_{2\omega_2}}\mathfrak{g}^{\otimes 3}$$

sends $v_{\omega_2+\omega_4}$ to

$$\begin{aligned}m_1 = & -v_{\omega_2}\otimes x_2^- v_{\omega_2}\otimes x_1^- x_3^- x_2^- v_{\omega_2} - v_{\omega_2}\otimes x_1^- x_2^- v_{\omega_2}\otimes x_3^- x_2^- v_{\omega_2} \\ & - v_{\omega_2}\otimes x_3^- x_2^- v_{\omega_2}\otimes x_1^- x_2^- v_{\omega_2} + 2v_{\omega_2}\otimes x_2^- v_{\omega_2}\otimes x_1^- x_3^- x_2^- v_{\omega_2} \\ & + v_{\omega_2}\otimes x_2^- x_1^- x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_2} + v_{\omega_2}\otimes v_{\omega_2}\otimes x_2^- x_1^- x_3^- x_2^- v_{\omega_2} \\ & - 3x_2^- v_{\omega_2}\otimes x_1^- x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_2} + x_2^- v_{\omega_2}\otimes v_{\omega_2}\otimes x_1^- x_3^- x_2^- v_{\omega_2} \\ & + 3x_1^- x_2^- v_{\omega_2}\otimes x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_2} + x_1^- x_2^- v_{\omega_2}\otimes v_{\omega_2}\otimes x_3^- x_2^- v_{\omega_2} \\ & + 3x_3^- x_2^- v_{\omega_2}\otimes x_1^- x_2^- v_{\omega_2}\otimes v_{\omega_2} + x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_2}\otimes x_1^- x_2^- v_{\omega_2} \\ & - 3x_1^- x_3^- x_2^- v_{\omega_2}\otimes x_2^- v_{\omega_2}\otimes v_{\omega_2} + x_1^- x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_2}\otimes x_2^- v_{\omega_2} \\ & + 6x_2^- x_1^- x_3^- x_2^- v_{\omega_2}\otimes v_{\omega_2}\otimes v_{\omega_2}.\end{aligned}$$

Furthermore, the image of $v_{\omega_2+\omega_4}$ under the composite map

$$V(\omega_2+\omega_4)\xrightarrow{\bar{f}_{\omega_1+\omega_3}}\mathfrak{g}\otimes V(\omega_1+\omega_3)\xrightarrow{1\otimes\bar{g}_{\omega_1+\omega_3}}\mathfrak{g}^{\otimes 3}$$

is

$$
\begin{aligned}
m_2 = &-2v_{\omega_2} \otimes v_{\omega_2} \otimes x_2^- x_1^- x_3^- x_2^- v_{\omega_2} + 2v_{\omega_2} \otimes x_2^- x_1^- x_3^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \\
&+ 2x_2^- v_{\omega_2} \otimes v_{\omega_2} \otimes x_1^- x_3^- x_2^- v_{\omega_2} - 2x_2^- v_{\omega_2} \otimes x_1^- x_3^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \\
&- 2x_1^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \otimes x_3^- x_2^- v_{\omega_2} + 2x_1^- x_2^- v_{\omega_2} \otimes x_3^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \\
&- 2x_3^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \otimes x_1^- x_2^- v_{\omega_2} + 2x_3^- x_2^- v_{\omega_2} \otimes x_1^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \\
&+ 2x_1^- x_3^- x_2^- v_{\omega_2} \otimes v_{\omega_2} \otimes x_2^- v_{\omega_2} - 2x_1^- x_3^- x_2^- v_{\omega_2} \otimes x_2^- v_{\omega_2} \otimes v_{\omega_2}.
\end{aligned}
$$

Finally, the composite map

$$
V(\omega_2 + \omega_4) \xrightarrow{\bar{f}_{\omega_4}} \mathfrak{g} \otimes V(\omega_4) \xrightarrow{1 \otimes \bar{g}_{\omega_4}} \mathfrak{g}^{\otimes 3}
$$

maps $v_{\omega_2+\omega_4}$ to $m_3 = v_{\omega_2} \otimes w_{\omega_4}$. In particular, this implies that all composite maps are non-zero. Furthermore, it can be shown that

$$
\begin{aligned}
m_1 &= 3s_1 + s_2 - 2s_3, \\
m_2 &= 2s_1 - s_2 + s_3, \\
2m_3 &= s_2 + s_3.
\end{aligned}
$$

In particular, m_1, m_2 and m_3 are linearly independent. Suppose that $x_1 m_1 + x_2 m_2 + x_3 m_3 \neq 0$ has the zero projection onto $S^2(\mathfrak{g}) \otimes \mathfrak{g}$. Using the above equations we conclude that $x_i \neq 0$, $i = 1, 2, 3$. $\square$

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