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Average norms of polynomials

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Abstract

We present an explicit formula for the average $\mathcal{L}_{2\alpha}$-norm over all the polynomials of degree n with coefficients in T, where T is a finite set of complex numbers and α is a positive integer.

1. Introduction

There are many old research questions concerning sets of polynomials with special coefficients. A number of these questions were suggested by Erdős and Szekeres; Hilbert; Littlewood; and Prouhet, Tarry, and Escott (see [B3, pp. 5–7]). For example, Prouhet, Tarry, and Escott asked for a polynomial with integer coefficients that is divisible by $(z-1)^n$ and has the smallest possible sum of the absolute values of the coefficients. Erdős and Szekeres asked for the minimum of $\|\prod_{j=1}^{n}(1-z^{a_j})\|_\infty$, where the a_j are positive integers, for a given n. The problem to find the maximum and the minimum norms of polynomials with restricted coefficients is an old and difficult problem. In the $\mathcal{L}_4$-norm this problem is often called Golay's "Merit Factor" problem. In the supremum norm this problem is due to Littlewood. These problems are at least fifty years old and still unsolved. In this paper we give a complete solution to the following problem (for particular cases, see [B3, p. 35]): find the average $\mathcal{L}_{2\alpha}$-norm over polynomials of degree n with coefficients in a given finite set T, where α is a positive integer.

Let $T=\{x_1,x_2,\ldots,x_d\}$ be any finite set of complex numbers. A polynomial $p(z)=a_nz^n+\cdots+a_1z+a_0$ is said to be a T*-polynomial* if $a_i\in T$ for all i, $0\leqslant i\leqslant n$. We denote by $\mathfrak{T}_T(n)$ the set of all T-polynomials of degree n. For example, if $T=\{0,1,2\}$ then the

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set of T-polynomials of degree 1 is given by $\mathfrak{T}_T(1) = \{z, 2z, z+1, 2z+1, z+2, 2z+2\}$. The cardinality of the set $\mathfrak{T}_T(n)$ is denoted by $N_T(n)$. Clearly, for all $n \geqslant 0$,

$$N_T(n) = \begin{cases} (d-1)d^n & 0 \in T \text{ and } n \geqslant 1, \\ d^{n+1} & \text{otherwise.} \end{cases}$$

A T-polynomial $p(z)$ is said to be a *Littlewood polynomial* if $T = \{-1, 1\}$, and $p(z)$ is a *polynomial of height h* if $T = \{-h, -h+1, \dots, h-1, h\}$). For example, $p(z) = z^2 - z + 1$ is a Littlewood polynomial of degree 2, and $p(z) = z^3 - 2z^2 - 1$ is a polynomial of height 2 and degree 3.

Let $p(z)$ be any T-polynomial of degree n. For any positive integer α, the $\mathcal{L}_\alpha$*-norm* on the boundary of the unit disk is defined by

$$\|p\|_\alpha = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^\alpha d\theta \right)^{1/\alpha}.$$

Let f, g, h be any three $\mathbb{R}$-polynomials. The (f, g, h)*-average* over T-polynomials of degree n is defined by

$$E_T(n; f, g, h) = \frac{1}{2\pi N_T(n)} \sum_{p \in \mathfrak{T}_T(n)} \int_0^{2\pi} h\left(e^{i\theta}\right) f\left(p\left(e^{i\theta}\right)\right) g\left(p\left(e^{-i\theta}\right)\right) d\theta, \tag{1}$$

for any $n \geqslant 1$. We denote by $e_T(n; s, t, m)$ the (z^s, z^t, z^m)-average over T-polynomials of degree n, where $m \in \mathbb{Z}$ and $s, t, n \geqslant 0$. We define the *average* $\mathcal{L}_\alpha$*-norm* over T-polynomials of degree n by

$$\begin{aligned} \mu_T^\alpha(n) = e_T(n; \alpha/2, \alpha/2, 0) &= \frac{1}{N_T(n)} \sum_{p \in \mathfrak{T}_T(n)} \|p\|_\alpha^\alpha \\ &= \frac{1}{2\pi N_T(n)} \sum_{p \in \mathfrak{T}_T(n)} \int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^\alpha d\theta, \end{aligned} \tag{2}$$

for any positive integer α. In the case $\alpha = 0$, we define $\mu_T^0(n) = 1$ for all $n \geqslant 0$. For simplicity, we define $\lambda^\alpha(n) = \mu_{\{-1,1\}}^\alpha(n)$ and $\varphi_h^\alpha(n) = \mu_{\{-h,-h+1,\dots,h-1,h\}}^\alpha(n)$, for all $n, \alpha, h \geqslant 0$.

We would like to find an explicit formula for $\mu_T^{2\alpha}(n)$, where T is a finite set of complex numbers and α is a positive integer (for particular cases, see [B3, p. 35]). While the Littlewood polynomials and polynomials of height h have attracted much attention (for example, see [B3,BC,NB]), the case of other sets T has not been treated. The Littlewood polynomials have been considered by several authors. In 1990, Newman and Byrnes [NB] found $\lambda^4(n) = 2n^2 + 3n + 1$ and in 2002, Borwein and Choi [BC] proved

$\lambda^6(n) = 6n^3 + 9n^2 + 4n + 1$ and $\lambda^8(n) = 24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n$. In the case of polynomials of height 1, Borwein [B2] proved

$$\varphi_1^2(n) = \frac{2}{3}(n+1), \qquad \varphi_1^4(n) = \frac{2}{9}\big(4n^2 + 7n + 3\big), \quad \text{and}$$
$$\varphi_1^6(n) = \frac{2}{9}\big(8n^3 + 18n^2 + 13n + 3\big).$$

More generally, Borwein [B2] found for any $h \geqslant 0$,

$$\varphi_h^2(n) = \frac{h(h+1)}{3}(n+1) \quad \text{and}$$
$$\varphi_h^4(n) = \frac{h(h+1)}{45}\big(10h(h+1)n^2 + \big(19h^2 + 19h - 3\big)n + 3\big(3h^2 + 3h - 1\big)\big).$$

In this paper we suggest a general approach to the study of the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree n, for any positive integer α and any finite set T of complex numbers, which allows one to get an explicit formula for $\mu_T^{2\alpha}(n)$. More precisely, we find an explicit formula for the generating function $e_T(x,u,v,w)$. Using this generating function we get explicit formulas for $e_T(n;s,t,m)$ in general, and $\mu_T^{2\alpha}(n) = e_T(n;\alpha,\alpha,0)$ in particular, where $m \in \mathbb{Z}$, $s,t,n \geqslant 0$, α is a positive integer, and T is a finite set of complex numbers.

The main result of this paper can be formulated as follows. We denote by $e_T(x,u,v,w)$ the generating function for the sequence $\{e_T(n;s,t,m)\}_{n,s,t,m}$, that is,

$$e_T(x,u,v,w) = \sum_{m\in\mathbb{Z}} \sum_{p,q,n\geqslant 0} e_T(n;s,t,m)x^n u^s v^t w^m.$$

Theorem 1.1. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be a finite set of complex numbers. Then the generating function $e_T(x,u,v,w)$ is given by*

$$\sum_{n\geqslant 1}\Bigg[\frac{1}{d^n}\sum_{j_1,\ldots,j_n=1}^{d} x^{n-1}\Big/\Big(\big(1 - x_{j_1}u - x_{j_2}uw^{-1} - \cdots - x_{j_n}uw^{-n+1}\big) \times \big(1 - \overline{x_{j_1}}v - \overline{x_{j_2}}vw - \cdots - \overline{x_{j_n}}vw^{n-1}\big)\Big)\Bigg].$$

Moreover, $e_T(n;s,t,m)$ is given by

$$\frac{1}{d^{n+1}} \sum_{j_1,\ldots,j_{n+1}=1}^{d} \sum_{s,t\geqslant 0} \sum_{\substack{k_1+\cdots+k_{n+1}=s\\ \ell_1+\cdots+\ell_{n+1}=t\\ \sum_{a=1}^{n+1}(a-1)(\ell_a-k_a)=m}} \prod_{a=1}^{n+1}\big(x_{j_a}^{t_a}\overline{x_{j_a}}^{r_a}\big)\binom{s}{k_1,\ldots,k_{n+1}}\binom{t}{\ell_1,\ldots,\ell_{n+1}}.$$

The paper is organized as follows. The proof of our main result, Theorem 1.1, is presented in Section 2. In Section 3 we present a general application for our results. In particular, we give explicit formulas up to $\alpha = 4$ where $\sum_{t\in T} t = 0$. Finally, in Section 4 we suggest several directions to generalize the results of the previous sections.

2. Proofs

Let us start by introducing a quantity that plays a crucial role in the proof of Theorem 1.1.

Theorem 2.1. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be a set of complex numbers, m be any integer, $n \geqslant 1$, and $s, t \geqslant 0$. Then*

$$e_T(n; s, t, m) = \frac{1}{d}\sum_{j=1}^{d}\sum_{k=0}^{s}\sum_{\ell=0}^{t} x_j^{s-k}\overline{x_j}^{\,t-\ell}\binom{s}{k}\binom{t}{\ell} e_T(n-1; k, \ell, m+k-\ell). \tag{3}$$

Proof. Let $z = \mathrm{e}^{i\theta}$ and $f_{s,t,m}(p(z)) = z^m p^s(z)\overline{p}^t(z)$, where $\overline{p}(z)$ is the conjugate polynomial of the polynomial $p(z) \in \mathfrak{T}_T(n)$. If $p(z) \in \mathfrak{T}_T(n)$ then there exists a unique polynomial $q(z) \in \mathfrak{T}_T(n-1)$ and a unique index j, $1 \leqslant j \leqslant d$, such that $p(z) = zq(z) + x_j$. So we have

$$\sum_{p(z)\in\mathfrak{T}_T(n)} f_{s,t,m}\big(p(z)\big) = z^m \sum_{j=1}^{d}\sum_{q(z)\in\mathfrak{T}_T(n-1)} \big(zq(z)+x_j\big)^s\big(\bar{z}\,\bar{q}(z)+\overline{x_j}\big)^t.$$

Using $(x+y)^k = \sum_{j=0}^{k}\binom{k}{j}x^j y^{k-j}$ and $\bar{z} = z^{-1}$ we get

$$\sum_{p(z)\in\mathfrak{T}_T(n)} f_{s,t,m}\big(p(z)\big) = \sum_{q(z)\in\mathfrak{T}_T(n-1)}\sum_{j=1}^{d}\sum_{k=0}^{s}\sum_{\ell=0}^{t} x_j^{s-k}\,\overline{x_j}^{\,t-\ell}\binom{s}{k}\binom{t}{\ell} f_{k,\ell,m+k-\ell}\big(q(z)\big).$$

Therefore, using (1) we get the desired result. □

To present recurrence (3) in terms of generating functions we need the following lemma.

Lemma 2.2. *Let $F(x, y) = \sum_{s,t\geqslant 0} d_{s,t}x^s y^t$ be a generating function for the sequence $\{d_{s,t}\}_{s,t\geqslant 0}$. Then*

$$\sum_{s,t\geqslant 0} x^s y^t\left(\sum_{k=0}^{s}\sum_{\ell=0}^{t} a^{s-k}b^{t-\ell}d_{k,\ell}\binom{s}{k}\binom{t}{\ell}\right) = \frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax}, \frac{y}{1-by}\right).$$

Proof. By direct calculations we have

$$\frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right)=\sum_{s,t\geqslant 0}a^{-s}b^{-t}d_{s,t}\frac{(ax)^s(by)^t}{(1-ax)^{s+1}(1-by)^{t+1}}.$$

Using $x^r/(1-x)^r=\sum_{n\geqslant 0}\binom{n}{r}x^r$ we get

$$\frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right)=\sum_{s,t\geqslant 0}\left(\sum_{k,\ell\geqslant 0}a^{k-s}b^{\ell-t}d_{s,t}\binom{k}{s}\binom{\ell}{t}x^k y^\ell\right);$$

equivalently,

$$\frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right)=\sum_{s,t\geqslant 0}x^s y^t\left(\sum_{k=0}^{s}\sum_{\ell=0}^{t}a^{s-k}b^{t-\ell}d_{k,\ell}\binom{s}{k}\binom{t}{\ell}\right),$$

as claimed. □

Now we are ready to prove our main result, namely Theorem 1.1.

Theorem 2.3. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be a finite set of complex numbers. Then the generating function $e_T(x,v,u,w)$ is given by*

$$\sum_{n\geqslant 1}\Bigg[\frac{1}{d^n}\sum_{j_1,\ldots,j_n=1}^{d}x^{n-1}\Big/\Big(\big(1-x_{j_1}u-x_{j_2}uw^{-1}-\cdots-x_{j_n}uw^{-n+1}\big)\times\big(1-\overline{x_{j_1}}v-\overline{x_{j_2}}vw-\cdots-\overline{x_{j_n}}vw^{n-1}\big)\Big)\Bigg].$$

Proof. If we multiply Eq. (3) by $x^n u^s v^t w^m$, and sum over all $m\in\mathbb{Z}$, $s,t\geqslant 0$, and $n\geqslant 1$, using Lemma 2.2, then we arrive at

$$\begin{aligned}&e_T(x,u,v,w)-\sum_{m\in\mathbb{Z}}\sum_{s,t\geqslant 0}e(0,s,t,m)u^s v^t w^m\\&\quad=\frac{x}{d}\sum_{j=1}^{d}\left[\frac{1}{(1-x_j u)(1-\overline{x_j}v)}\,e_T\left(x,\frac{u}{w(1-x_j u)},\frac{wv}{1-\overline{x_j}v},w\right)\right].\end{aligned}$$

On the other hand, by the definitions we have that $e_T(0,s,t,m)=(1/d)\sum_{j=1}^{d}\delta_m x_j^s\overline{x_j}^t$ for any $m\in\mathbb{Z}$ and $s,t\geqslant 0$, where $\delta_m=1$ if $m=0$, otherwise $\delta_m=0$. So

$$\sum_{m\in\mathbb{Z}}\sum_{s,t\geqslant 0}e_T(0,s,t,m)u^s v^t w^m=\frac{1}{d}\sum_{j=1}^{d}\frac{1}{(1-x_j u)(1-\overline{x_j}v)}.$$

Therefore, by combining the above two equations we get that

$$e_T(x,u,v,w) = \frac{1}{d}\sum_{j=1}^{d} \frac{1}{(1-x_j u)(1-\overline{x_j}v)}\left[1 + x e_T\left(x, \frac{u}{w(1-x_j u)}, \frac{wv}{1-x_j v}, w\right)\right]. \tag{4}$$

An infinite number of applications of this identity completes the proof. □

Remark 2.4. It follows from Theorem 2.3 that the generating function $e_T(x,u,v,w)$ is symmetric under the translation $(u,v,w,T) \to (v,u,w^{-1},\overline{T})$, where $\overline{T} = \{\overline{x_1},\dots,\overline{x_d}\}$, that is, $e_T(x,u,v,w) = e_{\overline{T}}(x,v,u,w^{-1})$.

Theorem 2.3 can be presented as follows.

Corollary 2.5. *Let $T = \{x_1, x_2, \dots, x_d\}$ be a finite set of complex numbers. Then the generating function $e_T(x,v,u,w)$ is given by*

$$\sum_{n\geqslant 1}\sum_{j_1,\dots,j_n=1}^{d}\sum_{s,t\geqslant 0}\sum_{\substack{k_1+\cdots+k_n=s\\ \ell_1+\cdots+\ell_n=t}}\prod_{a=1}^{n}\left(x_{j_a}^{t_a}\overline{x_{j_a}}^{r_a}\right)\binom{s}{k_1,\dots,k_n}\binom{t}{\ell_1,\dots,\ell_n} \times \frac{x^{n-1}u^s v^t w^{(\sum_{a=1}^{n}(a-1)(r_a-t_a))}}{d^n}.$$

Proof. Using Theorem 2.3 we get that the generating function $e_T(x,v,u,w)$ is given by

$$\sum_{n\geqslant 1}\left[\sum_{j_1,j_2,\dots,j_n=1}^{d}\sum_{s,t\geqslant 0}\left(x_{j_1} + x_{j_2}w^{-1} + \cdots + x_{j_n}w^{-n+1}\right)^s \times \left(\overline{x_{j_1}} + \overline{x_{j_2}}w + \cdots + \overline{x_{j_n}}w^{n-1}\right)^t \frac{x^{n-1}u^s v^t}{d^n}\right].$$

The rest is easy to check by using the identity

$$(a_1 + \cdots + a_n)^s = \sum_{k_1+\cdots+k_n=s}\binom{s}{k_1,\dots,k_n}\prod_{j=1}^{n} a_j^{k_j}. \qquad \square$$

We denote by $\mu_T^\alpha(x)$ the generating function for the sequence $\{\mu_T^\alpha(n)\}_{n\geqslant 0}$, that is, $\mu_T^\alpha(x) = \sum_{n\geqslant 0}\mu_T^\alpha(n)x^n$. Corollary 2.5 gives the generating function $\mu_T^{2\alpha}(x)$ for any given finite set T and positive integer α.

Example 2.6. Using Corollary 2.5 for $\alpha = 1$ we get that

$$\mu_T^2(x) = \sum_{n\geqslant 1} \sum_{j_1,\dots,j_n=1}^{d} \left(\sum_{k=1}^{n} \left|x_{j_k}\right|^2 \right) \frac{x^{n-1}}{d^n} = \sum_{n\geqslant 1} \left(n d^{n-1} \sum_{j=1}^{d} |x_j|^2 \right) \frac{x^{n-1}}{d^n},$$

so it is easy to see that

$$\mu_T^2(x) = \frac{1}{d(1-x)} \sum_{k=1}^{n} |x_{j_k}|^2 + x\mu_T^2(x),$$

hence

$$\mu_T^2(x) = \sum_{n\geqslant 0} \left(\frac{n+1}{d} \sum_{j=1}^{d} |x_j|^2 \right) x^n.$$

In particular, we have that $\lambda^2(n) = n+1$ and $\varphi_1^2(n) = \frac{2}{3}(n+1)$.

Corollary 2.5 provides a finite algorithm for finding the average $\mu_T^{2\alpha}(n)$ for given n, T, and α. This algorithm has been implemented in MAPLE (see [M, Procedure genmu()]).

Remark 2.7. To apply Corollary 2.5 we have to consider d^n possibilities for j_i and $\binom{n+\alpha-1}{\alpha}^2$ possibilities for k_i and ℓ_i, namely we have to consider $d^n \binom{n+\alpha-1}{\alpha}^2$ possibilities. Thus, the approach in Corollary 2.5 would be very slow for n large.

3. Exact formulas

In this section we suggest an another approach for finding an explicit formula for $\mu_T^{2\alpha}(n)$. First of all, we denote by $e_T^{s,t}(x,w)$ the $(s+t)$-derivative of the generating function $e_T(x,u,v,w)$ with respect to u exactly s times and then with respect to v exactly t times at $u=v=0$, that is,

$$e_T^{s,t} = e_T^{s,t}(x,w) = \frac{\partial^{s+t}}{\partial u^s \partial v^t} e_T(x,u,v,w)\bigg|_{u=v=0}.$$

For any $s,t \geqslant 0$, we define $A_T^{s,t} = \sum_{j=1}^{d} x_j^s \overline{x_j}^t$. Now let us consider Eq. (4). This equation provides a finite algorithm, which we call the μ-*algorithm*, for finding $e_T(n;s,t,m)$ in general, and $\mu_T^{2\alpha}(n)$ in particular, since $s!\,t!\,e_T(n;s,t,m)$ is the coefficient of $w^m x^n$ in $e_T^{s,t}(x,w)$, and $\mu_T^{2\alpha}(n) = e_T(n;\alpha,\alpha,0)$. Therefore, the μ-algorithm with input α and output $\mu_T^{2\alpha}(x)$ can be carried out as follows:

(1) Apply the derivative operator with respect to u exactly s times and then with respect to v exactly t times on Eq. (4) for all s, t, where $0 \leqslant s,t \leqslant \alpha$.

(2) Find $e_T^{s,t}$ for all s, t, $0 \leqslant s, t \leqslant \alpha$, by solving the system which is obtained from step (1).
(3) Find $\mu_T^{2\alpha}(x)$, which is the free coefficient of w in $e_T^{\alpha,\alpha}(x, w)$.

This algorithm has been implemented in MAPLE (see [M, Procedure findav()]), and yields explicit results for given α. Below we present several explicit calculations.

3.1. Formula for $\mu_T^2(n)$

Let us start by apply the μ-algorithm for $\alpha = 1$. The first step of the μ-algorithm gives

$$
\begin{aligned}
&e_T^{0,0} = 1 + x e_T^{0,0}, \qquad e_T^{0,1} = \frac{1}{d} A_T^{0,1} + \frac{x}{d} A_T^{0,1} e_T^{0,0} + \frac{x}{w} e_T^{0,1},\\
&e_T^{1,0} = \frac{1}{d} A_T^{1,0} + \frac{x}{d} A_T^{1,0} e_T^{0,0} + \frac{x}{w} e_T^{1,0},\\
&e_T^{1,1} = \frac{1}{d} A_T^{1,1} + \frac{x}{d} A_T^{1,1} e_T^{0,0} + \frac{xw}{d} A_T^{1,0} e_T^{0,1} + \frac{x}{dw} A_T^{0,1} e_T^{1,0} + x e_T^{1,1}.
\end{aligned}
$$

Equivalently (the second step of the μ-algorithm),

$$
\begin{aligned}
&e_T^{0,0} = \frac{1}{1-x}, \qquad e_T^{1,0} = \frac{A_T^{1,0}}{d(1-x)(1-xw^{-1})}, \qquad e_T^{0,1} = \frac{A_t^{0,1}}{d(1-x)(1-xw)},\\
&e_T^{1,1} = \frac{1}{d(1-x)^2} A_T^{1,1} + \left(\frac{xw}{d^2(1-x)^2(1-xw)} + \frac{xw^{-1}}{d^2(1-x)^2(1-xw^{-1})} \right) A_T^{1,0} A_T^{0,1}.
\end{aligned}
$$

Therefore, the third step of the μ-algorithm gives $\mu_T^2(x)$, which is the free coefficient of w in $e_T^{1,1}(x, w)$. Hence, we get the following result.

Corollary 3.1. *We have*

$$
\mu_T^2(x) = \frac{1}{d(1-x)^2} A_T^{1,1} = \frac{1}{d(1-x)^2} \sum_{j=1}^{d} |x_j|^2.
$$

3.2. Formula for $\mu_T^4(n)$

Again, using the μ-algorithm (see [M, Procedure findav()]) for $\alpha = 2$ we get the following result.

Corollary 3.2. *We have*

$$
\begin{aligned}
\mu_T^4(x) = {}& \frac{1}{d(1-x)^2} A_T^{2,2} + \frac{4x}{d^2(1-x)^3} \left(A_T^{1,1}\right)^2\\
&+ \frac{2x^2(1+x)^2}{d^3(1-x^2)^3} \Big[\left(A_T^{1,0}\right)^2 A_T^{0,1} + \left(A_T^{0,1}\right)^2 A_T^{1,0} \Big]
\end{aligned}
$$

$$+\frac{8x^3}{d^4(1-x)^4(1+x)}\big(A_T^{1,0}\big)^2\big(A_T^{0,1}\big)^2.$$

For example,

$$\varphi^4(n)=\frac{h(h+1)}{45}\big(10h(h+1)n^2+(19h^2+19h-3)n+3(3h^2+3h-1)\big)$$

and $\lambda^4(n)=2n^2+3n+1$, for all $n\geqslant 0$.

3.3. *Formula for $\mu_T^{2\alpha}(n)$ where $A_T^{1,0}=0$*

Similarly to the previous subsection, our results can be extended to the case of $\mu_T^6(n)$. Since the answers become very cumbersome, we present here only the simplest case when $A_T^{1,0}=\sum_{j=1}^d x_j=0$. Thus, if we apply our approach for finding $\mu_T^{2\alpha}(n)$ where $A_T^{1,0}=0$ (so $A_T^{0,1}=0$), then we get the following results.

Corollary 3.3. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be such that $\sum_{j=1}^d x_j=0$. Then*

(i) $$\mu_T^2(x)=\frac{1}{d(1-x)^2}A_T^{1,1}.$$

(ii) $$\mu_T^4(x)=\frac{1}{d(1-x)^2}A_T^{2,2}+\frac{4x}{d^2(1-x)^3}\big(A_T^{1,1}\big)^2.$$

(iii) $$\mu_T^6(x)=\frac{1}{d(1-x)^2}A_T^{3,3}+\frac{18x}{d^2(1-x)^3}A_T^{1,1}A_T^{2,2}+\frac{36x^2}{d^3(1-x)^4}\big(A_T^{1,1}\big)^3.$$

(iv) $$\begin{aligned}\mu_T^8(x)={}&\frac{1}{d(1-x)^2}A_T^{4,4}+\frac{32x}{d^2(1-x)^3}A_T^{1,1}A_T^{3,3}+\frac{36x^2}{d^2(1-x)^4}\big(A_T^{2,2}\big)^2\\&+\frac{432x^2}{d^3(1-x)^4}A_T^{2,2}\big(A_T^{1,1}\big)^2\\&+\frac{72x^4(3-2x-2x^2+3x^3-x^4)}{d^4(1-x)^4(1+x)}\big(A_T^{0,2}\big)^2\big(A_T^{2,0}\big)^2+\frac{576x^3}{d^4(1-x)^5}\big(A_T^{1,1}\big)^4\\&+\frac{48x^3}{d^3(1-x)^2(1-x^3)}\big(A_T^{0,3}A_T^{2,0}A_T^{2,1}+A_T^{3,0}A_T^{0,2}A_T^{1,2}\big)\\&+\frac{72x^2}{d^3(1-x)^2(1-x^2)}\big(\big(A_T^{1,2}\big)^2A_T^{2,0}+\big(A_T^{2,1}\big)^2A_T^{0,2}\big)\\&+\frac{6x^2}{d^3(1-x)^2(1-x^2)}\big(\big(A_T^{2,0}\big)^2A_T^{0,4}+\big(A_T^{0,2}\big)^2A_T^{4,0}\big).\end{aligned}$$

4. Further results

In this section we suggest several directions to generalize the results of the previous sections. The first of these directions is to obtain an exact formula for the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree n with weight z^m for given α, n, and m. Let us define

$$\mu_T^\alpha(n;m) = e_T(n;\alpha/2,\alpha/2,m) = \frac{1}{2\pi N_T(n)} \sum_{p\in\mathfrak{T}_n} \int_0^{2\pi} \mathrm{e}^{im\theta} \left|p\left(\mathrm{e}^{i\theta}\right)\right|^\alpha \mathrm{d}\theta. \tag{5}$$

Clearly, $\mu_T^\alpha(n) = \mu_T^\alpha(n;0)$ for all n, and $\alpha > 0$. Using Theorem 1.1 we get the following result.

Theorem 4.1. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be any finite set of complex numbers. The generating function $\sum_{n\geqslant 0}\sum_{m\in\mathbb{Z}} \mu_T^{2\alpha}(n;m)x^n w^m$ is given by*

$$\sum_{n\geqslant 1}\sum_{j_1,\ldots,j_n=1}^{d} \sum_{\substack{k_1+\cdots+k_n=\alpha\\ \ell_1+\cdots+\ell_n=\alpha}} \prod_{a=1}^{n} \left(x_{j_a}^{t_a}\overline{x_{j_a}}^{r_a}\right)\binom{\alpha}{k_1,\ldots,k_n}\binom{\alpha}{\ell_1,\ldots,\ell_n} \times \frac{x^{n-1}w^{(\sum_{a=1}^{n}(a-1)(r_a-t_a))}}{d^n}.$$

By the definitions, it is clear that $\alpha!^2\mu_T^{2\alpha}(n;m)$ is the coefficient of $x^n w^m$ in $E_T^{\alpha,\alpha}(x,w)$ (see Section 3). Therefore, in a way similar to that in Section 3 we obtain the following result.

Corollary 4.2. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be any finite set of complex numbers. Then*

$$\mu_T^2(n;m) = \begin{cases} \dfrac{n+1-|m|}{d^2} A_T^{0,1} A_T^{1,0} & m \neq 0,\ -n \leqslant m \leqslant m,\\ \dfrac{n+1}{d} A_T^{1,1} & m = 0,\\ 0 & \textit{otherwise}. \end{cases}$$

The second of these directions is to consider the general case to find an explicit formula for $e_T(n,s,t,m)$ for any given s,t. For example, the following is true.

Corollary 4.3. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be any finite set of complex numbers. Then for any $n \geqslant 0$ and $m \in \mathbb{Z}$,*

$$e_T(n;1,2,m) = \begin{cases} \dfrac{1}{d} A_T^{1,2} & 0 \leqslant m \leqslant n,\\ 0 & \textit{otherwise}, \end{cases} \quad \textit{and}$$

$$e_T(n;2,1,m)=\begin{cases}\frac{1}{d}A_T^{2,1} & -n\leqslant m\leqslant 0,\\ 0 & \textit{otherwise}.\end{cases}$$

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Further reading

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