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Selection in the Saffman-Taylor Finger Problem and the Taylor-Saffman Bubble Problem Without Surface Tension

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Abstract—We consider the Saffman-Taylor problem describing the displacement of one fluid by another having a smaller viscosity, in a porous medium or in a Hele-Shaw configuration, and the Taylor-Saffman problem of a bubble moving in a channel containing moving fluid. Each problem is known to possess a family of solutions, the former corresponding to propagating fingers and the latter to propagating bubbles, with each member characterized by its own velocity and each occupying a different fraction of the porous channel through which it propagates. To select the correct member of the family of solutions, the conventional approach has been to add surface tension σ and then take the limit $\sigma \to 0$. We propose a selection criterion that does not rely on surface tension arguments.

Keywords—Saffman-Taylor, Finger, Bubble interface, Front.

We first consider the Saffman-Taylor (ST) [1] finger problem. Thus, we consider

$$\vec{v} = F\nabla P, \tag{1}$$

$$\nabla \cdot \vec{v} = 0, \tag{2}$$

in the channel $(X, Y) : -\infty < X < \infty,\ -1 < Y < 1$. Here $\vec{v}$ denotes the fluid velocity, P the pressure, and $F = k/\mu$ the filtration coefficient, where k is the permeability and μ the viscosity. A front (interface) separates one fluid from the other. In addition to uniformly propagating planar fronts, there exist fronts in the form of a finger, which are curved near the tip and parallel to the channel walls far behind the tip, and which propagate with constant velocity u. Let the subscripts 1 and 2 denote the regions 1 and 2, which refer to the displacing and the displaced fluid, respectively. Thus, the filtration coefficient $F = F_1(F_2)$ in region 1 (region 2). We assume that $F_2 > F_1$, in which case planar fronts are unstable [1]. Therefore, we consider finger fronts $X = X_f(Y)$. On the front, we have

$$P_1 = P_2, \qquad v_n = u_n, \tag{3}$$

where $\vec{u} = (u, 0)$ and the subscript n denotes the normal component.

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We do not prescribe an upstream boundary condition at $X = -\infty$. Rather, we prescribe the downstream fluid velocity as

$$\vec{v} = (1, 0), \qquad X \to \infty. \tag{4}$$

We introduce the potential ϕ and stream function ψ by

$$\phi = -FP, \tag{5}$$

$$v_X = \frac{\partial \phi}{\partial X} = \frac{\partial \psi}{\partial Y}, \qquad v_Y = \frac{\partial \phi}{\partial Y} = -\frac{\partial \psi}{\partial X}. \tag{6}$$

Both ϕ and ψ satisfy Laplace's equation in regions 1 and 2. In region 2, the solution is given by

$$\phi_2 = v_2^0 X, \qquad \psi_2 = v_2^0 Y, \tag{7}$$

where v_2^0 is an as yet unknown constant. The pressure jump across the front is zero, so that

$$\phi_2 = \frac{F_2}{F_1}\phi_1, \qquad X = X_f. \tag{8}$$

In terms of the potential function ϕ, condition (3) is

$$\frac{\partial \phi_2}{\partial n} = u_n, \qquad \frac{\partial \phi_1}{\partial n} = u_n. \tag{9}$$

Recalling the relations

$$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial \tau}, \qquad \frac{u_n}{u} = \frac{\partial Y}{\partial \tau},$$

where $\frac{\partial}{\partial \tau}$ denotes the tangential derivative, condition (9) may be written as

$$\psi_1 = uY, \qquad \psi_2 = uY. \tag{10}$$

It follows from (7) and (10) that

$$u = v_2^0. \tag{11}$$

In region 1, the velocity at $\pm\infty$ must be independent of Y. Therefore, we seek a solution satisfying

$$\phi_1 \sim X, \quad \frac{\partial \psi_1}{\partial Y} = 1, \quad X \to \infty, \qquad \phi_1 \sim v_1^0 X, \quad \frac{\partial \psi_1}{\partial Y} = v_1^0, \quad X \to -\infty, \tag{12}$$

where the constant v_1^0 is as yet undetermined.

The walls of the channel are streamlines

$$\psi(Y = \pm 1) = \pm 1. \tag{13}$$

We now determine the constants v_1^0 and v_2^0. From (7), (8), and (12), it follows that

$$v_2^0 = \frac{F_2}{F_1} v_1^0. \tag{14}$$

Far behind its tip, the finger occupies the region $-\lambda < Y < \lambda$, where the constant λ is to be determined. The difference between the stream functions at the wall ($Y = 1$) and at the fro' ($Y = \lambda$), according to (13) and (10) is

$$\psi_1(1) - \psi_1(\lambda) = 1 - u\lambda.$$

On the other hand, this quantity may be computed from (12) as

$$\psi_1(1) - \psi_1(\lambda) = v_1^0(1 - \lambda). \tag{16}$$

Using (11), (14), and (15), we obtain

$$u = \frac{(F_2/F_1)}{1 + \lambda[(F_2/F_1) - 1]}, \qquad v_1^0 = \frac{1}{1 + \lambda[(F_2/F_1) - 1]}, \qquad v_2^0 = \frac{(F_2/F_1)}{1 + \lambda[(F_2/F_1) - 1]}. \tag{17}$$

We note that the denominator of (17) is always positive in the parameter region $F_2/F_1 > 1$, where planar fronts are unstable. It also follows from (17) that the velocity u of the finger is always greater than the velocity of a planar front. In order to determine the shape of the front, it is convenient to introduce new potential and stream functions

$$\Phi_1 = \phi_1 - v_* X, \qquad \Psi_1 = \psi_1 - v_* Y, \tag{18}$$

where the constant v_* is determined by the condition on the front

$$\Phi_1(X_f) = 0. \tag{19}$$

It follows from (7) and (8) that

$$v_* = \frac{F_1}{F_2} v_2^0. \tag{20}$$

Condition (10) then takes the form

$$\Psi_1 = \psi_1 - v_* Y = UY, \qquad U \equiv \frac{(F_2/F_1) - 1}{1 + \lambda[(F_2/F_1) - 1]}. \tag{21}$$

At the walls ($Y = \pm 1$), we have

$$\Psi_1 = \psi_1(1) - v_* = \lambda U \equiv V. \tag{22}$$

Thus, we must find conjugate harmonic functions Φ_1 and Ψ_1 which satisfy conditions (21),(22), on the front and on the walls, respectively, and

$$\Phi_1 \simeq VX, \qquad X \to \infty, \tag{23}$$

with condition (19) employed to determine the shape of the front. As shown by Saffman and Taylor [1], the problem is solved by considering the inverse harmonic functions $X(\Phi_1, \Psi_1)$, $Y(\Phi_1, \Psi_1)$, with

$$Y = \frac{\Psi_1}{V} + \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n\pi\Phi_1}{V}\right) \cos\left(n\frac{\pi\Psi_1}{V}\right), \tag{24}$$

and $X(\Phi_1, \Psi_1)$ conjugate to $Y(\Phi_1, \Psi_1)$. Using the usual Fourier method, the coefficients B_n are determined by (21) as

$$B_n = -\frac{2}{\pi}\frac{(-1)^n}{n}(1 - \lambda). \tag{25}$$

Thus, the conjugate harmonic functions $Y(\Phi_1, \Psi_1)$ and $X(\Phi_1, \Psi_1)$ are determined as

$$Y = \frac{\Psi_1}{V} + \frac{2(\lambda - 1)}{\pi} \tan^{-1}\left(\frac{\sin(\pi\Psi_1/V)}{\cos(\pi\Psi_1/V) + \exp(\pi\Phi_1/V)}\right), \tag{26}$$

$$X = \frac{\Phi_1}{V} - \frac{\lambda - 1}{\pi} \ln\left(\frac{1 + 2\cos(\pi\Psi_1/V)\exp(-\pi\Phi_1/V) + \exp(-2\pi\Phi_1/V)}{4}\right).$$

The shape of the finger is obtained by setting $\Phi_1 = 0$ in (26) (cf.(19)), so that

$$X_f = \frac{1-\lambda}{\pi} \ln\left(\frac{1}{2}\left(1+\cos\left(\frac{\pi Y}{\lambda}\right)\right)\right). \tag{27}$$

The width 2λ of the finger is as yet undetermined, so that we have a one-parameter family of finger solutions. However, experiments [1] indicate that the finger corresponding to $\lambda = 1/2$ is preferred over other members of the family. Numerical computations (e.g., [2–4]) of the problem in which surface tension σ is first added to the model and then the limit $\sigma \to 0$ is taken also select the value $\lambda = 1/2$. We will show that solutions with $\lambda = 1/2$ should be selected, independent of the value of the parameter F_2/F_1. Assuming this result, we can write (17) as

$$u = \frac{2(F_2/F_1)}{1+(F_2/F_1)}. \tag{28}$$

We see that the finger front propagates faster than the planar front ($u > 1$). Near the planar front stability boundary ($F_2/F_1 = 1$), the velocity u is close to 1. However, penetrating more deeply into the instability region, the finger velocity may exceed the planar velocity by a factor of 2 (as $F_2/F_1 \to \infty$).

We note that the Saffman-Taylor problem also arises in the study of filtration combustion [5], when propagation of the combustion front is limited by the rate of delivery of gaseous oxidizer. Then, the problem for the fluid field decouples from that for the temperature and the concentrations of the reactive components. The problem for the fluid field can then be reduced to the problem considered by Saffman and Taylor. In the filtration combustion problem, surface tension cannot be added, since the front is merely the interface between the burned and unburned regions.

We now present a criterion to solve the selection problem without adding surface tension.

First, we note the important role of fluid motion in the transverse direction Y. The finger propagating in the strip $-\infty < X < \infty$, $-\lambda \le Y \le \lambda$, necessarily pushes fluid in the transverse direction because the amount of fluid $V = U\lambda$ which is displaced by the finger $X = X_f(Y)$ is greater than or equal to the amount to fluid $V\lambda$ removed from the strip at $X = +\infty$. The transverse fluid motion at $X = \pm\infty$ is negligible so that an essentially localized pulse of transverse fluid motion accompanies the propagating finger. The integral

$$I_0 = \int_\Lambda |v_{Y1}|\, dX\, dY$$

represents a quantitative measure of the total transverse fluid motion in the strip Λ bounded by the lines $Y = \pm\lambda$ and the interface X_f.

Employing the relations

$$v_{Y1} = \frac{\partial \phi_1}{\partial Y} = \frac{\partial \Phi_1}{\partial Y} = -\frac{\partial \Psi_1}{\partial X}, \qquad \Psi_1(X = X_f) = UY, \quad \Psi_1(X=\infty) = VY,$$

we evaluate I_0 as

$$I_0 = 2\int_{-\infty}^{+\infty}\int_0^\lambda \frac{\partial \Psi_1}{\partial X}\, dX\, dY = V\lambda(1-\lambda). \tag{29}$$

We define the quantity $\Delta L \equiv I_0/I_p$, with $I_p = \int_{-\lambda}^{\lambda} v_{Y1}\, dY$, where the integral is evaluated near the tip, where v_{Y1} achieves a maximum. We note that the maximum is proportional to U, and estimate that $I_p \sim 2U\lambda$. We observe that ΔL achieves a maximum at $\lambda = 1/2$. The integral I_0 describes the total amount of v_{Y1} in the area of the strip, while the integral I_p describes the amount of v_{Y1} in a cross section of the strip. Therefore, the ratio $\Delta L = I_0/I_p$ may be thought of as an effective width (in X) of the region in which there is transverse fluid motion. The propagating finger is a traveling wave which induces a localized wave of transverse fluid motion. The effective extent of this localized wave is given by ΔL which depends on λ, or equivalently, on the wave velocity u. We propose that the width of a traveling wave in a dissipative medium must necessarily increase with its propagation velocity. Mathematically, this is expressed by the inequality

$$\frac{d\Delta L}{du} = \frac{d\Delta L}{d\lambda}\frac{d\lambda}{du} \geq 0. \tag{30}$$

The derivative $\frac{d\lambda}{du}$ determined by (17) is always negative. Therefore, we demand that $\frac{d\Delta L}{d\lambda} < 0$. This condition rules out those members of the family of fingers corresponding to $\lambda < 1/2$ since $\frac{d\Delta L}{d\lambda} > 0$ for them. Thus, only fingers corresponding to $\lambda \geq 1/2$ remain. It is interesting to note the introduction of surface tension into the Saffman-Taylor problem also limits solutions to those corresponding to $\lambda \geq 1/2$ [2–4,6–9]. Among the solutions in the range $\lambda \geq 1/2$, we select the finger corresponding to $\lambda = 1/2$, whose propagation velocity is largest, as a function of λ and u. We observe that the selected velocity lies at the edge of the interval of allowed velocities, similar to selection in the Kolmogorov, Petrovsky, Piskunov problem [10]. Note that the largest velocity corresponds to an extremum of ΔL. Thus, our selection criterion is to extremize the functional ΔL, which singles out the finger with $\lambda = 1/2$. The criterion may be interpreted as the selection of the traveling wave solution with largest velocity from among all traveling waves allowed by (30).

We have verified that our selection criterion is also valid for the Taylor-Saffman problem [11] of an air bubble ($F_2 = \infty, V = 1, U = u$), moving in a channel containing a moving viscous fluid. For this problem, Taylor and Saffman derived a one-parameter family of solutions characterized by the width λ of the bubble and its speed U. In contrast to the finger problem, the bubble occupies a finite area S in the channel. The family of bubbles found by Taylor and Saffman is given by $S = (16/\pi)(U - 1/U^2)\tanh^{-1}[\tan^2((\pi U\lambda)/4)]$. Also, in contrast to the air finger problem, the product λU is no longer fixed at $\lambda U = 1$. To select the correct member of the family, Taylor and Saffman proposed the mathematical criterion that the product λU is a minimum, which selected the bubble corresponding to $U = 2$ and $\lambda = \lambda(S) = 2/\pi \tan^{-1}(\sqrt{\tanh(\pi S/4)})$. It is interesting to note that as $S \to \infty$, $\lambda \to 1/2$, which is the result for the finger problem. However, the principle of minimum λU cannot be directly applied to the air finger problem since $\lambda U \equiv 1$. In contrast, our criterion applies to both problems. In the TS problem, $I_0 = (U - 1)\lambda^2$ and $I_p \sim 2U\lambda$, so that ΔL is a function of the product $U\lambda$ only (for fixed S). As in the finger problem, others later introduced σ and took the limit $\sigma \to 0$ to verify the result in [11]. We observe that our selection criterion reduces to the condition that λU be a minimum, so that, in a sense, it may be thought of as the physical principle sought by Taylor and Saffman, which underlies their mathematical criterion. Finally, we note that after the completion of this paper, we learned of the result [12] which also showed that $\lambda = 1/2$ is selected for the ST problem, without appealing to surface tension arguments.

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