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Note

On the second eigenvalue of a graph

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Abstract

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It is shown that the second largest eigenvalue of the adjacency matrix of any d-regular graph G containing two edges the distance between which is at least $2k+2$ is at least $2\sqrt{d-1}-(2\sqrt{d-1}-1)/(k+1)$.

Let $G=(V, E)$ be a simple, undirected graph with maximum degree d, and let $Q=Q_G=(q_{u,v})$ be the Laplace matrix of G whose rows and columns are indexed by the vertices of G, where $q_{u,u}$ is the degree of u for every vertex u of G, $q_{u,v}=-1$ for every two distinct adjacent vertices u and v of G and $Q_{u,v}=0$ otherwise. Notice that $Q=D-A$, where D is the diagonal matrix $(\deg(u))_{u\in V}$, and A is the adjacency matrix of G. Notice also that $Q=C^tC$, where C is the matrix whose rows are indexed by the edges of G and whose columns are indexed by its vertices, in which each row corresponding to the edge $e=\{u, v\}$, $(u<v)$, has a 1 in the column corresponding to u, *a* -1 *in that corresponding to* v and 0 in every other place. Therefore Q is a symmetric, positive semi-definite matrix. Its smallest eigenvalue is 0, corresponding to the constant eigenvector. Let $\lambda=\lambda(G)$ denote its second smallest eigenvalue.

There are several known results that relate λ to various structural properties of the graph G. In particular, there is a strong correspondence between λ and the expansion properties of G. See, e.g. [1–2]. In view of this correspondence, it is interesting to study the maximum possible value of λ for a graph with a given maximum degree and a given number of vertices. Alon and Boppana ([1], see also [4]) proved that for any fixed d and for any infinite family of graphs G with maximum degree d,

$$\limsup \lambda(G) \leq d - 2\sqrt{d-1}.$$

This bound is sharp (at least when $d-1$ is a prime congruent to 1 modulo 4), as shown by the construction of Lubotzky, Phillips and Sarnak [4], obtained, independently, by Margulis [5]. In fact, in [1] it is conjectured that almost all d-regular graphs G on n vertices satisfy

$$\lambda(G) \geq d - 2\sqrt{d-1} - o(1)$$

(as n tends to infinity). This conjecture is still open, but Friedman, Kahn and Szemeredi [3] showed that for almost all d-regular graphs G on n vertices

$$\lambda(G) \geq d - 2\sqrt{d-1} - \log d - o(1),$$

as n tends to infinity.

In this note we obtain an upper bound for $\lambda(G)$ in terms of the maximum degree of G and its diameter. This result implies the one of Alon and Boppana stated above. Its advantage is that it does not contain any hidden constants and is therefore convenient to apply for any given graph. By the *distance* between two edges e and f of a graph we mean the minimum number of edges in a path that connects an end point of e and an end point of f. Our bound is given in the following theorem.

Theorem 1. *Let G be a graph in which there are two edges of distance at least $2k+2$, and let d be the maximum degree of G. Then*

$$\lambda(G) \leq d - 2\sqrt{d-1} + (2\sqrt{d-1} - 1)/(k+1).$$

We note that the assertion of the theorem holds for non-simple graphs as well (with the obvious generalization of the Laplace matrix for nonsimple graphs). Since the diameter of any graph on n vertices with maximum degree d is at least

$$\frac{\log n}{\log(d-1)} - O(1),$$

this theorem implies the Alon-Boppana result mentioned above. It also implies that the second largest eigenvalue of the adjacency matrix of any d-regular graph containing two edges the distance between which is at least $2k+2$ is at least

$$2\sqrt{d-1}\left(1 - \frac{1}{k+1}\right) + \frac{1}{k+1}.$$

The proof of the theorem. Let $G = (V, E)$ be the given graph, with maximum degree d, and let v_1, v_2 and u_1, u_2 be two edges at distance at least $2k+2$ between them. Let Q denote the Laplace matrix of G and let $\lambda = \lambda(G)$ be its second smallest eigenvalue. Define $V_0=\{v_1, v_2\}$ and let V_i be the set of all vertices of distance i from V_0 $(1 \leq i \leq k)$. Similarly, define $U_0=\{u_1, u_2\}$, and let U_i be the set of all vertices of distance i from U_0, $(1 \leq i \leq k)$. Observe that the union of all the sets V_i is disjoint from the union of all the sets U_j, and there are no edges that connect a vertex in the first union to a vertex in the second. Observe also that $|V_i| \leq (d-1)|V_{i-1}|$ for all $1 \leq i \leq k$, and a similar inequality holds for the sets U_j.

Let a and b be two reals, to be chosen later, such that a is positive and b is negative. Define a function f from V to the set of reals as follows: $f(v)=a(d-1)^{-i/2}$ for $v \in V_i$, $0 \leq i \leq k$, $f(v)=b(d-1)^{-i/2}$ for $v \in U_i$, $0 \leq i \leq k$, and $f(v)=0$ otherwise. Observe that a and b can be chosen so that $\sum_{v \in V} f(v)=0$. Let us fix a and b satisfying this equation. By the variational definition of the second smallest eigenvalue, λ is simply the minimum value of the quantity $(Qf, f)/(f, f)$, taken over all nonzero functions f satisfying $\sum_{v \in V} f(v)=0$. Thus, for the specific f defined above we have $(Qf, f)/(f, f) \geq \lambda$. However $(f, f)=A_1+B_1$, where

$$A_1=a^2 \sum_{i=0}^{k} \frac{|V_i|}{(d-1)^i}$$

and B_1 is a similar expression arising from the sets U_j.

Since $(Qf, f)=\sum_{uv \in E}(f(u)-f(v))^2$, and since there are no edges joining a vertex in V_i to a vertex of U_j and there are at most $d-1$ edges joining a vertex of V_i to a vertex of V_{i+1}, we have $(Qf, f)=A_2+B_2$, where

$$A_2 \leq a^2\left(\sum_{i=0}^{k-1} |V_i|(d-1)\left(\frac{1}{(d-1)^{i/2}}-\frac{1}{(d-1)^{(i+1)/2}}\right)^2+|V_k| \frac{d-1}{(d-1)^k}\right),$$

and B_2 is bounded by a similar expression arising from the sets U_j.

We need to establish an upper bound for $(A_2+B_2)/(A_1+B_1)$. Obviously any number greater or equal than each of the two quantities A_2/A_1 and B_2/B_1 will be such a bound. As these two quantities are analogous, we obtain a bound for the first; clearly the same bound holds for the second as well.

By the upper bound given above for A_2 we have

$$\begin{aligned} A_2 \leq a^2\Bigg(&\sum_{i=0}^{k-1} \frac{|V_i|}{(d-1)^i}(d-2\sqrt{d-1})+\frac{|V_k|}{(d-1)^k} \\ &\times(d-2\sqrt{d-1})+(2\sqrt{d-1}-1) \frac{|V_k|}{(d-1)^k}\Bigg) \\ \leq\ & (d-2\sqrt{d-1}) A_1+(2\sqrt{d-1}-1) \frac{A_1}{k+1}, \end{aligned}$$

where the last inequality holds since the sequence $|V_i|/(d-1)^i$ is a non-increasing sequence.

We have thus shown that

$$\frac{A_2}{A_1} \leq d - 2\sqrt{d-1} + (2\sqrt{d-1} - 1)\frac{1}{k+1}.$$

The same bound holds for B_2/B_1 and hence for λ, completing the proof of the theorem. □

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