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Garside groups are strongly translation discrete ☆

Sang Jin Lee

Department of Mathematics, Konkuk University, Gwangjin-gu, Seoul 143-701, South Korea

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Abstract

In this paper we show that all Garside groups are strongly translation discrete, that is, the translation numbers of non-torsion elements are strictly positive and for any real number r there are only finitely many conjugacy classes of elements whose translation numbers are less than or equal to r. It is a consequence of the inequality "$\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1$" for a positive integer n and an element g of a Garside group G, where $\inf_s$ denotes the maximal infimum for the conjugacy class. We prove the inequality by studying the semidirect product $G(n) = \mathbb{Z} \ltimes G^n$ of the infinite cyclic group $\mathbb{Z}$ and the cartesian product G^n of a Garside group G, which turns out to be a Garside group. We also show that the root problem in a Garside group G can be reduced to a conjugacy problem in $G(n)$, hence the root problem is solvable for Garside groups.

Keywords: Garside group; Braid group; Artin group; Translation number; Root problem

1. Introduction

The notion of translation numbers of finitely generated groups was first introduced by Gersten and Short in [GS91]. This concept comes from the action of the fundamental group of a compact Riemannian manifold of non-positive curvature on the universal cover of this manifold.

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E-mail address: sangjin@konkuk.ac.kr.

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Definition 1.1. Let G be a finitely generated group and X a finite set of semigroup generators for G. The *translation number* with respect to X of a non-torsion element $g \in G$ is defined by

$$t_X(g) = \lim_{n\to\infty} \frac{|g^n|_X}{n},$$

where $|\cdot|_X$ denotes the shortest word length in the alphabet X.

For $g \in G$, let $[g]$ denote the conjugacy class of g in G. As noted in [GS91], the translation number is constant on each conjugacy class in G. The following definitions were suggested by Conner [Con00] and Kapovich [Kap97].

Definition 1.2. A finitely generated group G is said to be

(1) *translation separable* (or *translation proper*) if for some (and hence for any) finite set X of semigroup generators for G the translation numbers of non-torsion elements are strictly positive;
(2) *translation discrete* if it is translation separable and for some (and hence for any) finite set X of semigroup generators for G the set $t_X(G)$ has 0 as an isolated point;
(3) *strongly translation discrete* if it is translation separable and for some (and hence for any) finite X of semigroup generators for G and for any real number r the number of conjugacy classes $[g]$ in G with $t_X(g) \leqslant r$ is finite.

There are good properties of groups which have some kind of translation discreteness. Gersten and Short [GS91] proved that every finitely generated nilpotent subgroup of a translation separable group is virtually abelian. Conner [Con00] proved that a translation separable solvable group of finite virtual cohomological dimension is metabelian by finite and that every solvable subgroup of finite virtual cohomological dimension in a translation discrete group is a finite extension of $\mathbb{Z}^m$.

Therefore, it would be interesting to study translation discreteness of groups. Biautomatic groups are translation separable by Gersten and Short [GS91]. Word hyperbolic groups are strongly translation discrete. Moreover, the translation numbers in a word hyperbolic group are rational and have bounded denominators. This follows from a claim of Gromov [Gro87, 5.2C] and was accurately proved by Swenson [Swe95, Corollary of Theorem 13]. Kapovich [Kap97] proved that C(4)-T(4)-P, C(3)-T(6)-P and C(6)-P small cancellation groups are strongly translation discrete. (These groups are biautomatic but not necessarily word hyperbolic by Gersten and Short [GS90].) Bestvina [Bes99] showed that the Artin groups of finite type are translation discrete.

The class of Garside groups provides a lattice-theoretic generalization of the braid groups and the Artin groups of finite type. Garside groups have been studied actively for the last ten years. Recently, Charney, Meier and Whittlesey [CMW04] proved that a Garside group is translation discrete under the condition that it has a tame Garside element, generalizing the work of Bestvina. See Section 7 for the definition of tameness. In this paper, we show

Main Theorem. (Theorem 7.2) *All Garside groups are strongly translation discrete.*

We remark that we do not need the existence of a tame Garside element. Moreover, even when restricted to the Artin groups of finite type, our result is stronger than that of Bestvina in [Bes99].

Our approach is different from that of Bestvina (and so that of Charney, Meier and Whittlesey). We show that if G is a Garside group, then so is the semidirect product $G(n) = \mathbb{Z} \ltimes G^n$ of the infinite cyclic group $\mathbb{Z}$ and the cartesian product G^n, where the action of $\mathbb{Z}$ on G^n is a coordinate change. (It is a special case of the crossed product of Picantin [Pic01a].) Studying the relationship between the Garside structure of $G(n)$ and that of G, we show that

$$\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1$$

for all $n \geqslant 1$, where $\inf_s$ denotes the maximal infimum for the conjugacy class. The main theorem follows from it.

Another result is a new solution to the root problem in Garside groups. The root problem in a group G is to solve the equation $x^n = g$ in G for x, given $g \in G$ and $n \geqslant 2$. Styšhnev [Sty78] showed that the problem is solvable for the braid groups, and Sibert [Sib02] obtained the same result for the Garside groups under the assumption that the positive conjugacy classes are all finite. See Section 8 for the exact description of his assumption. Roughly, Styšhnev and Sibert showed the following.

> Let G be a Garside group and G^+ its positive monoid. Given $g \in G$ and $n \geqslant 2$, there exists a finite set $C \subset G^+$ such that $x^n = g$ has a solution in G if and only if $x^n = c$ has a solution in G^+ for some $c \in C$.

Note that we can solve the equation $x^n = c$ in G^+ using a sort of exhaustive search. In this paper we show that the root problem in a Garside group G can be reduced to a conjugacy problem in $G(n)$. Our solution does not need any extra condition.

2. Garside monoids and groups

In the late sixties, Garside [Gar69] solved the word and conjugacy problems in the braid groups studying the positive braid monoids. His theory has been generalized and improved by several mathematicians [BS72,Thu92,EM94,BKL98,DP99,BKL01,FG03,Pic01b,Geb05]. Recently, Dehornoy and Paris [DP99] introduced the notion of Garside groups, which is a lattice-theoretic generalization of the braid groups and the Artin groups of finite type. Here we collect relevant information about the word and conjugacy problems in Garside groups. See [DP99, Deh02,FG03,Pic01b,Geb05] for details.

Let M be a monoid. Let *atoms* (or 'indivisible elements') be the elements $a \in M$ such that $a \neq 1$ and if $a = bc$ then either $b = 1$ or $c = 1$. For $a \in M$, let $\|a\|$ be the supremum of the lengths of all expressions of a in terms of atoms. The monoid M is said to be *atomic* if it is generated by its atoms and $\|a\| < \infty$ for any $a \in M$. In an atomic monoid M, there are partial orders $\leqslant_L$ and $\leqslant_R$: $a \leqslant_L b$ if $ac = b$ for some $c \in M$; $a \leqslant_R b$ if $ca = b$ for some $c \in M$. For $a \in M$, let $L(a) = \{b \in M\colon\ b \leqslant_L a\}$ and $R(a) = \{b \in M\colon\ b \leqslant_R a\}$.

Definition 2.1. An atomic monoid M is called a *Garside monoid* if

(1) M is finitely generated;
(2) M is left and right cancellative;
(3) $(M, \leqslant_L)$ and $(M, \leqslant_R)$ are lattices;

(4) there exists an element Δ, called a *Garside element*, such that $L(\Delta)$ and $R(\Delta)$ are the same and they form a set of generators for M.

The elements of $L(\Delta)$ are called the *simple elements*. Let $\mathcal{D}$ be the set of simple elements, that is, $\mathcal{D} = L(\Delta)$. Let $\wedge_L$ and $\vee_L$ (respectively, $\wedge_R$ and $\vee_R$) denote the gcd and lcm with respect to $\leqslant_L$ (respectively, $\leqslant_R$).

Garside monoids satisfy Ore's conditions, and thus embed in their groups of fractions. A *Garside group* is defined to be the group of fractions of a Garside monoid. When M is a Garside monoid and G the group of fractions of M, we identify the elements of M and their images in G and call them the *positive elements* of G. M is called the *positive monoid* of G, often denoted G^+. The partial orders $\leqslant_L$ and $\leqslant_R$, and thus the lattice structures in the positive monoid G^+ can be extended to the Garside group G as follows: $g \leqslant_L h$ (respectively, $g \leqslant_R h$) for $g, h \in G$ if $gc = h$ (respectively, $cg = h$) for some $c \in G^+$. Let $\tau: G \to G$ be the inner automorphism of G defined by $\tau(g) = \Delta^{-1} g \Delta$.

Theorem 2.2. *Let G be a Garside group, G^+ its positive monoid and Δ a Garside element.*

(1) $\tau(G^+) = G^+$.
(2) *There is an integer e such that Δ^e is central in G.*
(3) *For $g \in G$, there are integers r and s such that $\Delta^r \leqslant_L g \leqslant_L \Delta^s$.*
(4) *For $g \in G$, there are unique simple elements $s_1, \ldots, s_k \in \mathcal{D} \setminus \{1, \Delta\}$ such that*

$$g = \Delta^r s_1 \cdots s_k,$$

and $(s_i s_{i+1} \cdots s_k) \wedge_L \Delta = s_i$ for $i = 1, \ldots, k$.

By (3) the following invariants are well defined: $\inf(g) = \max\{r \in \mathbb{Z}\colon\ \Delta^r \leqslant_L g\}$; $\sup(g) = \min\{s \in \mathbb{Z}\colon\ g \leqslant_L \Delta^s\}$; $\operatorname{len}(g) = \sup(g) - \inf(g)$. The expression in (4) is called the *normal form* of g. If $\Delta^r s_1 \cdots s_k$ is the normal form of g, then $\inf(g) = r$ and $\sup(g) = r + k$.

We remark that τ and the invariants $\inf(g)$, $\sup(g)$ and $\operatorname{len}(g)$ depend on the choice of the Garside element. (In Section 3, we discuss non-uniqueness of Garside elements.) Throughout the paper, when we mention a Garside monoid/group without specifying its Garside element, we always assume that a Garside element is chosen and fixed.

Definition 2.3. For $g \in G$, $[g]$ denotes its conjugacy class. We define $\inf_s(g) = \max\{\inf(h)\colon h \in [g]\}$, $\sup_s(g) = \min\{\sup(h)\colon\ h \in [g]\}$ and $\operatorname{len}_s(g) = \sup_s(g) - \inf_s(g)$.

Definition 2.4. For $g \in G$, the set $[g]^S = \{h \in [g]\colon\ \inf(h) = \inf_s(g),\ \sup(h) = \sup_s(g)\}$ is called the *super-summit set* of g.

Definition 2.5. Let $\Delta^r s_1 \cdots s_k$ be the normal form of $g \in G$. Then $\mathbf{c}(g) = \Delta^r s_2 \cdots s_k \tau^{-r}(s_1)$ and $\mathbf{d}(g) = \Delta^r \tau^r(s_k) s_1 \cdots s_{k-1}$ are called the *cycling* and *decycling* of g.

Theorem 2.6. *Let G be a Garside group and $g \in G$.*

(1) *If $\mathbf{c}^k(g) = g$ for some $k \geqslant 1$, then* $\inf(g) = \inf_s(g)$.
(2) *If $\mathbf{d}^k(g) = g$ for some $k \geqslant 1$, then* $\sup(g) = \sup_s(g)$.

(3) *If* $h \in [g]^S$*, then* $\mathbf{c}(h), \mathbf{d}(h), \tau(h) \in [g]^S$.
(4) *If* $h \in [g]^S$*, then* $\tau(\mathbf{c}(h)) = \mathbf{c}(\tau(h))$ *and* $\tau(\mathbf{d}(h)) = \mathbf{d}(\tau(h))$.
(5) $\mathbf{c}^k(\mathbf{d}^l(g)) \in [g]^S$ *for some* $k, l \geqslant 0$.
(6) $[g]^S$ *is finite and non-empty.*
(7) *For* $h, h' \in [g]^S$*, there is a finite sequence* $h = h_0 \to h_1 \to \cdots \to h_m = h'$ *such that for* $i = 1, \ldots, m$, $h_i \in [g]^S$ *and* $h_i = s_i^{-1} h_{i-1} s_i$ *for some* $s_i \in \mathcal{D}$.

Recently, Franco and González-Meneses [FG03] improved the algorithm to generate the super-summit set $[g]^S$ from an element of $[g]^S$. Gebhardt [Geb05] defined the *ultra-summit set* of $g \in G$ as $[g]^U = \{h \in [g]^S \colon \mathbf{c}^k(h) = h \text{ for some } k > 0\}$ and showed that it satisfies most important properties of the super-summit set. (Theorem 2.6 holds when we replace the super-summit set $[g]^S$ with the ultra-summit set $[g]^U$.)

We will often use the following lemma later. It is easy to prove and so we omit the proof.

Lemma 2.7. *Let* G *be a Garside group and* $g \in G$.

$$\mathbf{d}(g) = \left(\tau^{-1}\left(\mathbf{c}\left(g^{-1}\right)\right)\right)^{-1}, \qquad \sup(g) = -\inf\left(g^{-1}\right), \qquad \sup_s(g) = -\inf_s\left(g^{-1}\right).$$

3. Minimal Garside element

In a Garside monoid, Garside elements are not unique. For example, if Δ is a Garside element, then Δ^m is also a Garside element for any $m \geqslant 1$. It is known by Dehornoy that there exists a unique Garside element whose $\|\cdot\|$-norm is minimal. See [Deh02, the discussion after Definition 1.11 in p. 275]. In this section, we give an another proof.

Lemma 3.1. *Let* G^+ *be a Garside monoid and* $a, b \in G^+$*. If* $L(a) = R(b)$*, then* $a = b$.

Proof. Since $a \in L(a) = R(b)$ and $b \in R(b) = L(a)$, $a \leqslant_R b$ and $b \leqslant_L a$. Therefore, $b = c_1 a$ and $a = bc_2$ for some $c_1, c_2 \in G^+$, and thus $b = c_1 a = c_1 b c_2$. Since

$$\|b\| = \|c_1 b c_2\| \geqslant \|c_1\| + \|b\| + \|c_2\| \geqslant \|b\|,$$

$\|c_1\| = \|c_2\| = 0$. Hence, $c_1 = c_2 = 1$, and thus $a = b$. □

Lemma 3.2. *If* Δ_1 *and* Δ_2 *are Garside elements of a Garside monoid* G^+*, then* $\Delta_1 \wedge_L \Delta_2 = \Delta_1 \wedge_R \Delta_2$ *and it is a Garside element of* G^+.

Proof. Note that for $a, b \in G^+$, $L(a \wedge_L b) = L(a) \cap L(b)$ and $R(a \wedge_R b) = R(a) \cap R(b)$. Since $L(\Delta_i) = R(\Delta_i)$ for $i = 1, 2$,

$$L(\Delta_1 \wedge_L \Delta_2) = L(\Delta_1) \cap L(\Delta_2) = R(\Delta_1) \cap R(\Delta_2) = R(\Delta_1 \wedge_R \Delta_2).$$

By Lemma 3.1, $\Delta_1 \wedge_L \Delta_2 = \Delta_1 \wedge_R \Delta_2$. Since the generating subsets of an atomic monoid are exactly those subsets that contain all atoms, the intersection of any collection of generating subsets is a generating subset. Therefore $L(\Delta_1 \wedge_L \Delta_2) = L(\Delta_1) \cap L(\Delta_2)$ is a generating subset of G^+, and thus $\Delta_1 \wedge_L \Delta_2$ is a Garside element. □

Lemma 3.3. *In a Garside monoid G^+, there is a unique Garside element that is minimal with respect to both $\leqslant_L$ and $\leqslant_R$ among all Garside elements of G^+.*

Proof. Since a Garside monoid is finitely generated, G^+ has countably many elements, and thus countably many Garside elements, say $\{\Delta_j\}_{j\geqslant 1}$. Let $\Delta^{(i)} = \Delta_1 \wedge_L \Delta_2 \wedge_L \cdots \wedge_L \Delta_i$. Since $\Delta^{(i+1)} \leqslant_L \Delta^{(i)}$ for all i, the atomicity of G^+ implies $\Delta^{(m)} = \Delta^{(m+1)} = \cdots$ for some $m \geqslant 1$. By Lemma 3.2, $\Delta^{(m)}$ is a Garside element. Since it is both the left and right gcd of all Garside elements, it is minimal with respect to both $\leqslant_L$ and $\leqslant_R$ among all Garside elements of G^+. Uniqueness is obvious. □

Definition 3.4. The unique Garside element of Lemma 3.3 is called the *minimal Garside element* of the Garside monoid G^+.

Lemma 3.5. *Let ϕ be a monoid automorphism of a Garside monoid G^+.*

(1) *ϕ permutes the atoms of G^+.*
(2) *For $a, b \in G^+$, $a \leqslant_L b$ if and only if $\phi(a) \leqslant_L \phi(b)$. The same is true for $\leqslant_R$.*
(3) *For $a, b \in G^+$, $\phi(a \wedge_L b) = \phi(a) \wedge_L \phi(b)$. The same is true for $\vee_L$, $\wedge_R$ and $\vee_R$.*
(4) *If $\phi(\Delta) = \Delta$ for a Garside element Δ, then ϕ permutes the simple elements.*
(5) *If Δ is the minimal Garside element of G^+, then $\phi(\Delta) = \Delta$.*

Proof. (1), (2) and (3) are easy. See [DP99, Lemma 9.1, p. 599]. (4) is immediate from (2). Let us prove (5). Because the automorphism ϕ respects the partial orders $\leqslant_L$ and $\leqslant_R$, $\phi(\Delta)$ is the minimal Garside element of G^+, hence $\phi(\Delta)$ is equal to Δ by the uniqueness of the minimal Garside element. □

The following theorem summarizes the discussions in this section.

Theorem 3.6. *In a Garside monoid G^+, there exists a unique Garside element Δ that is minimal with respect to both $\leqslant_L$ and $\leqslant_R$ among all Garside elements of G^+. Furthermore, for any monoid automorphism ϕ of G^+, $\phi(\Delta) = \Delta$.*

4. Semidirect product of Garside monoids

Let G and H be Garside groups. Let G (and hence G^+) act on H^+ on the right via a homomorphism $\rho: G \to \mathrm{Aut}(H^+)$, where $\mathrm{Aut}(H^+)$ is the group of automorphisms of the monoid H^+. We use the notation b^g to denote the action of $g \in G$ on $b \in H^+$ via ρ, that is, $b^g = \rho(g)(b)$. Recall that the semidirect product $G^+ \ltimes_\rho H^+$ has the underlying set $\{(a, b)\colon\ a \in G^+,\ b \in H^+\}$ and the product

$$(a_1, b_1)(a_2, b_2) = \bigl(a_1 a_2, b_1^{a_2} b_2\bigr), \quad a_1, a_2 \in G^+ \text{ and } b_1, b_2 \in H^+.$$

Let $G \ltimes_\rho H$ denote the semidirect product of the Garside groups G and H via the homomorphism $G \to \mathrm{Aut}(H)$ that is the composition of ρ with the canonical monomorphism $\mathrm{Aut}(H^+) \to \mathrm{Aut}(H)$. If there is no confusion, we omit ρ from the notation of semidirect products. The main goal of this section is to prove the following theorem.

Theorem 4.1. *If G and H are Garside groups and $\rho : G \to \mathrm{Aut}(H^+)$ is a homomorphism, then $G^+ \ltimes_\rho H^+$ is a Garside monoid and its group of fractions is $G \ltimes_\rho H$. Moreover, if Δ_G and Δ_H are Garside elements of G and H, and Δ_H is fixed under the action of G, then (Δ_G, Δ_H) is a Garside element of $G^+ \ltimes_\rho H^+$ and the set of simple elements is $\{(a,b)\colon a \leqslant_L \Delta_G,\ b \leqslant_L \Delta_H\}$.*

In fact, Theorem 4.1 is already known by Picantin. He showed that the crossed product of Garside monoids is a Garside monoid [Pic01a, Proposition 3.12]. Our semidirect product is a special case of his crossed product. Because his construction is more general than ours, the proof in his paper is longer and more complicated. So we include the proof. Theorem 4.1 is a consequence of the following proposition.

Proposition 4.2. *Let $G^+ \ltimes H^+$ and $G \ltimes H$ be the semidirect products of Garside monoids and groups via a homomorphism $\rho : G \to \mathrm{Aut}(H^+)$. Let $(a_1, b_1), (a_2, b_2) \in G^+ \ltimes H^+$.*

(1) *$G^+ \ltimes H^+$ is (left and right) cancellative.*
(2) *$G^+ \ltimes H^+$ is atomic.*
(3) *$(a_1, b_1) \leqslant_L (a_2, b_2)$ if and only if $a_1 \leqslant_L a_2$ and $b_1^{a_1^{-1}} \leqslant_L b_2^{a_2^{-1}}$.*
(4) *$(a_1, b_1) \leqslant_R (a_2, b_2)$ if and only if $a_1 \leqslant_R a_2$ and $b_1 \leqslant_R b_2$.*
(5) *$(a_1, b_1) \wedge_L (a_2, b_2) = (a_1 \wedge_L a_2, (b_1^{a_1^{-1}} \wedge_L b_2^{a_2^{-1}})^{a_1 \wedge_L a_2})$.*
(6) *$(a_1, b_1) \vee_L (a_2, b_2) = (a_1 \vee_L a_2, (b_1^{a_1^{-1}} \vee_L b_2^{a_2^{-1}})^{a_1 \vee_L a_2})$.*
(7) *$(a_1, b_1) \wedge_R (a_2, b_2) = (a_1 \wedge_R a_2, b_1 \wedge_R b_2)$.*
(8) *$(a_1, b_1) \vee_R (a_2, b_2) = (a_1 \vee_R a_2, b_1 \vee_R b_2)$.*
(9) *Let Δ_G and Δ_H be Garside elements of G and H. If Δ_H is fixed under the action of G, then (Δ_G, Δ_H) is a Garside element of $G^+ \ltimes H^+$ and the set of simple elements is $\{(a,b)\colon a \leqslant_L \Delta_G,\ b \leqslant_L \Delta_H\}$.*
(10) *$G^+ \ltimes H^+$ is a Garside monoid and its group of fractions is $G \ltimes H$.*

Proof. (1) is obvious.

(2) The atoms of $G^+ \ltimes H^+$ are of the form either $(a, 1)$ for an atom $a \in G^+$ or $(1, b)$ for an atom $b \in H^+$. Therefore, $G^+ \ltimes H^+$ is generated by its atoms. Since $\|(a,b)\| = \|a\| + \|b\| < \infty$ for $(a,b) \in G^+ \ltimes H^+$, $G^+ \ltimes H^+$ is atomic.

(3) If $(a_1, b_1) \leqslant_L (a_2, b_2)$, then $(a_2, b_2) = (a_1, b_1)(a_3, b_3) = (a_1 a_3, b_1^{a_3} b_3)$ for some $(a_3, b_3) \in G^+ \ltimes H^+$. Since $a_2 = a_1 a_3$, $a_1 \leqslant_L a_2$. Since $b_2 = b_1^{a_3} b_3 = b_1^{a_1^{-1} a_2} b_3$, $b_1^{a_1^{-1} a_2} \leqslant_L b_2$, and thus $b_1^{a_1^{-1}} \leqslant_L b_2^{a_2^{-1}}$. Conversely, if $a_1 \leqslant_L a_2$ and $b_1^{a_1^{-1}} \leqslant_L b_2^{a_2^{-1}}$, then $a_2 = a_1 a_3$ and $b_2^{a_2^{-1}} = b_1^{a_1^{-1}} b_3$ for some $(a_3, b_3) \in G^+ \ltimes H^+$. Since $(a_1, b_1)(a_3, b_3^{a_2}) = (a_1 a_3, b_1^{a_3} b_3^{a_2}) = (a_2, b_1^{a_1^{-1} a_2} b_3^{a_2}) = (a_2, (b_1^{a_1^{-1}} b_3)^{a_2}) = (a_2, b_2)$, $(a_1, b_1) \leqslant_L (a_2, b_2)$.

(4) can be proved similarly to (3).

(5)

$$
\begin{aligned}
&(a_3, b_3) \leqslant_L (a_1, b_1) \quad \text{and} \quad (a_3, b_3) \leqslant_L (a_2, b_2) \\
&\quad \Leftrightarrow \quad a_3 \leqslant_L a_1, \qquad a_3 \leqslant_L a_2, \qquad b_3^{a_3^{-1}} \leqslant_L b_1^{a_1^{-1}}, \qquad b_3^{a_3^{-1}} \leqslant_L b_2^{a_2^{-1}} \quad \text{by (3)} \\
&\quad \Leftrightarrow \quad a_3 \leqslant_L (a_1 \wedge_L a_2) \quad \text{and} \quad b_3^{a_3^{-1}} \leqslant_L \bigl(b_1^{a_1^{-1}} \wedge_L b_2^{a_2^{-1}}\bigr) \quad \text{by the definition of } \wedge_L \\
&\quad \Leftrightarrow \quad (a_3, b_3) \leqslant_L \bigl(a_1 \wedge_L a_2, \bigl(b_1^{a_1^{-1}} \wedge_L b_2^{a_2^{-1}}\bigr)^{(a_1 \wedge_L a_2)}\bigr) \quad \text{by (3).}
\end{aligned}
$$

(6), (7) and (8) can be proved similarly to (5).

(9) $L((\Delta_G, \Delta_H)) = R((\Delta_G, \Delta_H))$ because

$$
\begin{aligned}
&(a,b) \leqslant_L (\Delta_G, \Delta_H) \\
&\quad \Leftrightarrow \quad a \leqslant_L \Delta_G, \qquad b^{a^{-1}} \leqslant_L (\Delta_H)^{\Delta_G^{-1}} \quad \text{by (3)} \\
&\quad \Leftrightarrow \quad a \leqslant_L \Delta_G, \qquad b \leqslant_L \Delta_H \quad \text{since } \Delta_H \text{ is fixed under the action of } G \\
&\quad \Leftrightarrow \quad a \leqslant_R \Delta_G, \qquad b \leqslant_R \Delta_H \quad \text{by definition of Garside elements} \\
&\quad \Leftrightarrow \quad (a,b) \leqslant_R (\Delta_G, \Delta_H) \quad \text{by (4).}
\end{aligned}
$$

In particular, $R((\Delta_G, \Delta_H)) = \{(a,b)\colon a \leqslant_R \Delta_G,\ b \leqslant_R \Delta_H\}$ contains all the atoms of $G^+ \ltimes H^+$, and thus it generates $G^+ \ltimes H^+$. Hence (Δ_G, Δ_H) is a Garside element and the set of simple elements is as in the statement.

(10) $G^+ \ltimes H^+$ is finitely generated since G^+ and H^+ are finitely generated; cancellative by (1); atomic by (2). $(G^+ \ltimes H^+, \leqslant_L)$ and $(G^+ \ltimes H^+, \leqslant_R)$ are lattices by (5), (6), (7) and (8). Since, at least, the minimal Garside element of H is fixed under the action of G, $G^+ \ltimes H^+$ has a Garside element by (9). Hence $G^+ \ltimes H^+$ is a Garside monoid. It is easy to see that $G \ltimes H$ is the group of fractions of $G^+ \ltimes H^+$. □

We remark that our semidirect product is different from that of Crisp and Paris in [CP05]. They studied the semidirect product of the form $(H * \cdots * H) \rtimes_\rho B_n$, where B_n is the n-braid group and $\rho : B_n \to \mathrm{Aut}(H * \cdots * H)$ is an Artin type representation.

Example 4.3. (1) For a Garside group G, the cartesian product $G^n = G \times \cdots \times G$ is a Garside group with a Garside element $(\Delta, \ldots, \Delta)$, where Δ is a Garside element of G.

(2) The n-braid group B_n acts on G^n by $(g_1, \ldots, g_n)^\alpha = (g_{\theta^{-1}(1)}, \ldots, g_{\theta^{-1}(n)})$, $\alpha \in B_n$, where θ is the induced permutation of α. The semidirect product $B_n \ltimes G^n$ is a Garside group. Note that if G is the m-braid group B_m, $B_n \ltimes (B_m)^n$ consists of reducible braids.

The second example is similar to the wreath product, which is a semidirect product of a cartesian product of a group and a subgroup of a symmetric group, where the action is the coordinate change. Since Garside groups are torsion-free, we use a braid group instead of a subgroup of a symmetric group.

Lemma 4.4. *Let $G \ltimes H$ be the semidirect product of Garside groups G and H via a homomorphism $\rho\colon G \to \mathrm{Aut}(H^+)$. Let Δ_G and Δ_H be Garside elements of G and H such that Δ_H is fixed under the action of G. For $(g,h) \in G \ltimes H$, the following hold, where (Δ_G, Δ_H) is the Garside element of $G \ltimes H$ used for* (3), (4) *and* (5).

(1) $(g,h)^{-1} = (g^{-1}, (h^{-1})^{g^{-1}})$.
(2) $(\Delta_G, \Delta_H)^k = (\Delta_G^k, \Delta_H^k)$ *for all $k \in \mathbb{Z}$.*
(3) $\tau((g,h)) = (\tau(g), \tau(h^{\Delta_G}))$.
(4) $\inf((g,h)) = \min\{\inf(g), \inf(h)\}$.
(5) $\sup((g,h)) = \max\{\sup(g), \sup(h)\}$.

Proof. (1) and (2) are obvious.

(3) $\tau((g,h)) = (\Delta_G^{-1}, \Delta_H^{-1})(g,h)(\Delta_G, \Delta_H) = (\Delta_G^{-1} g \Delta_G, \Delta_H^{-1} h^{\Delta_G} \Delta_H) = (\tau(g), \tau(h^{\Delta_G}))$.

(4) By Proposition 4.2(4), $(\Delta_G^k, \Delta_H^k) \leqslant_R (g,h)$ if and only if $\Delta_G^k \leqslant_R g$ and $\Delta_H^k \leqslant_R h$. Since $(\Delta_G, \Delta_H)^k = (\Delta_G^k, \Delta_H^k)$, $\inf(g,h) = \min\{\inf(g), \inf(h)\}$.

(5) can be proved similarly to (4). □

We close this section showing that if Δ_G and Δ_H are the minimal Garside elements of G and H, respectively, then (Δ_G, Δ_H) is the minimal Garside element of $G \ltimes H$. Because we do not have a proof using only the techniques developed in this paper, we use the characterization of Garside elements by Dehornoy [Deh02, Proposition 1.10].

Let us say a subset S of a Garside monoid *LC-closed* if the following holds.

- If $a, b \in S$, then $a \wedge_L b$, $a \wedge_R b$, $a \vee_L b$, $a \vee_R b$, $a^{-1}(a \vee_L b)$, $(a \vee_R b)a^{-1} \in S$.
- If $a \in S$ and $b \leqslant_L a$ or $b \leqslant_R a$, then $b \in S$.

(LC stands for lattice operations and complements. The elements $a^{-1}(a \vee_L b)$ and $(a \vee_R b)a^{-1}$ are called the right complement of a in b and the left complement of a in b, respectively [Deh02].) For a subset A of a Garside monoid, let *LC-closure* of A be the smallest LC-closed set containing A. Then the following holds.

- Δ is a Garside element and $\mathcal{D}$ is the set of simple elements corresponding to Δ if and only if $\mathcal{D}$ is an LC-closed finite generating set of the Garside monoid and Δ is the right lcm of the elements of $\mathcal{D}$ [Deh02, Proposition 1.10].
- Δ is the minimal Garside element and $\mathcal{D}$ is the set of simple elements corresponding to Δ if and only if $\mathcal{D}$ is the LC-closure of the set of atoms and Δ is the right lcm of all elements of $\mathcal{D}$ [Deh02, discussion after Definition 1.11, p. 275].

Lemma 4.5. *If Δ_G and Δ_H are the minimal Garside elements of G and H, respectively, then (Δ_G, Δ_H) is the minimal Garside element of $G \ltimes H$.*

Proof. Let Δ_G, Δ_H and Δ be the minimal Garside elements of G, H and $G \ltimes H$, respectively. We show that (Δ_G, Δ_H) is equal to Δ.

Because (Δ_G, Δ_H) is a Garside element of $G \ltimes H$ and Δ is the minimal Garside element of $G \ltimes H$, we have $\Delta \leqslant_R (\Delta_G, \Delta_H)$.

Let $A = \{(a,1)\colon a \text{ is an atom of } G^+\}$ and $B = \{(1,b)\colon b \text{ is an atom of } H^+\}$. It is easy to see that the LC-closure of A is $\{(s,1)\colon s \leqslant \Delta_G\}$, observing that for $a_1, a_2 \in G^+$ $(a_1,1) \wedge_L (a_2,1) = (a_1 \wedge_L a_2, 1)$ and the same is true for $\vee_L$, $\wedge_R$ and $\vee_R$. In particular, the LC-closure of A contains $(\Delta_G, 1)$. Similarly, the LC-closure of B contains $(1, \Delta_H)$. Therefore the LC-closure of $A \cup B$ contains $(\Delta_G, 1) \vee_R (1, \Delta_H) = (\Delta_G, \Delta_H)$. Because $A \cup B$ is the set of atoms of $G \ltimes H$, Δ is the right lcm of the element of the LC-closure of $A \cup B$. Hence $(\Delta_G, \Delta_H) \leqslant_R \Delta$.

Consequently, (Δ_G, Δ_H) is equal to Δ. □

5. The product $G(n) = \mathbb{Z} \ltimes G^n$

Definition 5.1. For a Garside group G, we define $G(n)$ as the semidirect product $\mathbb{Z} \ltimes G^n$, where $\mathbb{Z} = \langle \delta \rangle$ acts on the cartesian product G^n by $(g_1, \ldots, g_n)^\delta = (g_n, g_1, \ldots, g_{n-1})$. The element $(\delta^k, (g_1, \ldots, g_n)) \in G(n)$ is denoted $\delta^k(g_1, \ldots, g_n)$.

The infinite cyclic group $\mathbb{Z} = \langle\delta\rangle$ is a Garside group with minimal Garside element δ. (Observe that $\mathbb{Z}$ is isomorphic to the 2-strand braid group B_2.) Therefore, $G(n)$ is a Garside group and if Δ is a Garside element of G, then $\delta(\Delta, \dots, \Delta)$ is a Garside element of $G(n)$. Throughout the entire section, the Garside elements δ, Δ and $\delta(\Delta, \dots, \Delta)$ are used for the respective groups.

Lemma 5.2. *In $G(n)$, the following hold.*

(1) $\inf(\delta^k(g_1, \dots, g_n)) = \min\{k, \inf(g_1), \dots, \inf(g_n)\}$.
(2) $\sup(\delta^k(g_1, \dots, g_n)) = \max\{k, \sup(g_1), \dots, \sup(g_n)\}$.
(3) $\tau(\delta^k(g, \dots, g)) = \delta^k(\tau(g), \dots, \tau(g))$.
(4) $\mathbf{c}(\delta^k(g, \dots, g)) = \begin{cases} \delta^k(\mathbf{c}(g), \dots, \mathbf{c}(g)) & \text{if } k \geqslant \inf(g), \\ \delta^k(\tau(g), \dots, \tau(g)) & \text{if } k < \inf(g). \end{cases}$
(5) $\mathbf{d}(\delta^k(g, \dots, g)) = \begin{cases} \delta^k(\mathbf{d}(g), \dots, \mathbf{d}(g)) & \text{if } k \leqslant \sup(g), \\ \delta^k(g, \dots, g) & \text{if } k > \sup(g). \end{cases}$

Proof. (1), (2) and (3) are immediate consequences of Lemma 4.4. We prove (4) for the case $k = \inf(g)$. The other cases of (4) and (5) can be proved similarly. Let $\alpha = \delta^k(g, \dots, g)$ and $g = \Delta^k sa$, where $s = \Delta \wedge_L (sa)$. Let $\mathbf{D} = \delta(\Delta, \dots, \Delta)$. Then, $\inf(\alpha) = k$ by (1) and

$$\alpha = \delta^k(g, \dots, g) = \delta^k\big(\Delta^k sa, \dots, \Delta^k sa\big) = \mathbf{D}^k(sa, \dots, sa).$$

Since $\mathbf{D} \wedge_L (sa, \dots, sa) = (s, \dots, s)$ by Proposition 4.2(3),

$$\begin{aligned} \mathbf{c}(\alpha) &= \mathbf{D}^k(a, \dots, a)\tau^{-k}((s, \dots, s)) = \mathbf{D}^k(a, \dots, a)\big(\tau^{-k}(s), \dots, \tau^{-k}(s)\big) \\ &= \mathbf{D}^k\big(a\tau^{-k}(s), \dots, a\tau^{-k}(s)\big) = \delta^k\big(\Delta^k a\tau^{-k}(s), \dots, \Delta^k a\tau^{-k}(s)\big) \\ &= \delta^k\big(\mathbf{c}(g), \dots, \mathbf{c}(g)\big). \qquad \square \end{aligned}$$

Corollary 5.3. *Let $\alpha = \delta^k(g, \dots, g) \in G(n)$.*

(1) *The ultra-summit set of α contains an element of the form $\delta^m(h, \dots, h)$.*
(2) *If g is contained in its super-summit set, then so is α.*
(3) *If g is contained in its ultra-summit set, then so is α.*

Proof. (1) There exists $l_1, l_2 \geqslant 0$ such that $\mathbf{c}^{l_1}\mathbf{d}^{l_2}(\alpha) \in [\alpha]^U$. By Lemma 5.2 (4) and (5), $\mathbf{c}^{l_1}\mathbf{d}^{l_2}(\alpha)$ is of the form $\delta^m(h, \dots, h)$.

(2) Note that for $h \in [g]^S$ and $l \geqslant 1$, $\inf(h) = \inf(\mathbf{c}^l(h)) = \inf(\mathbf{d}^l(h)) = \inf(\tau^l(h))$.

If $k \geqslant \inf(g)$, then $k \geqslant \inf(\mathbf{c}^l(g))$ for all $l \geqslant 1$, hence $\mathbf{c}^l(\alpha) = \delta^k(\mathbf{c}^l(g), \dots, \mathbf{c}^l(g))$ by Lemma 5.2(4). Since $\inf(\mathbf{c}^l(\alpha)) = \min\{k, \inf(\mathbf{c}^l(g))\} = \min\{k, \inf(g)\} = \inf(\alpha)$ for all $l \geqslant 1$, we obtain $\inf(\alpha) = \inf_s(\alpha)$.

Similarly, if $k < \inf(g)$, then $k < \inf(\mathbf{c}^l(g))$ for all $l \geqslant 1$, hence $\mathbf{c}^l(\alpha) = \delta^k(\tau^l(g), \dots, \tau^l(g))$ by Lemma 5.2(4). Since $\inf(\mathbf{c}^l(\alpha)) = \min\{k, \inf(\tau^l(g))\} = \min\{k, \inf(g)\} = \inf(\alpha)$ for all $l \geqslant 1$, we obtain $\inf(\alpha) = \inf_s(\alpha)$.

In both cases, we have $\inf(\alpha) = \inf_s(\alpha)$. Similarly, we obtain that $\sup(\alpha) = \sup_s(\alpha)$, hence α is contained in its super-summit set.

(3) If $k < \inf(g)$, then for any $l \geqslant 1$, $\mathbf{c}^l(\alpha) = \delta^k(\tau^l(g), \dots, \tau^l(g))$ as in (2). Note that Δ^e is central for some $e \geqslant 1$. Then $\tau^e(g) = g$, hence $\mathbf{c}^e(\alpha) = \alpha$ and $\alpha \in [\alpha]^U$.

If $k \geqslant \inf(g)$, then for any $l \geqslant 1$, $\mathbf{c}^l(\alpha) = \delta^k(\mathbf{c}^l(g), \dots, \mathbf{c}^l(g))$ as in (2). Because g is contained in its ultra-summit set, $\mathbf{c}^m(g) = g$ for some $m \geqslant 1$, hence $\mathbf{c}^m(\alpha) = \alpha$ and $\alpha \in [\alpha]^U$. □

Lemma 5.4. *If $k \equiv 1 \pmod n$, then $\delta^k(g_1, \dots, g_n)$ and $\delta^k(h_1, \dots, h_n)$ are conjugate in $G(n)$ if and only if $g_1 \cdots g_n$ and $h_1 \cdots h_n$ are conjugate in G.*

Proof. Note that $\delta^k(g_1, \dots, g_m, g_{m+1}, 1, \dots, 1)$ and $\delta^k(g_1, \dots, g_{m-1}, g_m g_{m+1}, 1, 1, \dots, 1)$ are conjugate for $1 \leqslant m \leqslant n-1$. (The latter can be obtained from the former by conjugating on the right by the element $(1, \dots, 1, g_{m+1}, 1, \dots, 1)$, where the non-trivial coordinate is in the mth position.) Therefore,

$$\delta^k(g_1, \dots, g_n) \quad \text{is conjugate to } \delta^k(g_1 \cdots g_n, 1, \dots, 1). \tag{$*$}$$

Assume that $g = g_1 \cdots g_n$ is conjugate to $h = h_1 \cdots h_n$ in G. Let $g = x^{-1}hx$. Since

$$\left(x^{-1}, \dots, x^{-1}\right)\delta^k(h, 1, \dots, 1)(x, \dots, x) = \delta^k(g, 1, \dots, 1),$$

$\delta^k(g, 1, \dots, 1)$ is conjugate to $\delta^k(h, 1, \dots, 1)$, and thus $\delta^k(g_1, \dots, g_n)$ and $\delta^k(h_1, \dots, h_n)$ are conjugate by $(*)$.

Conversely, assume that $\delta^k(g_1, \dots, g_n)$ is conjugate to $\delta^k(h_1, \dots, h_n)$ in $G(n)$. Let $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_n$. Then $\delta^k(g, 1, \dots, 1)$ is conjugate to $\delta^k(h, 1, \dots, 1)$ by $(*)$. Let $\delta^k(g, 1, \dots, 1) = \gamma^{-1}\delta^k(h, 1, \dots, 1)\gamma$ for some $\gamma = \delta^m(x_1, \dots, x_n)$. Assume $m \not\equiv 0 \pmod n$. Since

$$\begin{aligned}
\delta^k(g, 1, \dots, 1) &= \gamma^{-1}\delta^k(h, 1, \dots, 1)\gamma \\
&= \left(x_1^{-1}, \dots, x_n^{-1}\right)\delta^{-m}\delta^k(h, 1, \dots, 1)\delta^m(x_1, \dots, x_n) \\
&= \delta^k\left(x_n^{-1}, x_1^{-1}, \dots, x_{n-1}^{-1}\right)(\underbrace{1, \dots, 1}_{m}, h, \underbrace{1, \dots, 1}_{n-m-1})(x_1, \dots, x_n) \\
&= \delta^k\left(x_n^{-1}x_1, x_1^{-1}x_2, \dots, x_{m-1}^{-1}x_m, x_m^{-1}hx_{m+1}, x_{m+1}^{-1}x_{m+2}, \dots, x_{n-1}^{-1}x_n\right),
\end{aligned}$$

$x_i = x_{i+1}$ for $i \neq m, n$; $g = x_n^{-1}x_1$; $x_m = hx_{m+1}$. (Here, x_k denotes x_j such that $1 \leqslant j \leqslant n$ and $k \equiv j \pmod n$.) Since $x_n g = x_1 = x_2 = \cdots = x_m = hx_{m+1} = hx_{m+2} = \cdots = hx_n$, $g = x_n^{-1}hx_n$. Therefore g is conjugate to h. The case $m \equiv 0 \pmod n$ can be proved similarly. □

6. $\inf_s$ and $\sup_s$ of powers

Theorem 6.1. *Let G be a Garside group. For $g \in G$ and $n \geqslant 1$,*

(1) $\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1$,
(2) $\sup_s(g) - 1 < \frac{\sup_s(g^n)}{n} \leqslant \sup_s(g)$,
(3) $\operatorname{len}_s(g) - 2 < \frac{\operatorname{len}_s(g^n)}{n} \leqslant \operatorname{len}_s(g)$.

Proof. We prove only (1), because (2) follows from (1) and Lemma 2.7, and (3) follows from (1) and (2). It is obvious that $\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n}$. Since $\inf_s(g)$ depends only on the conjugacy class of g, we may assume that g is contained in its super-summit set. Let $\inf_s(g) = \inf(g) = r$. Choose an integer k such that $k > r$ and $k \equiv 1 \pmod{n}$. Let $\alpha = \delta^k(g, \dots, g) \in G(n)$. Then α is contained in its super-summit set by Corollary 5.3(2). By Lemma 5.2(1),

$$\inf_s(\alpha) = \inf(\alpha) = \min\bigl\{k, \inf(g)\bigr\} = \min\{k, r\} = r. \tag{$*$}$$

Assume that $\inf_s(g^n) \geqslant n(\inf_s(g) + 1) = n(r + 1)$. Then g^n is conjugate to $\Delta^{n(r+1)}a$ for some $a \in G^+$. Since $k \equiv 1 \pmod{n}$, $\alpha = \delta^k(g, \dots, g)$ and $\beta = \delta^k(\Delta^{r+1}, \dots, \Delta^{r+1}, \Delta^{r+1}a)$ are conjugate by Lemma 5.4. By Lemma 5.2(1),

$$\inf_s(\alpha) \geqslant \inf(\beta) = \min\{k, r + 1\} = r + 1,$$

which contradicts ($*$). Therefore $\inf_s(g^n) < n(\inf_s(g) + 1)$. □

We remark that the inequalities in Theorem 6.1 do not hold for inf, sup or len. For example, consider the 3-braid group $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. It is a Garside group with minimal Garside element $\Delta = \sigma_1\sigma_2\sigma_1$. Let $g = \sigma_1^{-(k+1)}\Delta\sigma_1^{k+1} = \sigma_1^{-k}\sigma_2\sigma_1^{k+2}$. Then $\inf(g) = -k$ and $\inf_s(g) = 1$, and for any $m \geqslant 1$, $\inf(g^{2m}) = 2m$ because $g^{2m} = \Delta^{2m}$ and Δ^2 is central. Therefore, $\frac{\inf(g^{2m})}{2m} = 1 \geqslant -k + 1 = \inf(g) + 1$ for any $k \geqslant 0$ and $m \geqslant 1$.

Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x and $\lceil x \rceil$ the smallest integer greater than or equal to x.

Corollary 6.2. *Let G be a Garside group, $g \in G$ and $n \geqslant 1$. Then $\inf_s(g)$ and $\sup_s(g)$ are uniquely determined by $(n, \inf_s(g^n), \sup_s(g^n))$ as follows*:

$$\inf_s(g) = \left\lfloor \frac{\inf_s(g^n)}{n} \right\rfloor \quad \textit{and} \quad \sup_s(g) = \left\lceil \frac{\sup_s(g^n)}{n} \right\rceil.$$

In particular, for any $n \geqslant 1$, $\inf_s(g) \geqslant 0$ if and only if $\inf_s(g^n) \geqslant 0$ and $\sup_s(g) \leqslant 0$ if and only if $\sup_s(g^n) \leqslant 0$.

Corollary 6.3. *If g^n is conjugate to g^{-n} for some $n \neq 0$, then $\inf_s(g) = -\sup_s(g)$.*

Proof. We may assume that $n \geqslant 1$. Since g^n is conjugate to g^{-n}, $\inf_s(g^n) = \inf_s(g^{-n}) = -\sup_s(g^n)$. By Corollary 6.2, we are done. □

7. Garside groups are strongly translation discrete

Generalizing Bestvina's work on the Artin groups of finite type, Charney, Meier and Whittlesey [CMW04] proved that Garside groups are translation discrete, assuming the existence of a tame Garside element. A Garside element Δ is said to be *tame* if there exists a constant $c > 0$ such that $\|\Delta^n\| \leqslant cn$ for all $n \geqslant 1$. This property has recently been explored in [Sib04], where general criteria are established that imply tameness and where it is conjectured that all Garside monoids have a tame Garside element. In this section, we prove the main theorem of this paper that all Garside groups are strongly translation discrete without any assumption.

Let G be a Garside group and $\mathcal{D}$ the set of simple elements. Abusing the notation, let $|g|_{\mathcal{D}}$ denote the shortest word length of $g \in G$ in terms of $\mathcal{D} \cup \mathcal{D}^{-1}$, in other words, $|g|_{\mathcal{D}}$ is $|g|_{\mathcal{D}\cup\mathcal{D}^{-1}}$

in the notation of Section 1. Similarly, let $t_{\mathcal{D}}(g)$ denote the translation number of $g \in G$ with respect to $\mathcal{D} \cup \mathcal{D}^{-1}$.

The shortest word length $|g|_{\mathcal{D}}$ is also known as *geodesic length*. It is known that [Cha95]

$$|g|_{\mathcal{D}} = \begin{cases} \sup(g) & \text{if } \inf(g) \geqslant 0, \\ -\inf(g) & \text{if } \sup(g) \leqslant 0, \\ \operatorname{len}(g) & \text{if } \inf(g) < 0 < \sup(g). \end{cases}$$

Theorem 7.1. *Let G be a Garside group and $\mathcal{D}$ the set of simple elements. If $g \in G$ is contained in its super-summit set, then for any $n \geqslant 1$*

$$|g|_{\mathcal{D}} - 2 < \frac{|g^n|_{\mathcal{D}}}{n} \leqslant |g|_{\mathcal{D}}.$$

In particular, $|g|_{\mathcal{D}} - 2 \leqslant t_{\mathcal{D}}(g) \leqslant |g|_{\mathcal{D}}$.

Proof. It is obvious that $\frac{|g^n|_{\mathcal{D}}}{n} \leqslant |g|_{\mathcal{D}}$. Since g is in its super-summit set, $\inf(g) = \inf_s(g)$, $\sup(g) = \sup_s(g)$ and $\operatorname{len}(g) = \operatorname{len}_s(g)$. If $\inf(g) \geqslant 0$, then $\inf(g^n) \geqslant n \inf(g) \geqslant 0$, and thus

$$\frac{|g^n|_{\mathcal{D}}}{n} = \frac{\sup(g^n)}{n} \geqslant \frac{\sup_s(g^n)}{n} > \sup_s(g) - 1 = \sup(g) - 1 = |g|_{\mathcal{D}} - 1.$$

If $\sup(g) \leqslant 0$, then $\sup(g^n) \leqslant n \sup(g) \leqslant 0$, and thus

$$\frac{|g^n|_{\mathcal{D}}}{n} = \frac{-\inf(g^n)}{n} \geqslant \frac{-\inf_s(g^n)}{n} > -\inf_s(g) - 1 = -\inf(g) - 1 = |g|_{\mathcal{D}} - 1.$$

If $\inf(g) < 0 < \sup(g)$, then $\inf_s(g^n) < 0 < \sup_s(g^n)$ by Corollary 6.2. Therefore, $\inf(g^n) < 0 < \sup(g^n)$ and

$$\frac{|g^n|_{\mathcal{D}}}{n} = \frac{\operatorname{len}(g^n)}{n} \geqslant \frac{\operatorname{len}_s(g^n)}{n} > \operatorname{len}_s(g) - 2 = \operatorname{len}(g) - 2 = |g|_{\mathcal{D}} - 2.$$

In all cases, $\frac{|g^n|_{\mathcal{D}}}{n} > |g|_{\mathcal{D}} - 2$. Therefore, $|g|_{\mathcal{D}} - 2 < \frac{|g^n|_{\mathcal{D}}}{n} \leqslant |g|_{\mathcal{D}}$ as desired. By taking $n \to \infty$, $|g|_{\mathcal{D}} - 2 \leqslant t_{\mathcal{D}}(g) \leqslant |g|_{\mathcal{D}}$. □

Theorem 7.2. *All Garside groups are strongly translation discrete.*

Proof. Let G be a Garside group and $\mathcal{D}$ the set of simple elements. Since Garside groups are biautomatic by Dehornoy [Deh02] and biautomatic groups are translation separable by Gersten and Short [GS91], G is translation separable. Therefore, it suffices to show that for any real number r, there are only finitely many conjugacy classes $[g]$ such that $t_{\mathcal{D}}(g) \leqslant r$. By Theorem 7.1, $|g|_{\mathcal{D}} \leqslant t_{\mathcal{D}}(g) + 2$ if $g \in [g]^S$. Since $t_{\mathcal{D}}(g)$ depends only on the conjugacy class of g,

$$\{[g]\colon\ t_{\mathcal{D}}(g) \leqslant r\} = \{[g]\colon\ t_{\mathcal{D}}(g) \leqslant r,\ g \in [g]^S\} \subset \{[g]\colon\ |g|_{\mathcal{D}} \leqslant r + 2\}.$$

Since there are only finitely many elements g such that $|g|_{\mathcal{D}} \leqslant r + 2$, the set $\{[g]\colon\ t_{\mathcal{D}}(g) \leqslant r\}$ is finite. □

Corollary 7.3. *Let G be a Garside group.*

(1) *Every solvable subgroup of G is finitely generated and virtually abelian.*
(2) *G cannot contain subgroups isomorphic to the additive group of rational numbers or the group of p-adic fractions $\mathbb{Q}_p = \{k/p^l \mid k \in \mathbb{Z},\ l \in \mathbb{N}\}$.*
(3) *For any finite set X of semigroup generators for G, $t_X(G)$ is a closed discrete set.*

Proof. The proof of (1) is the same as for Bestvina's Corollaries 4.2 and 4.4 in [Bes99]. (2) is a property of translation discrete groups, and (3) is a property of strongly translation discrete groups. See [Kap97] for (2) and (3). □

8. The root problem in a Garside group

Generalizing the work of Styšhnev [Sty78] on the braid groups, Sibert [Sib02] showed that the root problem is solvable for Garside groups under a certain condition, called the finiteness of positive conjugacy classes. Two elements a and b of a Garside monoid G^+ are said to be *positively conjugate* if there exists $c \in G^+$ satisfying $ac = cb$. Note that two elements of G^+ are positively conjugate if and only if they are conjugate in G, because for any element g of G, there exists an integer m such that Δ^m is central and $\Delta^m g \in G^+$. The assumption of Sibert is that the positive conjugacy classes are all finite. It is known that if a Garside monoid has a tame Garside element, then it satisfies the finiteness condition of positive conjugacy class [Sib04].

In this section, we present a new solution to the root problem in Garside groups and show that the root problem is solvable for Garside groups without any assumption.

Theorem 8.1. *Let G be a Garside group and $g \in G$.*

(1) *There are only finitely many $n \in \mathbb{N}$ such that g has an nth root.*
(2) *For each $n \in \mathbb{N}$, there are only finitely many conjugacy classes of nth roots of g.*

Proof. (1) is a direct consequence of the obvious property $t_X(g^n) = n\, t_X(g)$ of translation numbers, together with the definition of translation discreteness. See [Kap97, p. 1853]. (2) is a direct consequence of (1), together with the definition of strong translation discreteness. □

Recall that the nth root of a braid is unique up to conjugacy by González-Meneses [Gon04]. However, this is not true for Garside groups. For example, the group

$$G = \langle x_1, \dots, x_m \colon x_1^{p_1} = x_2^{p_2} = \cdots = x_m^{p_m} \rangle, \quad p_i \geqslant 2,$$

is a Garside group with a Garside element $\Delta = x_1^{p_1}$. See [DP99, Example 4]. If the exponents $p_1, \dots, p_m$ have a common divisor $n \geqslant 2$, then there are at least m conjugacy classes of nth roots of Δ.

Theorem 8.2. *Let G be a Garside group and $g \in G$. The element g has an nth root if and only if the ultra-summit set of $\delta(g, 1, \dots, 1) \in G(n)$ contains an element of the form $\delta(h, \dots, h)$. In this case, if $\delta(g, 1, \dots, 1) = \gamma^{-1}\delta(h, \dots, h)\gamma$ for some $\gamma = \delta^k(x_1, \dots, x_n)$, then $g = (x_n^{-1} h x_n)^n$.*

Proof. The first statement is immediate from Corollary 5.3 and Lemma 5.4. Suppose $\delta(g,1,\dots,1)=\gamma^{-1}\delta(h,\dots,h)\gamma$ for $\gamma=\delta^k(x_1,\dots,x_n)$. Since

$$\delta(g,1,\dots,1)=\gamma^{-1}\delta(h,\dots,h)\gamma=\delta\bigl(x_n^{-1}hx_1,x_1^{-1}hx_2,\dots,x_{n-1}^{-1}hx_n\bigr),$$

$g=x_n^{-1}hx_1$ and $x_i=hx_{i+1}$ for $i=1,\dots,n-1$. Therefore, $x_1=hx_2=h^2x_3=\cdots=h^{n-1}x_n$, and thus $g=x_n^{-1}hx_1=x_n^{-1}h(h^{n-1}x_n)=x_n^{-1}h^nx_n=(x_n^{-1}hx_n)^n$. □

Corollary 8.3. *The root problem is solvable for the Garside groups.*

We remark that Theorem 8.2 does not require the condition of finite positive conjugacy class of [Sib02]. The following algorithm solves the root problem. It seems more efficient than the algorithm of Styšhnev [Sty78] and Sibert [Sib02], because it uses the conjugacy algorithm in $G(n)$ instead of an exhaustive search in G^+.

Root Extraction Algorithm. Given a Garside group G, $g\in G$ and an integer $n\geqslant 2$:

1. Set $\alpha=\delta(g,1,\dots,1)\in G(n)$ and compute the ultra-summit set $[\alpha]^U$ of α.
2. For each $\beta\in[\alpha]^U$, test whether β is of the form $\delta(h,\dots,h)$ for some $h\in G$.
   If so, compute $\gamma=\delta^k(x_1,\dots,x_n)$ such that $\alpha=\gamma^{-1}\beta\gamma$. Then $g=(x_n^{-1}hx_n)^n$.
3. If there is no element in $[\alpha]^U$ of the form $\delta(h,\dots,h)$, then g has no nth root.

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