Transversal Heteroclinic and Homoclinic Orbits in Singular Perturbation Problems*

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In this paper we give a geometric construction of heteroclinic and homoclinic orbits for singularly perturbed differential equations. By using methods from invariant manifold theory we show that transversal intersection of stable and unstable manifolds of the reduced problem implies the existence of transversal heteroclinic or homoclinic orbits of the singularly perturbed problem. We derive analytical conditions for transversality. We show how these results can be used to prove the existence of heteroclinic and homoclinic orbits in singularly perturbed problems which depend on additional parameters. We describe a configuration which implies transversal intersection of the stable and unstable manifolds of periodic orbits and the associated chaotic dynamics.

1. INTRODUCTION

We consider singularly perturbed systems of differential equations
\[ \begin{align*}
\dot{x} &= f(x, y, \varepsilon) \\
\varepsilon \dot{y} &= g(x, y, \varepsilon)
\end{align*} \tag{1.1} \]

with \( \varepsilon \in (-\varepsilon_0, \varepsilon_0), \varepsilon_0 > 0 \) small, and \((x, y) \in M\), an open subset \( M \subseteq \mathbb{R}^{n+k}\). We assume that \((f, g) \in C^r(M \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{n+k})\), with \( r \) sufficiently large. For \( \varepsilon \neq 0 \) these equations define a smooth dynamical system on \( M \). We consider \( x \) and \( y \) as functions of the variable \( t \). Problems of the form (1.1) arise frequently in applications. Their main feature is the existence of two different timescales in the problem, i.e., the slow timescale \( t \) and the fast
timescale $\tau := t/\varepsilon$. By transforming the slow system (1.1) to the fast variable $\tau$ we obtain the equivalent fast system

$$
\begin{align*}
\dot{x}' &= \varepsilon f(x, y, \varepsilon) \\
\dot{y}' &= g(x, y, \varepsilon).
\end{align*}
$$

(1.2)

The variable $x$ is usually called the slow variable and the variable $y$ is called the fast variable. By setting $\varepsilon = 0$ in (1.1) and (1.2) we obtain two essentially different problems, the reduced problem

$$
\begin{align*}
\dot{x} &= f(x, y, 0) \\
0 &= g(x, y, 0)
\end{align*}
$$

(1.3)

and the layer problem

$$
\begin{align*}
\dot{x}' &= 0 \\
\dot{y}' &= g(x, y, 0).
\end{align*}
$$

(1.4)

Under suitable assumptions the equation $g(x, y, 0) = 0$ defines a manifold $\mathcal{S}$ on which the reduced problem (1.3) defines a dynamical system. On the other hand $\mathcal{S}$ is a manifold of equilibria for the layer problem (1.4). The reduced problem essentially captures the slow dynamics and the layer problem the fast dynamics. By appropriately combining results on the dynamics of these two limiting problems one obtains results on the dynamics of the singularly perturbed problem (1.1) for small values of $\varepsilon$. The use of methods from dynamical systems theory for this purpose goes back some time, see e.g. [7, 8, 14]. A particularly elegant and useful approach in understanding the relations between the singularly perturbed problem (1.1) and the limiting ($\varepsilon = 0$) problems (1.3) resp. (1.4) is furnished by the theory of invariant manifolds for singularly perturbed problems developed in [7]. Recently methods from homoclinic bifurcation theory have been used in the investigation of transition layers for singularly perturbed problems [11]. Our analysis of the existence of homoclinic and heteroclinic orbits is based on the combined use of invariant manifold theory and methods from homoclinic and heteroclinic bifurcation theory. Methods similar to ours are used in the analysis of singularly perturbed boundary value problems [17, 18] which extend earlier results obtained in [23].

Due to its importance in the following we describe the invariant manifold theory from [7] in some detail in Section 2. To some extent the results proved there show that singularly perturbed problems are not “all that singular” if viewed in the right way. An actual advantage over regular
perturbation problems is the decoupling of problem (1.1) into the lower-dimensional problems (1.3) and (1.4) for $\varepsilon = 0$.

In Section 3 we build on this invariant manifold approach to prove the existence of heteroclinic and homoclinic orbits in systems of the form (1.1). More specifically we develop a method to prove transversal intersection of the stable and unstable manifolds of normally hyperbolic invariant manifolds. The normally hyperbolic invariant manifolds arise from the dynamics of the reduced problem as, e.g., hyperbolic fixed points or hyperbolic periodic orbits of the reduced problem. The connecting orbits are provided by the fast dynamics described by the layer problem (1.4). This result is an extension of the results in [25] where conditions for the existence of orbits heteroclinic to hyperbolic fixed points have been given.

In Section 4 we use methods going back to [3] and further developed in [21] to give analytical conditions for the necessary transversality conditions. These Melnikov Integral type conditions depend only on the reduced problem and the layer problem.

One of the main sources for singularly perturbed problems is traveling wave problems for reaction–diffusion equations or for viscous approximations of hyperbolic conservation laws (see, e.g., [24]). We illustrate our results in Section 5 where we construct the heteroclinic orbits for the traveling wave problem of the Fitzhugh–Nagumo equations. A detailed application of our method to construct heteroclinic orbits is given in [9], where the existence of viscous profiles for all magnetohydrodynamic shock waves (see [24]) is proved.

Another application is given in Section 6 where we use the method to prove the existence of transversal orbits homoclinic to hyperbolic periodic orbits and transversal heteroclinic cycles. Assume the existence of an orbit of the layer problem (1.4) homoclinic to a point $p$ on a hyperbolic periodic orbit of the reduced problem (1.3). We show that a transversal homoclinic orbit of (1.1) exists for small $\varepsilon$, if the homoclinic orbit of the layer problem breaks as the point $p$ moves on the periodic orbit. This result then implies chaotic behavior in a neighbourhood of the periodic orbit. Similar results for heteroclinic cycles are illustrated for a specific example which has been analyzed previously (by different methods) in [15] and (by similar methods) in [11].

2. GEOMETRIC SINGULAR PERTURBATION THEORY

In this section we formulate the main result from [7] where global center, center–stable, and center–unstable manifolds for systems of the form (1.1) are constructed. The starting point of this analysis is the fast
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system (1.2) supplemented by the trivial equation \( \epsilon' = 0 \) which gives the system

\[
x' = \varepsilon f(x, y, \varepsilon) \\
y' = g(x, y, \varepsilon). \\
\epsilon' = 0
\]

(2.1)

defined on \( \mathcal{M} \times (-\varepsilon_0, \varepsilon_0) \). Let \( \mathcal{S} \) be a \( C^r \) manifold of solutions of the equation

\[
g(x, y, 0) = 0.
\]

(2.2)

We need the following facts concerning the linearization \( LF(m, 0) \) of \( F := (\varepsilon f, g, 0) \) at points \( (m, 0) \in \mathcal{S} \times (-\varepsilon_0, \varepsilon_0) \). An easy calculation shows that \( \lambda = 0 \) is a trivial eigenvalue of \( LF(m, 0) \) of algebraic multiplicity \( n + 1 \). The remaining eigenvalues are called nontrivial eigenvalues. We assume that the numbers of nontrivial eigenvalues in the left half plane, on the imaginary axis, and in the right half plane are \( k^s, k^c, \) and \( k^u \). The corresponding stable-, center, and unstable eigenspaces \( E^s_m, E^c_m, \) and \( E^u_m \) have dimensions \( k^s, n + 1 + k^c, \) and \( k^u \), respectively.

Let \( \mathcal{S}_R \) be the open set where the nontrivial eigenvalues are nonzero. The manifold \( \mathcal{S}_R \) can be characterized as

\[
\mathcal{S}_R = \{(x, y) \in \mathcal{S} : \text{rank } D_y g(x, y, 0) = k\};
\]

thus we can parametrize \( \mathcal{S}_R \) locally by solving the equation \( g(x, y, 0) = 0 \), according to the implicit function theorem, locally for \( y = y(x) \). Note that we do not require that \( \mathcal{S}_R \) be connected; it may be the union of two (or more) manifolds separated by submanifolds of singular points where nontrivial eigenvalues are zero. Let \( \mathcal{S}_H \subset \mathcal{S}_R \) be the open set where all the nontrivial eigenvalues have nonzero real part; i.e., compact sets \( K \subset \mathcal{S}_H \) are normally hyperbolic invariant manifolds of the layer problem (1.4).

For a more detailed discussion of the geometrical situation we refer to [7], where additionally the more general case of vector fields \( X_\varepsilon \) defined on manifolds is discussed. The following remarks show how the singularly perturbed problem (1.1) fits in this more general framework. In the setting of Eq. (1.2) resp. (2.1) the vector field \( X_\varepsilon \) resp. \( X_\varepsilon \times \{0\} \) is given by \( (\varepsilon f, g) \) resp. \( (f, g, 0) \). The flow on \( M \times (-\varepsilon_0, \varepsilon_0) \) induced by \( X_\varepsilon \times \{0\} \) is denoted by "\( \cdot \tau \)". We have to distinguish between the flow induced by \( X_\varepsilon \times \{0\} \) in \( M \times (-\varepsilon_1, \varepsilon_1) \) and the flow induced by \( X_\varepsilon \) in \( M \) for fixed \( \varepsilon \). However, since the flow stays in hyperplanes \( \varepsilon = \text{const.} \) this should cause no confusion. Because \( X_0 \) vanishes identically on \( \mathcal{S} \), \( T_m \mathcal{S} \) is in the kernel of the
linearization $LX_0$. The subspace $T_m \mathcal{S}$ is invariant under the linearization $LX_0$, and so $LX_0$ induces a linear map

$$QX_0(m): T_m \mathcal{M}/T_m \mathcal{S} \to T_m \mathcal{M}/T_m \mathcal{S}$$
onumber

on the quotient space. The eigenvalues of $QX_0(m)$ are the nontrivial eigenvalues. The map $\pi^\mathcal{S}$ is the projection $T\mathcal{M}|_{\mathcal{S}^R} \to T\mathcal{S}_R$ defined by the splitting $T\mathcal{M}|_{\mathcal{S}^R} = T\mathcal{S}_R \oplus \mathcal{N}$, where $\mathcal{N}$ is the complement of $T\mathcal{S}_R$ invariant under $LX_0$.

The manifolds $\mathcal{C}^s$, $\mathcal{C}$, and $\mathcal{C}^u$ are defined in the usual way; i.e., they are locally invariant manifolds containing $K \times \{0\}$ and tangent to the corresponding center-stable, center, and center-unstable eigenspaces $E^c_m \oplus E^s_m$, $E_m^c$, and $E_m^c \oplus E_m^u$ of the linearization $LX_0|\{0\}$ at all points $(m, 0) \in K \times \{0\}$. The dimensions of $\mathcal{C}^s$, $\mathcal{C}$, and $\mathcal{C}^u$ are $n + 1 + k^c + k^s$, $n + 1 + k^c$, and $n + 1 + k^c + k^u$, respectively, where we keep in mind that $k^s + k^c + k^u = k$ holds.

For more details on these definitions we refer to [7] and to the literature on invariant manifolds, e.g. [6, 13]. However, we repeat the definition of a family of stable resp. unstable manifolds for $\mathcal{C}^s$ resp. $\mathcal{C}^u$ because we will make use of it later. Let $\mathcal{C}^s$ be a center-stable manifold for $X_0 \times \{0\}$ near $K$. We say that a family $\{\mathcal{F}^s(p): p \in \mathcal{C}^s\}$ is a $C^2$ family of $C^1$ stable manifolds for $\mathcal{C}^s$ near $K$ if

(i) $\mathcal{F}^s(p)$ is a $C^1$ manifold for each $p \in \mathcal{C}^s$.

(ii) $p \in \mathcal{F}^s(p)$, for each $p \in \mathcal{C}^s$.

(iii) $\mathcal{F}^s(p)$ and $\mathcal{F}^s(q)$ are disjoint or identical, for each $p$ and $q \in \mathcal{C}^s$.

(iv) $\mathcal{F}^s(m, 0)$ is tangent to $E^s_m$ at $(m, 0)$, for each $m \in K$.

(v) $\{\mathcal{F}^s(p): p \in \mathcal{C}^s\}$ is a positively invariant $C^2$ family of manifolds.

The property positively invariant in (v) means that

$$\mathcal{F}^s(p) \cdot \tau \subset \mathcal{F}^s(p \cdot \tau)$$

for all $p \in \mathcal{C}^s$ and all $\tau \geq 0$ such that $p \cdot [0, \tau] \in \mathcal{C}^s$. The family of unstable manifolds $\{\mathcal{F}^u(p): p \in \mathcal{C}^u\}$ is defined similarly. The importance of the families of stable resp. unstable manifolds $\mathcal{F}^s$ resp. $\mathcal{F}^u$ is that they provide a foliation of the center-stable resp. center-unstable manifolds $\mathcal{C}^s$ resp. $\mathcal{C}^u$, i.e., $\mathcal{C}^s = \{\mathcal{F}^s(p): p \in \mathcal{C}\}$ and similarly for $\mathcal{C}^u$.

The following invariant manifold theorem (Theorem 9.1 in [7]) describes the flow induced by (2.1) near $\mathcal{S} \times \{0\}$ for small $\varepsilon$.

**Theorem 2.1.** Let $\mathcal{M}$ be a $C^{r+1}$ manifold, $1 \leq r < \infty$. Let $X_\varepsilon$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be a $C^r$ family of vector fields on $\mathcal{M}$, and let $\mathcal{S}$ be a $C^r$ submanifold of $\mathcal{M}$ consisting entirely of equilibrium points of $X_0$. Let $k^s$, $k^c$, and
be fixed integers, and let $K \subset \mathcal{S}$ be a compact subset such that $QX_0(m)$ has $k^a$ eigenvalues in the left half plane, $k^c$ eigenvalues on the imaginary axis, and $k^u$ eigenvalues in the right half plane, for all $m \in K$. Then

(F₁) There is a $C'$ center–stable manifold $\mathcal{C}'$ for $X_x \times 0$ near $K$. There is a $C''$ center–unstable manifold $\mathcal{C}''$ for $X_x \times 0$ near $K$. There is a $C'$ center manifold $\mathcal{C}$ for $X_x \times 0$ near $K$.

(F₂) There is a $C'^{-1}$ family $\{ \mathcal{F}'(p) : p \in \mathcal{C}' \}$ of $C'$ stable manifolds for $\mathcal{C}'$ near $K$. If $p \in A \times \{ \varepsilon \}$, then $\mathcal{F}'(p) \in A \times \{ \varepsilon \}$. Each manifold $\mathcal{F}'(p)$ intersects $\mathcal{C}$ transversally, in exactly one point. There is a $C'^{-1}$ family $\{ \mathcal{F}''(p) : p \in \mathcal{C}'' \}$ of $C'$ unstable manifolds for $\mathcal{C}''$ near $K$. If $p \in A \times \{ \varepsilon \}$, then $\mathcal{F}''(p) \in A \times \{ \varepsilon \}$. Each manifold $\mathcal{F}''(p)$ intersects $\mathcal{C}$ transversally, in exactly one point.

(F₃) Let $K_s < 0$ be larger than the real parts of the eigenvalues of $QX_0(m)$ in the left half plane, for all $m \in K$. Then there is a constant $C_s$ such that if $p \in \mathcal{C}'$ and $q \in \mathcal{F}'(p)$, then

$$d(p \cdot \tau, q \cdot \tau) \leq C_s e^{K_s \tau} d(p, q)$$

for all $\tau \geq 0$ such that $p \cdot [0, \tau] \subset \mathcal{C}'$. Let $K_u > 0$ be smaller than the real parts of the eigenvalues of $QX_0(m)$ in the right half plane, for all $m \in K$. Then there is a constant $C_u$ such that if $p \in \mathcal{C}''$ and $q \in \mathcal{F}''(p)$, then

$$d(p \cdot \tau, q \cdot \tau) \leq C_u e^{K_u \tau} d(p, q)$$

for all $\tau \leq 0$ such that $p \cdot [\tau, 0] \subset \mathcal{C}''$.

(F₄) If $K \subset \mathcal{S}_H$, define for $(m, \varepsilon) \in \mathcal{C}$, $X_R(m) := \pi'(\partial / \partial \varepsilon) X_\varepsilon(m)|_{\varepsilon = 0}$

$$X_\varepsilon(m, \varepsilon) := \begin{cases} \varepsilon^{-1} X_\varepsilon(m) \times \{ 0 \} & \text{if } \varepsilon \neq 0 \\ X_R(m) \times \{ 0 \} & \text{if } \varepsilon = 0 \end{cases}$$

Then $X_\varepsilon$ is a $C'^{-1}$ vector field on $\mathcal{C}$ near $K \times \{ 0 \}$.

The assertion (F₄) of Theorem 2.1 implies that the vector field $\varepsilon^{-1} X_\varepsilon$, which corresponds to the slow time scale problem (1.1) with solution operator $\cdot t$, can be extended smoothly to $\varepsilon = 0$ in $\mathcal{C}$ near $K \times \{ 0 \}$. In the case $K \subset \mathcal{S}_H$ this can be used to reduce a singular perturbation problem to a regular perturbation problem of the reduced equations in $\mathcal{C}$.

The importance of this observation is that [7, p. 91] "any structure in $\mathcal{S}_H$ which persists under regular perturbations persists under singular perturbation" by restriction of the flow of (2.1) to the center manifold. Normally hyperbolic invariant manifolds of the reduced problem persist as normally hyperbolic invariant manifolds of the singularly perturbed problem; furthermore, it is possible to characterize their local stable and
unstable manifolds be using the foliations of the center-stable and center-unstable manifolds $\mathcal{C}^s$ resp. $\mathcal{C}^u$. This idea has been carried out in [7] for the case of hyperbolic fixed points and for hyperbolic periodic orbits of the reduced problem (Theorems 12.2, 13.2 in [7]). We combine these results in the slightly more general form of

**Theorem 2.2.** Let $\mathcal{M}$ be a $C^{r+1}$ manifold, $2 \leq r < \infty$. Let $X_\varepsilon$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be a $C^r$ family of vector fields on $\mathcal{M}$, and let $\mathcal{N}$ be a $C^r$ submanifold of $\mathcal{M}$ consisting entirely of equilibrium points of $X_0$. Let $\mathcal{N} \subset \mathcal{M}$ be a $j$-dimensional compact normally hyperbolic invariant manifold of the reduced vector field $X_{\varepsilon, R}$ with a $j + j^s$-dimensional local stable manifold $\mathcal{W}^s$ and a $j + j^u$-dimensional local unstable manifold $\mathcal{W}^u$. Then there exists $\varepsilon_i > 0$ such that

(i) There exists a $C^{r-1}$ family of manifolds $\{\mathcal{N}_\varepsilon : \varepsilon \in (-\varepsilon_i, \varepsilon_i)\}$ such that $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_\varepsilon$ is a normally hyperbolic invariant manifold of $X_\varepsilon$.

(ii) There are $C^{r-1}$ families of $(j + j^s + k^s)$-dimensional and $(j + j^u + k^u)$-dimensional manifolds $\{\mathcal{N}^s_\varepsilon : \varepsilon \in (-\varepsilon_i, \varepsilon_i)\}$ and $\{\mathcal{N}^u_\varepsilon : \varepsilon \in (-\varepsilon_i, \varepsilon_i)\}$ such that for $\varepsilon > 0$ the manifolds $\mathcal{N}^s_\varepsilon$ and $\mathcal{N}^u_\varepsilon$ are local stable and unstable manifolds of $\mathcal{N}_\varepsilon$.

(iii) For $\varepsilon > 0$ the local stable and unstable manifolds $\mathcal{N}^s_\varepsilon$ and $\mathcal{N}^u_\varepsilon$ are given by

$$\mathcal{N}^s_\varepsilon = \bigcup_{p \in \mathcal{W}^s_\varepsilon} \mathcal{F}^s_\varepsilon(p), \quad \mathcal{N}^u_\varepsilon = \bigcup_{p \in \mathcal{W}^u_\varepsilon} \mathcal{F}^u_\varepsilon(p),$$

where $\mathcal{F}^s_\varepsilon(p)$ resp. $\mathcal{F}^u_\varepsilon(p)$ are the projections of $\mathcal{F}(p)$ resp. $\mathcal{F}^u(p)$ from $\mathcal{M} \times (-\varepsilon_i, \varepsilon_i)$ into $\mathcal{M}$, and $\mathcal{W}^s_\varepsilon$ resp. $\mathcal{W}^u_\varepsilon$ are the local stable resp. unstable manifolds of $\mathcal{N}_\varepsilon$ for the flow restricted to the center manifold $\mathcal{C}$ for fixed $\varepsilon$.

**Proof.** We just sketch the proof since it can be found essentially in [7]. The conditions from Theorem 2.1 are satisfied if we choose $K = \mathcal{N}$ and we conclude the existence of global center, center-stable, and center-unstable manifolds with the properties $(F_1-)(F_4)$. The global invariant manifold theorem (see [13]) implies the existence of the invariant manifold $\mathcal{N}_\varepsilon$ together with its stable and unstable manifolds $\mathcal{W}^s_\varepsilon$ resp. $\mathcal{W}^u_\varepsilon$ in the center manifold $\mathcal{C}$ for small $\varepsilon$. Thus, the manifolds $\mathcal{N}^s_\varepsilon$ and $\mathcal{N}^u_\varepsilon$ are well defined and their dimensions are $j + j^s + k^s$ and $j + j^u + k^u$, respectively.

The definition of a family of invariant manifolds implies that $\mathcal{N}^s_\varepsilon$ resp. $\mathcal{N}^u_\varepsilon$ is a positively resp. negatively invariant manifold. Thus it suffices to show that

$$\lim_{\tau \to \infty} d(\mathcal{N}_\varepsilon, q \cdot \tau) = 0 \quad \text{for all} \quad q \in \mathcal{N}^s_\varepsilon. \quad (2.3)$$
By construction there exists \( p \in \mathcal{W}^s_\varepsilon \) such that \( q \in \mathcal{F}^s(p) \) holds. The invariance of the family \( \{ \mathcal{F}_\varepsilon(p) : p \in \mathcal{W}^s_\varepsilon \} \) and the estimate

\[
d(\mathcal{N}_\varepsilon, q \cdot \tau) \leq d(\mathcal{N}_\varepsilon, p \cdot \tau) + d(p \cdot \tau, q \cdot \tau)
\]

imply (2.3) because both expressions on the right hand side converge to zero exponentially for \( \tau \to \infty \), the first one since \( p \) is in the stable manifold \( \mathcal{W}^s_\varepsilon \) of \( \mathcal{N}_\varepsilon \) for the flow restricted to the center manifold, the second one because of the estimate \( (F_3) \) in Theorem 2.1. The analogous argument for the unstable manifold proves the theorem.

For \( \varepsilon < 0 \) the local stable and unstable manifolds can be defined similarly to (iii) by interchanging \( \mathcal{F}^s \) and \( \mathcal{F}^u \).

### 3. Heteroclinic and Homoclinic Orbits

Let us repeat the definition of transversal intersection of manifolds

**Definition 3.1.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be submanifolds of a manifold \( \mathcal{M} \). The manifolds \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) intersect transversally at a point \( p \in \mathcal{M}_1 \cap \mathcal{M}_2 \) iff

\[
T_p \mathcal{M} = T_p \mathcal{M}_1 + T_p \mathcal{M}_2
\]

holds, where \( T_p \mathcal{M} \) denotes the tangent space of the manifold \( \mathcal{M} \) and similarly for \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \).

and the definition of heteroclinic and homoclinic orbits

**Definition 3.2.** Let \( \mathcal{N} \), \( \mathcal{N}_1 \), and \( \mathcal{N}_2 \) be invariant manifolds of a dynamical system. The orbit of a point \( p \) is heteroclinic to \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) if \( p \) lies in the unstable manifold of \( \mathcal{N}_1 \) and in the stable manifold of \( \mathcal{N}_2 \). The orbit of a point \( p \) is homoclinic to \( \mathcal{N} \) if \( p \) lies in both the unstable and the stable manifold of \( \mathcal{N} \). The heteroclinic or homoclinic orbit is called transversal if the stable and unstable manifolds intersect transversally.

In the case of singularly perturbed systems (1.1) the existence of heteroclinic and homoclinic orbits can be based on constructing singular heteroclinic and homoclinic orbits. A singular orbit consists of orbits of the reduced problem and orbits of the layer problem, and connects invariant manifolds of the reduced problem. The idea behind this is to prove that the singular orbit perturbs into a heteroclinic or homoclinic orbit of the singularly perturbed problem for small \( \varepsilon \). In special cases the persistence of singular orbits has been proved by Conley Index methods based on the construction of isolating blocks around the singular orbits (see [2, 24]).
In this paper we take a more geometric approach by combining the invariant manifold theory outlined in Section 2 and methods from homoclinic and heteroclinic bifurcation theory.

From here on we assume $\mathcal{S} = \mathcal{S}_H$; i.e., we stay away from turning points. However, it is essential in the following that the manifold $\mathcal{S}$ may consist of several branches, two of which are given by

$$\mathcal{S}_1 = \{(x, y_1(x)) : x \in U_1 \subset \mathbb{R}^n\}, \quad \mathcal{S}_2 = \{(x, y_2(x)) : x \in U_2 \subset \mathbb{R}^n\}. \quad (3.1)$$

We assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ overlap in their $x$-coordinates, i.e., that $U := U_1 \cap U_2 \neq \emptyset$ holds. The assumptions of Theorem 2.1 are satisfied in each of them; however, the dimensions $k_i^u$ and $k_i^s$ are allowed to be different for $i = 1$ and $i = 2$. Let $\mathcal{N}_i$ and $\mathcal{N}_j$ denote two normally hyperbolic invariant manifolds of the reduced problem. We consider the case of heteroclinic orbits; the results on homoclinic orbits follow then by setting $\mathcal{N}_1 = \mathcal{N}_2$. We distinguish two cases.

In the first case $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$ holds; i.e., the invariant manifolds $\mathcal{N}_i$ and $\mathcal{N}_j$ lie in the same branch of the manifold of the reduced problem. The singular orbit is a heteroclinic orbit of the reduced problem; i.e., the unstable manifold $W^u_1$ intersects the stable manifold $W^s_2$ as shown schematically in Fig. 1. In all the graphics shown in this paper single
arrows represent the slow flow of the reduced problem (1.3) and double arrows represent the fast flow of the layer problem (1.4). When looking at these graphics one has to remember that in general they are schematic illustrations of higher-dimensional situations.

The second case is \( \mathcal{N}_1 \subseteq \mathcal{S}_1 \) and \( \mathcal{N}_2 \subseteq \mathcal{S}_2 \); i.e., the invariant manifolds lie in different branches of the slow manifold \( \mathcal{S} \). In this case the singular orbit consists of three pieces, an orbit of the reduced problem in the unstable manifold \( \mathcal{W}^u \) which is connected to an orbit of the reduced problem in the stable manifold \( \mathcal{W}^s_2 \) by a heteroclinic orbit of the layer problem. This configuration is shown in Fig. 2 and Fig. 3.

Our starting point is the observation that in both cases the existence of the singular heteroclinic orbit is equivalent to the nonempty intersection of the manifolds \( \mathcal{N}^u_1 = \bigcup_{p \in \mathcal{W}^u_1} \mathcal{F}^u(p) \) and \( \mathcal{N}^s_2 = \bigcup_{p \in \mathcal{W}^s_2} \mathcal{F}^s(p) \) which we call the singular unstable manifold of \( \mathcal{N}_1 \) and the singular stable manifold of \( \mathcal{N}_2 \), respectively. We omit the subscript zero from now on since we consider mostly the case \( \epsilon = 0 \); however, dependence on \( \epsilon \) is always shown explicitly. Remember that for \( \epsilon = 0 \) the unstable and stable fibers \( \mathcal{F}^u \) resp. \( \mathcal{F}^s \) at the point \( p = (x, y(x)) \in \mathcal{S} \) are the unstable and stable manifolds of the hyperbolic fixed point \( y(x) \) of the equation

\[
y' = g(x, y, 0).
\]

Now we are ready to state the main result of this section.
THEOREM 3.1. Let the manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the assumptions of Theorem 2.2. Assume that the manifolds intersect transversally along the singular heteroclinic orbit. Then there exists $\varepsilon_1 > 0$ such that there exists a transversal heteroclinic orbit of the singularly perturbed system (1.1) connecting the manifolds $\mathcal{N}_{1,\varepsilon}$ and $\mathcal{N}_{2,\varepsilon}$ for $0 < \varepsilon < \varepsilon_1$.

Proof. The proof of Theorem 2.2 shows that the singular unstable manifold $\mathcal{N}_1^u$ perturbs in a $C^{r-1}$ manner to $\mathcal{N}_{1,\varepsilon}^u$, the unstable manifold of $\mathcal{N}_{1,\varepsilon}$ for small $\varepsilon > 0$ and that the singular stable manifold $\mathcal{N}_2^s$ perturbs to $\mathcal{N}_{2,\varepsilon}^s$, the stable manifold of $\mathcal{N}_{2,\varepsilon}$ for small $\varepsilon > 0$. The stability of transversal intersection implies the theorem.

By identifying the manifolds $\mathcal{N} = \mathcal{N}_1 = \mathcal{N}_2$ Theorem 3.1 implies the existence of transversal homoclinic orbits. Obviously, this is only possible if the dimension $j$ of $\mathcal{N}$ satisfies $j \geq 1$. 
THEOREM 3.2. Let the manifold $\mathcal{N}$ satisfy the assumptions of Theorem 2.2. Assume that the manifolds $\mathcal{N}^u = \bigcup_{p \in \mathcal{W}^u} \mathcal{F}^u(p)$, $\mathcal{N}^s = \bigcup_{p \in \mathcal{W}^s} \mathcal{F}^s(p)$ intersect transversally in the points of the singular homoclinic orbit. Then there exists $\varepsilon_1 > 0$ such that there exists a transversal homoclinic orbit of the singularly perturbed system (1.1) connecting the manifold $\mathcal{N}_\varepsilon$ to itself for $0 < \varepsilon < \varepsilon_1$.

This result generalizes the persistence of transversal intersection of stable and unstable manifolds under small regular perturbations to singular perturbation problems. The main difficulty in applying Theorem 3.1 is to check the transversality assumption of the theorem. This problem will be investigated in Section 4; however, we would like to make the following remarks.

In the first case, transversality of the intersection of the manifolds $\mathcal{N}^u_1$ and $\mathcal{N}^s_2$ at a point $p = (x, y(x)) \in \mathcal{S}$ is equivalent to transversality of the intersection of $\mathcal{W}^u_1$ and $\mathcal{W}^s_2$ in the manifold $\mathcal{S}$ since

$$T_p \mathcal{N}^u_1 = T_p \mathcal{W}^u_1 + E^u_p, \quad T_p \mathcal{N}^s_2 = T_p \mathcal{W}^s_2 + E^s_p,$$

where $E^u_p$ and $E^s_p$ are the stable and unstable eigenspaces at $p$ which are complementary. Thus, transversality is completely determined by the reduced problem in this case. This situation is shown in Fig. 1.

In the second case, where $\mathcal{N}^u_1$ and $\mathcal{N}^s_2$ lie in different branches of the manifold $\mathcal{S}$, transversality is determined by the interaction of the slow and fast dynamics. One possible configuration is shown in Fig. 2 where transversality is mainly due to the dynamics of the layer problem. A situation where transversality is mostly due to the reduced problem is shown in Fig. 3.

A possible generalization of Theorem 3.1 is to relax the condition of normal hyperbolicity of $\mathcal{N}^u_1$ and $\mathcal{N}^s_2$ to the conditions

(I_1): $\mathcal{N}^u_1$ is a compact manifold with boundary overflowing invariant for the reduced vector field $X_R$ and satisfies the assumptions of the unstable manifold theorem for overflowing invariant manifolds [6, Theorem 4].

(I_2): $\mathcal{N}^s_2$ is a compact manifold with boundary inflowing invariant for the reduced vector field $X_R$ and satisfies the assumptions of the stable manifold theorem for inflowing invariant manifolds.

Under these assumptions there exist an unstable manifold $\mathcal{W}^u_1$ of $\mathcal{N}^u_1$ and a stable manifold $\mathcal{W}^s_2$ of $\mathcal{N}^s_2$ for the reduced problem which are stable under small smooth perturbations. Hence, the conclusions of Theorem 2.2
hold for the unstable manifold \( N_1^u \) of \( N_1 \) and for the stable manifold \( N_2^s \) of \( N_2 \) as well. This implies

**Corollary 3.1.** The assertion of Theorem 3.1 remains valid if the assumption of normal hyperbolicity of the manifolds \( N_1 \) and \( N_2 \) is replaced by the assumptions (I,1) and (I,2).

This generalization is important in the analysis of singularly perturbed problems which depend smoothly on additional parameters \( \mu \in \mathbb{R}^m \), i.e., systems of the form

\[
\begin{align*}
\dot{x} &= f(x, y, \varepsilon, \mu) \\
\varepsilon \dot{y} &= g(x, y, \varepsilon, \mu)
\end{align*}
\]  

(3.2)

which can be rewritten as

\[
\begin{align*}
\dot{x} &= f(x, y, \varepsilon, \mu) \\
\varepsilon \dot{y} &= g(x, y, \varepsilon, \mu) \\
\dot{\mu} &= 0
\end{align*}
\]  

(3.3)

on the extended phase space \( \mathcal{M} \subset \mathbb{R}^{n+k+m} \). Let \( V \subset \mathbb{R}^m \) be a compact set with piecewise smooth boundary \( \partial V \). Suppose that the reduced problem corresponding to system (3.2) has a family of invariant \( j_i \)-dimensional manifolds \( \{ N_i(\mu) : \mu \in V \}, \ i = 1, 2 \) for which the assumptions of Theorem 2.2 are satisfied. Then the manifolds

\[
N_i := \{ (N_i(\mu), \mu) : \mu \in V \}, \quad i = 1, 2
\]

are \((j_i + m)\)-dimensional invariant manifolds with boundary for the reduced problem corresponding to system (3.3). Moreover, the manifolds \( N_i, i = 1, 2 \) satisfy the hyperbolicity assumptions of Theorem 2.2. However, \( N_i \) is neither overflowing invariant nor inflowing invariant due to the trivial equation \( \dot{\mu} = 0 \). The standard technique of modifying the equation \( \dot{\mu} = 0 \) appropriately near \( \partial N_i \) can be used to make \( N_i \) overflowing invariant and \( N_2^s \) inflowing invariant (see [6]). This allows us to apply Corollary 3.1 to prove the existence of heteroclinic orbits connecting \( N_1 \) and \( N_2 \).

The same technique can be used to prove the existence of orbits homoclinic to \( \mathcal{N} := \{ (N(\mu), \mu) : \mu \in V \} \) by choosing manifolds

\[
\mathcal{N}', \mathcal{N}_1 \subset \mathcal{N}_2 \subset \mathcal{N}
\]

and by modifying the equation \( \dot{\mu} = 0 \) near the boundaries \( \partial N_1 \) and \( \partial N_2 \) such that the vectorfield restricted to \( \mathcal{N}' \) remains unchanged, \( N_1 \) is overflowing invariant, and \( N_2 \) is inflowing invariant. Then Corollary 3.1 can be
used to prove the existence of heteroclinic orbits connecting $\mathcal{N}_1$ and $\mathcal{N}_2$. If these heteroclinic orbits connect points in $\mathcal{N}'$ they are actually homoclinic orbits of the unmodified problem.

The motivation for this is that nontransversal intersections of stable and unstable manifolds associated with system (3.2) for fixed $\mu_0$ may correspond to transversal intersections in the extended system (3.3) due to the breaking of the connecting orbits as $\mu$ varies near $\mu_0$. This allows us then to conclude the existence of $\epsilon_1 > 0$ and a smooth family

$$\{\mathcal{H}_\epsilon : \mathcal{H}_\epsilon \subset \mathbb{R}^m, \epsilon \in [0, \epsilon_1]\}$$

such that system (3.2) has the desired heteroclinic or homoclinic orbits at the parameters $\mu \in \mathcal{H}_\epsilon$ for $\epsilon$ small. We will come back to this in Section 4 and Section 5.

We conclude this section with the remark that we do not consider singular orbits which connect locally invariant manifolds of the reduced problem. This implies that we do not consider the case where the singular orbit passes through branches $\mathcal{S}_r$ of the reduced problem without being in the stable or unstable manifold of some invariant set of the reduced problem.

### 4. Transversality Condition

In general the crucial assumption of transversal intersection of the manifolds $\mathcal{N}_1^u$ and $\mathcal{N}_2^s$ in Theorem 3.1 is hard to verify for a given problem. We now turn to a systematic analysis of this question.

We have shown that in the first case of a homoclinic or heteroclinic orbit which lies in one branch of the manifold $\mathcal{S}$ transversality is determined by the reduced problem. Thus, it is not a problem of singular perturbation theory any more and one has to deal with it on a case to case basis. If the reduced problem is low-dimensional the transversality condition can be verified easily. Another situation in which the transversality condition is trivially satisfied is that either $\mathcal{N}_1^u$ is an unstable invariant manifold of the reduced problem or $\mathcal{N}_2^s$ is a stable invariant manifold of the reduced problem. In these cases Theorem 3.1 implies the existence of transversal heteroclinic or homoclinic orbits of the singularly perturbed problem (1.1) for small $\epsilon$ independent of the dimension $k$ of the fast variable $y$. For an application of these ideas see [9], where the problem of finding transversal heteroclinic orbits of an originally six-dimensional system is reduced to finding transversal heteroclinic orbits of two-dimensional or even one-dimensional reduced problems.

In the second case the intersection of $\mathcal{N}_1^u$ and $\mathcal{N}_2^s$ is equivalent to the
existence of $x_0 \in U$ such that the points $p_1 = (x_0, y_1(x_0)) \in \mathcal{W}_1^u$ and $p_2 = (x_0, y_2(x_0)) \in \mathcal{W}_2^s$ are connected by a heteroclinic orbit of the layer problem (1.4). In particular $y_1(x_0)$ and $y_2(x_0)$ are connected by a heteroclinic orbit $y_0(\tau)$ of the $k$-dimensional system parametrized by $x$

$$y' = g(x, y, 0) \quad (4.1)$$

at $x = x_0$. The intersection of $\mathcal{N}_1^u$ and $\mathcal{N}_2^s$ at a point $p := (x_0, y_0(\tau))$ is transversal iff

$$T_p \mathcal{N}_1^u + T_p \mathcal{N}_2^s = \mathbb{R}^{n+k} \quad (4.2)$$

holds. The equation

$$\dim(T_p \mathcal{N}_1^u + T_p \mathcal{N}_2^s) = \dim(T_p \mathcal{N}_1^u) + \dim(T_p \mathcal{N}_2^s) - \dim(T_p \mathcal{N}_1^u \cap T_p \mathcal{N}_2^s) \quad (4.3)$$

and counting dimensions imply that the intersection is transversal iff

$$\dim(T_p \mathcal{N}_1^u \cap T_p \mathcal{N}_2^s) = d \quad (4.4)$$

holds, where $d := j_1 + j_1^u + j_2 + j_2^s + k_2 - n - k$. Thus, $d \geq 1$ is a necessary condition for transversality because

$$\dim(T_p \mathcal{N}_1^u \cap T_p \mathcal{N}_2^s) \geq 1$$

holds. By using ideas developed in [21] we show how the transversality relates to the bifurcation of heteroclinic orbits of the system (4.1) as the parameter $x$ varies. In the following we assume that the intersection of the $k_1^u$-dimensional unstable fiber $\mathcal{F}^u(p_1)$, which is the unstable manifold of $y_1(x_0)$ of system (4.1), and the $k_2^s$-dimensional stable fiber $\mathcal{F}^s(p_2)$, which is the stable manifold of $y_2(x_0)$ of system (4.1), is one-dimensional. Furthermore, we assume that the intersection $T_p \mathcal{F}^u(p_1) \cap T_p \mathcal{F}^s(p_2)$ is one-dimensional. This is equivalent to the assumption that $y_0^\prime$ is the only bounded solution of the variational equation

$$y' = D_y g(x_0, y_0(\tau), 0) y. \quad (4.5)$$

Then we have

**Theorem 4.1.** Let the manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the assumptions of Theorem 2.2. Let $\varphi(\mathcal{W}_1^u)$ and $\varphi(\mathcal{W}_2^s)$ denote the $x$-coordinates of the manifolds $\mathcal{W}_1^u$ resp. $\mathcal{W}_2^s$. Then the manifolds $\mathcal{N}_1^u$ and $\mathcal{N}_2^s$ intersect transversally in the points of the heteroclinic orbit $(x_0, y_0(\tau))$ if and only if there
exist exactly $d - 1$ linearly independent solutions $\xi \in T_{x_0} \phi(W^u_1) \cap T_{x_0} \phi(W^s_2)$ of the equation

$$(M, \xi) = 0, \quad (4.6)$$

where $M \in \mathbb{R}^n$ is defined by

$$M := \int_{-\infty}^{\infty} (\psi(\tau) \cdot D_x g(x_0, y_0(\tau), 0)) \, d\tau. \quad (4.7)$$

The function $\psi$ is the (up to a scalar multiple) unique bounded solution of the adjoint equation

$$\psi' = -[D_y g(x_0, y_0(\tau), 0)]^T \psi. \quad (4.8)$$

In the theorem $(\cdot, \cdot)$ denotes the scalar product and $^T$ denotes transposition of a matrix.

**Proof.** We have to show that equality (4.4) holds if and only if the conditions of the theorem are satisfied. Let $\Phi_\tau$ denote the flow defined by the layer problem (1.4). The linearization of the flow $\Phi_\tau$ is denoted by $D\Phi_\tau$. Assume that $p = (x_0, y_0)$ is a point on the heteroclinic orbit and that

$$(\xi_0, \eta_0) \in T_p N^u_1 \cap T_p N^s_2 \quad (4.9)$$

holds. We have $(\xi_0, \eta_0) \in T_p N^u_1$ iff $D\Phi_\tau(x_0, y_0)(\xi_0, \eta_0)$ is bounded for $\tau \to -\infty$ and $(\xi_0, \eta_0) \in T_p N^s_2$ iff $D\Phi_\tau(x_0, y_0)(\xi_0, \eta_0)$ is bounded for $\tau \to \infty$. Thus the relation (4.9) holds if and only if $D\Phi_\tau(x_0, y_0)(\xi_0, \eta_0)$ is bounded on $\mathbb{R}$. The function $(\xi(\tau), \eta(\tau)) := D\Phi_\tau(x_0, y_0)(\xi_0, \eta_0)$ is the solution of the variational equation

$$\begin{align*}
\xi' &= 0 \\
\eta' &= A(\tau) \eta + B(\tau) \xi
\end{align*} \quad (4.10)$$

with the initial value $\xi(0) = \xi_0$, $\eta(0) = \eta_0$ and the matrices

$$A(\tau) = D_x g(x_0, y_0(\tau), 0), \quad B(\tau) = D_x g(x_0, y_0(\tau), 0).$$

The first equation in (4.10) implies that

$$\xi(\tau) = \xi_0 \in T_{x_0} \phi(W^u_1) \cap T_{x_0} \phi(W^s_2)$$

holds. Thus, $(\xi_0, \eta(\tau))$ is a solution of (4.10) if and only if $B(\tau) \xi_0 \in \mathcal{R}(\mathcal{L})$ holds, where $\mathcal{R}(\mathcal{L})$ denotes the range of the operator $\mathcal{L}$ defined by

$$\mathcal{L} : C^1(\mathbb{R}, \mathbb{R}^k) \to C^0(\mathbb{R}, \mathbb{R}^k)$$

$$(\mathcal{L} \eta)(\tau) = \eta' - A(\tau) \eta. \quad (4.11)$$
It is shown in [21] that this is equivalent to

$$
\left( \int_{-\infty}^{\infty} \psi(\tau) \cdot B(\tau) \, d\tau, \xi_{0} \right) = 0
$$

(4.12)

for every bounded solution \( \psi \) of the adjoint equation (4.8).

Thus, the assumption that there are \( d-1 \) linearly independent solutions \( \xi \in T_{x_{0}} \varphi(\mathcal{N}^{-}\uparrow) \cap T_{x_{0}} \varphi(\mathcal{N}^{-}\downarrow) \) of the equation \( (M, \xi) = 0 \) and the fact that one solution is always given by \( (0, y'_{0}(\tau)) \) imply Eq. (4.4) because there are at most \( d \) linearly independent bounded solutions of (4.10). If there are more than \( d-1 \) solutions of the equation \( (M, \xi) = 0 \) there exist more than \( d \) linearly independent solutions of (4.10) and the criterion (4.4) for transversality is violated.

The following result will be used later. In the case \( k = 2 \) the solution \( \psi \) of the adjoint equation (4.8) is given by

$$
\psi(\tau) = \exp \left[ -\int_{0}^{\tau} \text{tr}(A(s)) \, ds \right] (-y'_{2}(\tau), y'_{1}(\tau)),
$$

(4.13)

where \( \text{tr}(A) \) denotes the trace of the matrix \( A \) (see [10, 21]).

If the intersection of \( \mathcal{F}^{u}(p_{1}) \) and \( \mathcal{F}^{s}(p_{2}) \) is transversal \( M = 0 \) holds because then all points in a neighborhood of \( p_{1} \) are connected to points in a neighborhood of \( p_{2} \) by a heteroclinic orbit of system (4.1). If this intersection is nontransversal then, generically, the vector \( M \) defined in (4.7) is the normal vector of a codimension one hypersurface \( \mathcal{H} \) in \( \varphi(\mathcal{S}_{1}) \) resp. \( \varphi(\mathcal{S}_{2}) \) along which a heteroclinic orbit of the layer problem (1.4) connecting the manifolds \( \mathcal{S}_{1} \) and \( \mathcal{S}_{2} \) exists. According to Theorem 4.1 the manifolds \( \mathcal{N}^{u}_{1} \) and \( \mathcal{N}^{s}_{2} \) intersect transversally iff \( \varphi(\mathcal{N}^{u}_{1}) \) and \( \varphi(\mathcal{N}^{s}_{2}) \) have \( d-1 \) common tangent vectors with \( \mathcal{H} \) at the point \( x_{0} \).

The transversality condition for homoclinic orbits is obtained by choosing \( \mathcal{N} = \mathcal{N}_{1} = \mathcal{N}_{2} \) in Theorem 4.1. In this case there exists a point \( (x_{0}, y(x_{0})) \in \mathcal{N} \) such that the system (4.1) has an orbit \( y_{0}(\tau) \) homoclinic to the point \( y(x_{0}) \).

**Corollary 4.1.** Let the manifold \( \mathcal{N} \) satisfy the assumptions of Theorem 2.2. Let \( \varphi(\mathcal{N}) \) denote the \( x \)-coordinates of the manifold \( \mathcal{N} \). Then the manifolds \( \mathcal{N}^{u} \) and \( \mathcal{N}^{s} \) intersect transversely in the points of the homoclinic orbit \( (x_{0}, y_{0}(\tau)) \) if and only if there exist exactly \( j-1 \) linearly independent solutions \( \xi \in T_{x_{0}} \varphi(\mathcal{N}) \) of the equation \( (M, \xi) = 0 \), where \( M \in \mathbb{R}^{n} \) is defined by Eq. (4.7).
Proof. The equation (4.4) implies that in this case $d = j$ holds. Now the corollary follows from Theorem 4.1 since

$$T_{x_0} \varphi(W^u) \cap T_{x_0} \varphi(W^s) = T_{x_0} \varphi(N)$$

(4.14)

holds.

Obviously, Theorem 4.1 remains valid under the weaker assumptions of Corollary 3.1. To illustrate Theorem 4.1 let us consider the situations shown in Fig. 2 and Fig. 3 as problems in $R^3$. In Fig. 2 we have a one-dimensional reduced problem on the two branches $S_1$ and $S_2$ and a two-dimensional layer problem, i.e., $n = 1$, $k = 2$, and $k^u_i = k^s_i = 1$, $i = 1, 2$. $N_i$ is an unstable fixed point of the reduced problem in $S_i$ and $N_2$ is a stable fixed point of the reduced problem in $S_2$; i.e., $j_1 = 0$, $j_2 = 1$, $j^s_1 = 0$ resp. $j_2 = 0$, $j^s_2 = 0$, $j^u_2 = 1$ holds. The equation (4.4) implies that $d = 1$ holds. Thus, Theorem 4.1 implies that $N_i^u$ and $N_2^s$ intersect transversally iff $M \neq 0$ holds.

In Fig. 3 we have a two-dimensional reduced problem on the two branches $S_1$ and $S_2$ and a one-dimensional layer problem, i.e., $n = 2$, $k = 1$, $k^u_1 = 1$, $k^s_1 = 0$, $k^s_2 = 0$, and $k_2^s = 1$ holds. $N_1$ resp. $N_2^s$ are saddles of the reduced problem in $S_1$ resp. $S_2$, i.e., $j_1 = 0$, $j_2 = 1$, $j^s_1 = 1$ resp. $j_2 = 0$, $j^s_2 = 1$, $j^u_2 = 0$, $j^s_2 = 0$, and $j^u_2 = 1$ and thus $d = 1$ holds. For this problem $M = (0, 0)$ because every point in $S_1$ is connected to a point in $S_2$ by a heteroclinic orbit of the layer problem. If $f(x_0, y_1(x_0))$ and $f(x_0, y_2(x_0))$ are linearly independent Theorem 4.1 implies that $N_1^u$ and $N_2^s$ intersect transversally because

$$T_{x_0} \varphi(W^u_1) \cap T_{x_0} \varphi(W^s_2) = 0$$

holds. If, however, $f(x_0, y_1(x_0))$ and $f(x_0, y_2(x_0))$ are linearly dependent then the intersection is nontransversal. In all the cases where we have transversality Theorem 3.1 implies the existence of a heteroclinic orbit of the corresponding singularly perturbed problem for small $\varepsilon$.

Theorem 3.1 and Theorem 3.2 together with Theorem 4.1 and Corollary 4.1 describe the least degenerate situation in which a singular heteroclinic or homoclinic orbit perturbs into a transversal heteroclinic or homoclinic orbit of the singularly perturbed problem (1.1) for small $\varepsilon$ since transversality exists already in the unperturbed problem, i.e., the intersection of $N_1^u$ and $N_2^s$ is transversal. If, however, the transversality condition is violated due to the existence of "too many" heteroclinic resp. homoclinic orbits along $W^u_1$ and $W^s_2$ resp. $N$ it is still possible that the stable and unstable manifolds for $\varepsilon \neq 0$ separate in a way which generates transversality. This situation is closer to the situation in the usual Melnikov method where for a regular perturbation problem the unperturbed problem has homoclinic orbits along all points of a hyperbolic periodic orbit (see
We will come back to this in Section 6 where we give a more detailed discussion of orbits homoclinic to periodic orbits.

We conclude this section by discussing how homoclinic and heteroclinic bifurcations in the reduced problem of a singular perturbation problem with additional parameters correspond to transversal intersection of stable and unstable manifolds of the reduced problem of the extended system. For simplicity we restrict ourselves to the case of orbits heteroclinic or homoclinic to hyperbolic fixed points.

Suppose the singularly perturbed problem is of the form (3.2) and there exists $\mu_0$ such that the reduced problem

$$\dot{x} = f(x, y(x), 0, \mu)$$

has a heteroclinic orbit $h_0(\mu_0) \subset \mathcal{S}$ parametrized by $(x_0(t), y(x_0(t), \mu_0))$ connecting the hyperbolic fixed points $N_1(\mu_0)$ and $N_2(\mu_0)$. We know from Theorem 3.1 that transversal heteroclinic orbits of the reduced problem persist, thus we consider now the case of a nontransversal heteroclinic orbit. By identifying $N_1$ and $N_2$ this allows us also to study orbits homoclinic to a hyperbolic fixed point.

We assume that the nontransversality is of the least degenerate type, i.e., the dimensions of the stable and unstable manifolds of $N_1$ and $N_2$ are equal, $j^s_1 = j^u_2$, $j^u_1 = j^s_2$, and the intersection

$$T_p W^s_1 \cap T_p W^u_2, \quad p = (x_0(t), y(x_0(t), \mu_0))$$

is one-dimensional. The last assumption is equivalent to the assumption that $\dot{x}_0$ is the only bounded solution of the linearized equation

$$\dot{x} = A(t) x,$$

where the matrix $A$ is defined by

$$A(t) := D_x f(x_0(t), y(x_0(t), \mu_0), 0, \mu_0).$$

It is important to note that the derivative $D_x$ in this formula has to be computed by the chain rule because $y = y(x, \mu)$ holds. The same remark holds for the derivative $D_\mu$ in Eq. (4.19) below. The hyperbolicity of $N_i$, $i = 1, 2$, implies that there exists a neighborhood $V \subset \mathbb{R}^m$ of $\mu_0$ and $m$-dimensional manifolds

$$\tilde{N}_i := \{(N_i(\mu), \mu) : \mu \in V\}, \quad i = 1, 2$$

of fixed points of the reduced problem

$$\dot{x} = f(x, y(x, \mu), 0, \mu)$$

$$\dot{\mu} = 0$$
of the extended system \((3.3)\). Our assumptions imply that the \((j_1^u + m)\)-dimensional unstable manifold \(\tilde{W}_1^u\) and the \((j_2^s + m)\)-dimensional stable manifold \(\tilde{W}_2^s\) of the extended system \((4.18)\) intersect along the heteroclinic orbit \((h_0(\mu_0), \mu_0)\).

**Theorem 4.2.** Under the above assumptions define the vector \(M \in \mathbb{R}^m\) by

\[
M := \int_{-\infty}^{\infty} (\psi(t) \cdot D_{\mu}f(x_0(t), y(x_0(t), \mu_0), 0, \mu_0)) \, dt,
\]

where the function \(\psi\) is the (up to a scalar multiple) unique bounded solution of the adjoint equation of \((4.16)\)

\[
\dot{\psi} = -A^T \psi.
\]

The manifolds \(\tilde{W}_1^u\) and \(\tilde{W}_2^s\) intersect transversally along the heteroclinic orbit \((h_0(\mu_0), \mu_0)\) if and only if \(M \neq 0\) holds.

If \(M \neq 0\) then there exist \(\varepsilon_1 > 0\) and a smooth family of codimension one hypersurfaces \(\mathcal{H}_\varepsilon \subset \mathbb{R}^m\) with \(\mu_0 \in \mathcal{H}_0\) such that for \(\mu \in \mathcal{H}_\varepsilon\) the singularly perturbed problem \((3.2)\) has a heteroclinic orbit \(h(\varepsilon)\) connecting the hyperbolic fixed points \(\mathcal{N}_1, (\mu)\) and \(\mathcal{N}_2, (\mu)\) for \(0 < \varepsilon < \varepsilon_1\).

If the \(i\)th component of the vector \(M = (\mu_1, \ldots, \mu_m)\) is different from zero the hypersurface \(\mathcal{H}_\varepsilon\) can be represented as the graph of a smooth function

\[
\mu_i = \mu_i(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_m, \varepsilon).
\]

**Proof:** The same proof as for Theorem 4.1 can be used to prove the existence of a codimension one hypersurface \(\mathcal{H}_0 \subset V\) with normal vector \(M\) at \(\mu_0 \in \mathcal{H}_0\) such that for \(\mu \in \mathcal{H}_0\) the fixed point \((\mathcal{N}_1, (\mu), \mu) \in \tilde{N}_1\) of the extended reduced problem \((4.18)\) is connected to the corresponding fixed point \((\mathcal{N}_2, (\mu), \mu) \in \tilde{N}_2\) by a transversal heteroclinic orbit \((h_0(\mu), \mu)\) of the extended reduced problem \((4.18)\). As described at the end of Section 3 the equation \(\dot{\mu} = 0\) can be modified near \(\partial V\) to make the manifold \(\tilde{N}_1\) overflowing invariant and the manifold \(\tilde{N}_2\) inflowing invariant. Thus, the unstable manifold \(\tilde{W}_1^u\) and the stable manifold \(\tilde{W}_2^s\) intersect transversally in the manifold \(\{(h_0(\mu), \mu) : \mu \in \mathcal{H}_0\}\) of heteroclinic orbits. Corollary 3.1 implies the existence of \(\varepsilon_1\) such that the unstable manifold \(\tilde{W}_{1, \varepsilon}^u\) and the stable manifold \(\tilde{W}_{2, \varepsilon}^s\) intersect transversally in the manifold \(\{(h_\varepsilon(\mu), \mu) : \mu \in \mathcal{H}_\varepsilon\}\) of heteroclinic orbits for \(0 < \varepsilon < \varepsilon_1\).

To prove that \(\mathcal{H}_\varepsilon\) can be represented as a graph we define a smooth function

\[
Q(\mu, \varepsilon) := \text{distance}\{\tilde{W}_{1, \varepsilon}^u, \tilde{W}_{2, \varepsilon}^s\}.
\]
Obviously, intersection of $\mathcal{W}^u_{1,\varepsilon}$ and $\mathcal{W}^s_{2,\varepsilon}$ corresponds to parameters which are solutions of the equation $Q(\mu, \varepsilon) = 0$. We know that $Q(\mu_0, 0) = 0$ holds and that $(\partial Q/\partial \mu)(\mu_0, 0) \neq 0$ holds because of the transversality of the intersection of the manifolds $\mathcal{W}^u_1$ and $\mathcal{W}^s_2$. Thus, we can solve the equation $Q(\mu, \varepsilon) = 0$ locally for $\mu_i = \mu_i(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_m, \varepsilon)$ by the implicit function theorem. This proves the theorem.

A detailed application of Theorem 4.2 is given in [26].

5. FITZHUGH–NAGUMO EQUATIONS

One of the most widely studied systems of singularly perturbed differential equations is the traveling wave problem for the Fitzhugh–Nagumo equations

$$u_t = u_{xx} + f(u) - w, \quad w_t = \varepsilon(u - \gamma w) \tag{5.1}$$

modeling the propagation of nerve impulses. In this system the function $f(u) = u(u-a)(u-1)$ is a cubic polynomial, and $0 < a < \frac{1}{2}$, $0 < \varepsilon \ll 1$, and $\gamma > 0$ are parameters. A traveling wave with wavespeed $c$ is a solution
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depending on the single variable \( \tau = x + ct \). The corresponding system of ordinary differential equations is

\[
\begin{align*}
    u' &= v \\
    v' &= cv - f(u) + w \\
    w' &= \frac{\varepsilon}{c} (u - \gamma w).
\end{align*}
\]  

We consider the case of \( \gamma \) sufficiently large such that the system (5.2) has three fixed points. However, we are only concerned with two of them, \( m_1 = (0, 0, 0) \) and \( m_2 = (u_2, 0, f(u_2)) \), as shown in Fig. 4. In the following we show how the construction of the heteroclinic orbit connecting the fixed points \( m_1 \) and \( m_2 \) fits in the general framework of the previous sections. Since the system (5.2) is well understood by now (see e.g. [2, 5, 12, 16, 19, 22]) it serves mostly as an illustrative example. Since it takes almost no extra work we show additionally that the system

\[
\begin{align*}
    u_t &= u_{xx} + f(u) - w, \\
    w_t &= \delta w_{xx} + \varepsilon (u - \gamma w)
\end{align*}
\]  

obtained by adding a small diffusion term to the second equation in (5.1) has traveling wave solutions close to the traveling wave solutions of (5.1) for \( 0 < \delta < \delta_1 \) and \( 0 < \varepsilon < \varepsilon_1 \).

In the recent paper [5] similar methods to ours have been used to prove the existence of parameters \( c, \gamma \) such that for \( \varepsilon \) small the system (5.2) satisfies the assumptions of the heteroclinic bifurcation theorem (see [4]) which has interesting consequences, i.e., the existence of infinitely many traveling fronts.

**THEOREM 5.1.** There exists \( \varepsilon_1 > 0 \) and a smooth family of wavespeeds \( \{ c_\varepsilon : \varepsilon \in [0, \varepsilon_1) \} \) such that the traveling wave problem (5.2) with \( c = c_\varepsilon \) has a heteroclinic orbit \( h_\varepsilon \) connecting the fixed points \( m_1 \) and \( m_2 \) for \( 0 < \varepsilon < \varepsilon_1 \).

**Proof.** For the reasons outlined at the end of Section 3 we extend the system (5.2) by adding the trivial equation

\[
c' = 0.
\]  

Obviously, system (5.2), (5.4) is of the form (1.2) with \( n = 2 \) corresponding to the two slow variables \( w, c \) and \( k = 2 \) corresponding to the fast variables \( u, v \). It is easy to see that system (5.2), (5.4) satisfies the assumptions of Theorem 2.1 as long as one stays away from the local maximum and minimum of the cubic function \( f(u) \). The reduced problem

\[
\begin{align*}
    \dot{w} &= (f^{-1}(w) - \gamma w)/c \\
    \dot{c} &= 0
\end{align*}
\]  

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is defined on the two-dimensional manifolds

\[ \mathcal{S}_i = \{(f_i^{-1}(w), 0, w, c), w \in I, c \in (\alpha, \beta)\}, \quad i = 1, 2, \quad (5.6) \]

where \( f_i^{-1}, i = 1, 2 \) denotes the two branches of the inverse function of \( f \). The interval \( I \) is chosen such that \((0, f(u_2)) \subset I \) holds (see Fig. 4, where the geometry for a fixed value of \( c \) is shown). The interval \((\alpha, \beta)\) will be specified later. The stable and unstable fibers \( \mathcal{F}^s \) and \( \mathcal{F}^u \) are one-dimensional, i.e., \( k_i^s = k_i^u = 1, i = 1, 2 \) holds. For fixed \( c \) the points \( m_1 \) and \( m_2 \) are stable fixed points of the reduced problem. Thus Theorem 2.2 implies that \( m_1 \) and \( m_2 \) are hyperbolic fixed points of system (5.2) with one-dimensional unstable and two-dimensional stable manifolds.

In the extended system (5.2), (5.4) the fixed points \( m_1 \) and \( m_2 \) of the system (5.2) correspond to one-dimensional manifolds

\[ \mathcal{N}_1 = \{(0, 0, 0, c), c \in (\alpha, \beta)\}, \quad \mathcal{N}_2 = \{(u_2, 0, f(u_2), c), c \in (\alpha, \beta)\} \quad (5.7) \]

of fixed points which are also fixed points of the reduced problem (5.5), i.e., \( j_1 = j_2 = 1 \) holds. However, this \( \varepsilon \)-independence of the fixed points is not essential in view of Theorem 2.2. In the following we use Corollary 3.1 to conclude the existence of a heteroclinic orbit for small \( \varepsilon \). Since \( m_1 \) and \( m_2 \) are stable fixed points of the reduced problem corresponding to system (5.2) the unstable and stable manifolds for the extended reduced problem (5.5) are given by

\[ \mathcal{W}^u_1 = \mathcal{N}_1, \quad \mathcal{W}^s_2 = \mathcal{S}_2. \quad (5.8) \]

More precisely we have to modify the equation \( c' = 0 \) appropriately near \( \mathcal{S}_1 \times \{\alpha\} \) and \( \mathcal{S}_1 \times \{\beta\} \) resp. \( \mathcal{S}_2 \times \{\alpha\} \) and \( \mathcal{S}_2 \times \{\beta\} \) to make \( \mathcal{N}_1 \) an overflowing resp. \( \mathcal{N}_2 \) an inflowing invariant manifold for the corresponding reduced problem. For this modified problem \( \mathcal{N}_1 \) resp. \( \mathcal{N}_2 \) then possesses an unstable resp. stable manifold \( \mathcal{W}^u_1 \) resp. \( \mathcal{W}^s_2 \) which coincides with the manifolds in (5.8) on a smaller interval \((\alpha', \beta') \subset (\alpha, \beta)\). Thus, it is not necessary to carry out the modification explicitly. We define the singular unstable and stable manifolds

\[ \mathcal{N}^u_1 = \bigcup_{p \in \mathcal{N}_1} \mathcal{F}^u(p), \quad \mathcal{N}^s_2 = \bigcup_{p \in \mathcal{S}_2} \mathcal{F}^s(p) \quad (5.9) \]

which are of dimension two resp. three. Thus, transversal intersection of \( \mathcal{N}^u_1 \) and \( \mathcal{N}^s_2 \) is possible and actually occurs because of the following lemma.

**Lemma 5.1.** There exists a \( c_0 = (1 - 2a)/\sqrt{2} \) such that for any interval
(\alpha, \beta) which contains \(c_0\) the manifolds \(\mathcal{N}^u_1\) and \(\mathcal{N}^s_2\) intersect transversally along the points of a singular heteroclinic orbit.

**Proof.** It is well known (see [19]) that for \(c_0 = (1 - 2a)/\sqrt{2}\) the layer problem

\[
\begin{align*}
    u' &= v \\
    v' &= cv - f(u) + w \\
    w' &= 0 \\
    c' &= 0
\end{align*}
\]

has a heteroclinic orbit \((u(\tau), v(\tau), 0, c_0)\) connecting the point \((0, 0, 0, c_0) \in \mathcal{N}_1\) to the point \((1, 0, 0, c_0) \in \mathcal{W}^s_2\), which implies that the manifolds \(\mathcal{N}^u_1\) and \(\mathcal{N}^s_2\) intersect along the singular heteroclinic orbit (see Fig. 4).

Now we use Theorem 4.1 to prove transversality. For this problem \(d = 1\) holds. Since we have

\[
\gamma_0, \gamma_\infty, \gamma_0 \gamma_\infty = \{(0, UT), (0, IT)\}
\]

the manifolds \(\mathcal{N}^u_1\) and \(\mathcal{N}^s_2\) intersect transversally iff the two-dimensional vector \(M = (m_1, m_2)\) defined by (4.7) has a nonzero second component, i.e., \(m_2 \neq 0\). In this problem the matrices \(A\) and \(B\) from (4.10) are given by

\[
A(\tau) = \begin{pmatrix} 0 & 1 \\ -f'(u) & c \end{pmatrix}, \quad B(\tau) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

where \(u\) and \(v\) are evaluated along the heteroclinic orbit. Equations (4.7), (4.13) with \(y_1 = u\) and \(y_2 = v\) and (5.11) imply that

\[
M = \int_{-\infty}^{\infty} \exp \left[ -\int_0^\tau \text{tr}(A(s)) \, ds \right] \, (-v'(\tau), v^2(\tau)) \, d\tau
\]

holds. This proves the lemma since the exponential function and \(v^2\) take positive values.

Thus, the assumptions of Corollary 3.1 are satisfied and we conclude that the manifolds \(\mathcal{N}^u_{1,\varepsilon}\) and \(\mathcal{N}^s_{2,\varepsilon}\) intersect transversally for \(0 < \varepsilon < \varepsilon_1\) which implies the existence of the heteroclinic orbit \(h_\varepsilon\).

To prove that we can actually parametrize the corresponding wavespeed by \(\varepsilon\), i.e., \(c = c_\varepsilon\), we define a smooth function

\[
Q(\varepsilon, c) := \text{distance}\{\mathcal{N}^u_{1,\varepsilon}, \mathcal{N}^s_{2,\varepsilon}\}
\]

Obviously, intersection of \(\mathcal{N}^u_{1,\varepsilon}\) and \(\mathcal{N}^s_{2,\varepsilon}\) corresponds to \(Q(\varepsilon, c) = 0\).
We conclude from Lemma 5.1 that \( Q(0, c_0) = 0 \) holds and that \( (\partial Q/\partial c)(0, c_0) \neq 0 \) holds because changing \( c \) breaks the singular heteroclinic orbit. Thus, we can solve the equation \( Q(\varepsilon, c) = 0 \) locally for \( c = c_\varepsilon \) by the implicit function theorem. This proves the theorem.

The same arguments can be used to prove the existence of a heteroclinic orbit of (5.2) connecting the fixed points \( m_2 \) and \( m_1 \) for small \( \varepsilon \). In general the speed of these two types of heteroclinic orbits is different, however, one can show that there exists a value \( \gamma_0 \) such that they have the same speed. This implies the existence of parameters \( c_\varepsilon \) and \( \gamma_\varepsilon \) for which a heteroclinic loop of (5.2) exists (see [5]).

The traveling wave problem for the Fitzhugh–Nagumo equations (5.3) with small diffusion in the second equation is now easy to analyze. In this case the traveling wave problem has the form

\[
\begin{align*}
\varepsilon' &= v \\
v' &= cv - f(u) + w \\
w' &= z \\
\delta z' &= -cz + \varepsilon(u - \gamma w)
\end{align*}
\tag{5.14}
\]

with the singular perturbation parameter \( \delta \) and the additional parameters \( c \) and \( \varepsilon \). The fixed points which we consider are given by \( m_1 = (0, 0, 0, 0) \) and \( m_2 = (u_2, 0, f(u_2), 0) \).

**Theorem 5.2.** There exist \( \varepsilon_1 > 0 \), \( \delta_1 > 0 \) and a smooth family of wavespeeds \( \{c_{\varepsilon, \delta} : \varepsilon \in (0, \varepsilon_1), \delta \in [0, \delta_1]\} \) with \( c_{\varepsilon, 0} = c_\varepsilon \) such that the traveling wave problem (5.14) with \( c = c_{\varepsilon, \delta} \) has a heteroclinic orbit \( h_{\varepsilon, \delta} \) connecting the fixed points \( m_1 \) and \( m_2 \) for \( 0 < \varepsilon < \varepsilon_1 \) and \( 0 < \delta < \delta_1 \).

**Proof.** For this problem we consider \( c \) and \( \varepsilon \) as parameters; hence the extended system is given by

\[
\begin{align*}
\varepsilon' &= v \\
v' &= cv - f(u) + w \\
w' &= z \\
\delta z' &= -cz + \varepsilon(u - \gamma w) \\
c' &= 0 \\
\varepsilon' &= 0
\end{align*}
\tag{5.15}
\]

This system is of the form (1.2) with the singular perturbation parameter \( \delta \) and \( n = 5 \) corresponding to the slow variables \( u, v, w, c, \varepsilon \) and \( k = 1 \).
corresponding to the fast variable $z$. The reduced problem for system (5.15) is just the traveling wave problem (5.2), (5.4) which we have considered before supplemented by the trivial equation $\varepsilon' = 0$. All the assumptions of Theorem 2.1 are satisfied, the manifold $\mathcal{S}$ of the reduced problem is particularly simple since it is the graph of the function $z = (\varepsilon/c)(u - \gamma w)$. Thus we are in the first case where the invariant manifolds and the connecting singular orbits lie completely in one branch of the manifold $\mathcal{S}$. We define the two-dimensional manifolds

$$\mathcal{N}_1 = \{(0, 0, 0, c, \varepsilon) : c \in (\alpha, \beta), \varepsilon \in (0, \varepsilon_1)\}$$

$$\mathcal{N}_2 = \{(u_2, 0, f(u_2), c, \varepsilon) : c \in (\alpha, \beta), \varepsilon \in (0, \varepsilon_1)\}$$

of fixed points of system (5.15) which are also fixed points of the corresponding $(\delta = 0)$ reduced problem. The constants $\alpha$, $\beta$, and $\varepsilon_1$ in (5.16) are the same as in Theorem 5.1. The corresponding unstable resp. stable manifolds of the reduced problem of system (5.15) are denoted by $\mathcal{W}_u^s$ resp. $\mathcal{W}_s^u$. As before we omit the necessary modifications of the equations to make $\mathcal{N}_1$ overflowing invariant and $\mathcal{N}_2$ inflowing invariant. The proof of Theorem 5.1 implies that the manifolds $\mathcal{W}_u^s$ and $\mathcal{W}_s^u$ intersect transversally along the manifold $\{(h_\varepsilon, c_\varepsilon, \varepsilon) : \varepsilon \in (0, \varepsilon_1)\}$ of heteroclinic orbits given by Theorem 5.1. Thus the singular unstable manifold $\mathcal{N}_u^s$ and the singular stable manifold $\mathcal{N}_s^u$ intersect transversally. Corollary 3.1 and similar arguments as at the end of the proof of Theorem 5.1 imply the theorem.

6. TRANSVERSAL ORBITS HOMOCLINIC TO PERIODIC ORBITS

If the stable and unstable manifolds of a hyperbolic periodic orbit intersect transversally the corresponding Poincaré map defined on a crosssection has a transversal homoclinic point. By the Smale–Birkhoff Homoclinic Theorem this implies chaotic shift like dynamics; in particular, the existence of a countable infinity of periodic points of arbitrary long period and the existence of a dense orbit (see, e.g., [10]). The main tool in detecting transversal homoclinic orbits is provided by Melnikov's method (see [10, 20, 21]).

For singularly perturbed systems (1.1) the methods presented in this paper provide another way to prove the existence of a transversal homoclinic orbit for small $\varepsilon$. Assume that the manifold $\mathcal{N}$ is a hyperbolic periodic orbit of the reduced problem (1.3) and further that there exists an orbit $(x_0, y_0(\tau))$ of the layer problem (1.4) homoclinic to a point $(x_0, y(x_0)) \in \mathcal{N}$. Then we have

**Theorem 6.1.** Let $\mathcal{N}$ be a hyperbolic periodic orbit satisfying the
assumptions of Theorem 2.2. Assume that there exists a singular homoclinic orbit \((x_0, y_0(\tau))\) in the one-dimensional intersection of the manifolds
\[
\mathcal{N}^u = \bigcup_{p \in \mathcal{W}^u} \mathcal{F}^u(p), \quad \mathcal{N}^s = \bigcup_{p \in \mathcal{W}^s} \mathcal{F}^s(p).
\]
Assume that \(y_0'\) is the only bounded solution of the variational equation
\[
y' = D_x g(x_0, y_0(\tau), 0) y
\]
and that the vector \(M \in \mathbb{R}^n\) defined by Eq. (4.7) satisfies
\[
(M, f(x_0, y(x_0))) \neq 0. \tag{6.1}
\]
Then there exists \(\varepsilon > 0\) such that there exists a transversal orbit of the singularly perturbed system (1.1) homoclinic to the hyperbolic periodic orbit \(\mathcal{N}_c\) for \(0 < \varepsilon < \varepsilon_1\).

Proof. The theorem follows by specializing Theorem 3.2 and Corollary 4.1 to this situation since for the reduced problem (1.3) the one-dimensional tangent space of the periodic orbit \(\mathcal{N}\) at the point \((x_0, y(x_0))\) is given by the vector \(f(x_0, y(x_0))\). □

Geometrically the transversality condition corresponds to the breaking of the homoclinic orbit as the base point \((x_0, y_0)\) moves along the periodic orbit.

The dynamics near the transversal homoclinic orbit should be particularly interesting due to the two different time scales in the problem. One expects that typical trajectories should exhibit long time intervals of slow variation close to the periodic orbit irregularly interrupted by fast spikelike motion close to the singular homoclinic orbit. Furthermore, the countably infinite nearby periodic orbits should be of relaxation oscillation type. Clearly, Theorem 6.1 holds for invariant tori as well if the obvious modifications are made. Another possible generalization is to use Theorem 3.1 and Theorem 3.2 to give conditions for the existence of transversal heteroclinic cycles which implies similar chaotic dynamics. See the example at the end of this section which is taken from [1, 15].

Now we discuss a particularly simple situation in which Theorem 6.1 applies and discuss its relationship to Melnikov's method. We consider the fast time scale problem (1.2) under the additional assumptions that \(n = 1\) and \(f(x + 2\pi, y, \varepsilon) = f(x, y, \varepsilon)\) holds, i.e., \(f\) is \(2\pi\)-periodic in \(x\), which allows us to consider the system (1.2) to be defined on \(\mathcal{M} = S^1 \times \mathbb{R}^k\). Assume that the equation \(g(x, y, 0) = 0\) has a one-dimensional manifold of solutions \(y = y(x)\). We assume further that \(y(x)\) is \(2\pi\)-periodic and that the manifold
\[
\mathcal{F} = \{(x, y(x)) : x \in S^1\} \tag{6.2}
\]
is a closed curve of hyperbolic fixed points of the layer problem (1.4). Thus, all the assumptions of Theorem 2.1 are satisfied by taking $N' = \mathcal{S}$. If the layer problem (1.4) has a homoclinic orbit Theorem 6.1 can be applied. In this situation the transversality condition is reduced to the scalar equation $M \neq 0$. Obviously, the transversality condition is violated if homoclinic orbits of the layer problem exist at a continuum of points in $\mathcal{S}$.

In the simplest application of Melnikov's method (see [10]) one considers systems of the form

\[ x' = 1, \]
\[ y' = g(y) \cdot \epsilon h(x, y) \]  

(6.3)

defined on the manifold $\mathcal{M} = S^1 \times \mathbb{R}^k$. The assumption that the equation $y' = g(y)$ has an orbit homoclinic to a hyperbolic fixed point $y_0$ implies that for $\epsilon = 0$ system (6.3) has orbits homoclinic to all points in $\mathcal{S} = S^1 \times \{y_0\}$. Thus, the intersection of the unstable and stable manifolds of $\mathcal{S}$ is nontransversal.

Instead of considering small perturbations which are represented by the term $\epsilon h(x, y)$ our method allows perturbations of arbitrary size provided that they are slow. The simplest possible case, which has been analyzed in [1] is given by

\[ x' = \epsilon \]
\[ y' = g(x, y, \epsilon). \]  

(6.4)

Theorem 6.1 describes the least degenerate situation, if the transversality condition is violated for $\epsilon = 0$ higher order terms determine transversality for $\epsilon \neq 0$.

It is easy to construct explicit examples of the form (6.4) which satisfy the assumptions of Theorem 6.1. The following problem, which gives rise to a transversal heteroclinic cycle, has been considered in [1, 15]. The equation

\[ y'' + y(y - a(\epsilon \tau))(1 - y) = 0 \]  

(6.5)

describes the motion in a slowly varying quartic potential for small $\epsilon$. We assume that $a$ is a smooth function, $a(0) = \frac{1}{2}, \ a'(0) \neq 0, \ a(t) \in (0, 1)$, and $a(t) = a(t + 2\pi)$. We rewrite Eq. (6.5) as a first order system

\[ x' = \epsilon \]
\[ y'_1 = y_2 \]
\[ y'_2 = -y_1(y_1 - a(x))(1 - y_1) \]  

(6.6)
which is exactly of the form (6.4) with $k = 2$. For fixed $x$ the system (6.5) is Hamiltonian and the fixed points $y = (0, 0)$ and $y = (1, 0)$ are saddles. It is easy to see that there exist an orbit homoclinic to the point $(0, 0)$ for $0 < a(x) < \frac{1}{2}$, an orbit homoclinic to the point $(1, 0)$ for $\frac{1}{2} < a(x) < 1$, and a heteroclinic cycle for $a(x) = \frac{1}{2}$.

As before we define the manifolds

$$\mathcal{N}_1 = S^1 \times \{(0, 0)\}, \quad \mathcal{N}_2 = S^1 \times \{(1, 0)\}$$

(6.7)

and the manifolds $\mathcal{N}_i^u$ and $\mathcal{N}_i^s$, $i = 1, 2$. The existence of the heteroclinic cycle for $a = \frac{1}{2}$ implies that $\mathcal{N}_i^u$ and $\mathcal{N}_j^s$, $i \neq j$ intersect whenever $a(x) = \frac{1}{2}$ holds. By our assumption this is true for $x = 0$. We conclude from Theorem 4.1 that these intersections are transversal iff $M \neq 0$ holds. Equations (4.7), (4.13), and (6.6) imply that

$$M = \int_{-\infty}^{\infty} (-y_2', y_1') \cdot (0, a'(0)(1 - y_1) y_1)^T \, d\tau$$

$$= \int_{-\infty}^{\infty} a'(0) y_2(1 - y_1) y_1 \, d\tau$$

(6.8)

holds. The assumption $a'(0) \neq 0$ implies $M \neq 0$ because $y_2$ and $(1 - y_1) y_1$ have constant signs along the two orbits which form the heteroclinic cycle. Thus the assumptions of Theorem 3.1 are satisfied and we conclude the existence of a transversal heteroclinic cycle connecting the periodic orbits $\mathcal{N}_{1, e}$ and $\mathcal{N}_{2, e}$ for small $e$.

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**References**

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