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Complex equiangular cyclic frames and erasures

Deepti Kalra

Department of Mathematics, University of Houston, 4800 Calhoun Road, Houston, TX 77204-3008, USA

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Abstract

We derive various interesting properties of complex equiangular cyclic frames for many pairs (n, k) using Gauss sums and number theory. We further use these results to study the random and burst errors of some special classes of complex equiangular cyclic (n, k) frames.

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1. Introduction

Two-uniform frames are of key importance in coding and decoding of vectors. The two-uniform frames have been discussed in [2] where it is proved that when such frames exist, they are optimal for two or more erasures. It is also shown in [2] that a frame is two-uniform if and only if it is equiangular in the terminology of [5].

It is known that equiangular (n, k)-frames, i.e. equiangular frames of n vectors for a k-dimensional Hilbert space, can only exist for certain pairs of integers (n, k). For real Hilbert spaces, necessary and sufficient conditions for the existence of real equiangular cyclic frames are expressed in terms of the existence of certain types of graphs [2]. It is shown in [8] that the existence of complex equiangular cyclic (n, k)-frames depends on the existence of certain difference sets.

Although in [8] the authors show that the necessary and sufficient condition for existence of an equiangular cyclic (n, k)-frame is the existence of a corresponding (n, k, λ) difference set, the

E-mail address: deepti@math.uh.edu

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construction of some types of equiangular cyclic (n, k)-frames shown in this paper reveal more important properties which are not reflected in [8]. Using the technique in this paper, we can prove results about the error operators for some equiangular cyclic(n, k)-frames. The construction in this paper, along with results for Gauss sums, give the precise information about the error operators and the corresponding correlation matrix.

In this paper we introduce Gauss sums in the field of frame theory. We observe that the construction of these equiangular cyclic (n, k)-frames involves a particular kind of Gauss sums and hence we use the theory developed for the same. We also look into the geometry of these frames and show that these frames form a spherical 1-design.

2. Basic concepts and definitions

We begin by the definition of a *frame* for a Hilbert space.

Definition 2.1. Let $\mathscr{H}$ be a real or complex Hilbert space and let $F = \{f_i\}_{i\in I}$ be a subset of $\mathbb{H}$, where I is an index set. Then F is called a frame for $\mathscr{H}$ provided that there are two positive integers A, B such that the inequalities

$$A\|x\|^2 \leqslant \sum_{j\in I} |\langle x, f_j\rangle|^2 \leqslant B\|x\|^2$$

hold for every $x \in \mathscr{H}$. Here $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors in $\mathscr{H}$.

If $A = B = 1$, then $\{f_i\}_{i\in I}$ is called a *Parseval* frame or *Unit Normalized Tight* frame or UNTF. A frame is called *uniform* or *equal-norm* provided there is a constant c such that $\|f_i\| = c$ for each $i \in I$.

Let $\mathbb{F}$ be a field of real or complex numbers. Let $\mathscr{F}(n, k)$ be the collection of all Parseval frames for a k-dimensional Hilbert space $\mathbb{F}^k$ consisting of n vectors. Such frames are called (n, k)-frames. The ratio of n/k is called the *redundancy* ratio of the (n, k)-frame.

It is known that a Parseval frame satisfies the Parseval identity,

$$x = \sum_{l\in I}\langle x, f_l\rangle f_l \quad \forall x \in \mathscr{H}.$$

In [2], certain frames are identified as being equivalent. Given frames $F = \{f_1, f_2, \ldots, f_n\}$ and $G = \{g_1, g_2, \ldots, g_n\}$, we say that they are *type I* equivalent if there exists a unitary (orthogonal, in the real case) matrix U such that $g_i = Uf_i$ for all i. If V and W are the analysis operators for F and G, respectively, then it is clear that F and G are *type I* equivalent if and only if $V = WU$ or equivalently, if and only if $VV^* = WW^*$. Thus, there is a one-to-one correspondence between $n \times n$ rank k projections and *type I* equivalence classes of (n, k)-frames. We say that two frames are *type II* equivalent if they are simply a permutation of the same vectors and *type III* equivalent if the vectors differ by multiplication with ± 1 in the real case and multiplication by complex numbers of modulus one in the complex case.

Let us now look at the case of losing m coefficients, *i.e.* the case of *m-erasures*. We define the error operator $E_{i_1,\ldots,i_m}$ as

$$E_{i_1,\ldots,i_m}(x) = x - \sum_{l\neq i_1,\ldots,i_m}\langle x, f_l\rangle f_l = \sum_{j=1}^{m}\langle x, f_{i_j}\rangle f_{i_j}.$$

The norm of this error operator is given by the operator norm of the $m \times m$ correlation matrix

$$\begin{bmatrix} \langle f_{i_1}, f_{i_1}\rangle & \langle f_{i_2}, f_{i_1}\rangle & \cdots & \langle f_{i_m}, f_{i_1}\rangle \\ \langle f_{i_1}, f_{i_2}\rangle & \langle f_{i_2}, f_{i_2}\rangle & \cdots & \langle f_{i_m}, f_{i_2}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_{i_1}, f_{i_m}\rangle & \langle f_{i_2}, f_{i_m}\rangle & \cdots & \langle f_{i_m}, f_{i_m}\rangle \end{bmatrix}.$$

If we lose only one coefficient, then it can be seen that the norm of the error operator is given by the norm of the corresponding frame vector. Therefore, in the case of a uniform frame, the error is constant for each coefficient.

In the case of losing two coefficients, ith and jth, the error operator $E_{i,j}$ is given by

$$E_{i,j}(x) = x - \sum_{l \neq i,j} \langle x, f_l\rangle f_l = \langle x, f_i\rangle f_i + \langle x, f_j\rangle f_j.$$

This condition is called *two-erasures*. A *two-uniform* frame is a frame which is uniform and $\|E_{i,j}\| = \text{constant}\ \forall i \neq j$.

We state the following characterization of two-uniform frames from [2].

Theorem 2.2. *Let $\{f_i\}_{i\in I}$ be a uniform (n, k)-frame. Then, it is two-uniform (equiangular) if and only if $|\langle f_i, f_j\rangle| = c_{n,k}$ for each $i, j \in I$ such that $i \neq j$ where*

$$c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}}.$$

A frame $\{f_i\}_{i\in I}$ such that $|\langle f_i, f_j\rangle| = \text{constant}$ for each $i, j \in I$ for $i \neq j$ is called an *equiangular* frame.

In [1], it is established that when $\{f_i\}_{i\in I}$ is a uniform (n, k)-frame, then each vector f_i, $i \in I$ is of length $\sqrt{\frac{k}{n}}$. Also, it is shown that two-uniform frames, when they exist, are optimal for two-erasures.

We now look at some basic concepts and theorems in number theory from [7].

We know that $a \in \mathbb{Z}_n$ such that $gcd(a, n) = 1$, is called a *quadratic residue* of an odd prime n if and only if $x^2 \equiv a \pmod{n}$ has a solution in $\mathbb{Z}_n$. Otherwise, a is called a *quadratic non-residue* of n. Note that if $a \equiv b \pmod{n}$, then a is a quadratic residue (non-residue) of n if and only if b is a quadratic residue (non-residue) of n. Therefore, we only look for residues in $\mathbb{Z}_n$. This is also called a *reduced residue* system. Note that the product of two quadratic residues or two quadratic non-residues is a quadratic residue in a reduced residue system of n.

Since $a^{n-1} \equiv 1 \pmod{n}$, then $a^{n-1} - 1 \equiv (a^{\frac{n-1}{2}} - 1)(a^{\frac{n-1}{2}} + 1) \equiv 0 \pmod{n}$.

Thus, $a^{\frac{n-1}{2}} \equiv 1 \pmod{n}$ or $a^{\frac{n-1}{2}} \equiv -1 \pmod{n}$.

We can now state the following result also known as the *Euler's* criterion.

Theorem 2.3. *Let n be an odd prime and $gcd(a, n) = 1$. Then,*

$$a^{\frac{n-1}{2}} = \left(\frac{a}{n}\right)_L = \begin{cases} 1 & a \text{ is a quadratic residue}, \\ -1 & a \text{ is a quadratic non-residue}, \end{cases}$$

where $(\ )_L$ is called the Legendre symbol.

It is easy to prove the following properties of Legendre symbol,

1. $\left(\frac{ab}{n}\right)_L = \left(\frac{a}{n}\right)_L \left(\frac{b}{n}\right)_L$.
2. $\left(\frac{1}{n}\right)_L = 1$.

Using modular arithmetics and the binomial theorem, we get

$$(n-a)^{\frac{n-1}{2}} \equiv (-1)^{\frac{n-1}{2}} a^{\frac{n-1}{2}} \pmod n.$$

Let $n = 2k + 1$ be such that n is prime and k is odd. For such k odd and n prime, we equivalently say $n \equiv 3$ (mod 4). Let a be in the reduced residue system of n. Thus by Euler's Criterion, a is a quadratic residue (non-residue) of n if and only if $(n-a)$ is a quadratic non-residue (residue) of n.

Theorem 2.4. *Let $n \equiv 3 \pmod 4$ be an odd prime. Then any reduced residue system $(\bmod\ n)$ contains $\frac{n-1}{2}$ quadratic residues and $\frac{n-1}{2}$ quadratic non-residues of n. One set of $\frac{n-1}{2}$ congruent quadratic residues is $\left\{1^2, 2^2, \ldots, \left(\frac{n-1}{2}\right)^2\right\}$.*

We now state the following result which provides a means for determining which primes have 2 as a quadratic residue.

Theorem 2.5. *For an odd prime n, we have*

$$\left(\frac{2}{n}\right)_L = \begin{cases} 1 & n \equiv \pm 1 \pmod 8, \\ -1 & n \equiv \pm 3 \pmod 8. \end{cases}$$

Let us now look at the case when $n \equiv 1$ (mod 4) such that $n = 4k^2 + 1$ is prime for odd k.

Definition 2.6. An element $a \in \mathbb{Z}_n$, such that $gcd(a, n) = 1$, is called a quartic(biquadratic) residue of an odd prime n if and only if $x^4 \equiv a \pmod n$ has a solution in $\mathbb{Z}_n$. Otherwise, a is called a quartic(biquadratic) non-residue of n.

Note that every quartic residue is a quadratic residue. Also, product of two quartic residues is a quartic residue. By [6], we know that a is a quartic residue of n if $a^{\frac{n-1}{4}} \equiv 1 \pmod n$.

Let us denote the set of quadratic residues by S_2 and the set of quartic residues by S_4. Then it can be shown that for primes $n \equiv 1$ (mod 4), $a \in S_2$ if and only if $(n-a) \in S_2$. Also, $a \in S_4$ if and only if $(n-a) \in S_2$. Therefore, $\overline{S}_4 = S_2 - S_4$ and $\{S_4, \overline{S}_4\}$ forms a partition of S_2 such that $|S_4| = |\overline{S}_4|$.

Let $a \notin S_2$. Then it can be checked that $\{aS_4, \bar{a}S_4\}$ forms a partition of ${S_2}^c$ with $x \in aS_4$ if and only if $(n-x) \in \bar{a}S_4$.

Hence ${\mathbb{Z}_n}^*$ can be partitioned into $\{S_4, \overline{S}_4, aS_4, \bar{a}S_4\}$ such that for every $x \in S_4$, we have $xS_4 = S_4$, $x\overline{S}_4 = \overline{S}_4$, $xaS_4 = aS_4$ and $x\bar{a}S_4 = \bar{a}S_4$.

Now let us look at the concept of difference sets.

Definition 2.7. A subset H of a finite (additive) Abelian group G is said to be a (n, k, λ)-difference set of G if for some fixed natural number λ, every nonzero element of G can be written as a difference of two elements of H in exactly λ ways, where $|G| = n$ and $|H| = k$.

The followings tabulation on difference sets is discussed in [7].

Type S (Singer difference sets). These are hyper planes in $PG(m, q)$, $q = p^r$. The parameters are

$$n = \frac{q^{m+1} - 1}{q - 1}, \quad k = \frac{q^m - 1}{q - 1}, \quad \lambda = \frac{q^{m-1} - 1}{q - 1}.$$

Type Q. Let $n = p^r \equiv 3 \pmod 4$. Then the quadratic residues of n form a difference set with parameters

$$n = p^r = 4t - 1, \quad k = 2t - 1, \quad \lambda = t - 1.$$

Type H_6. Let $n = 4x^2 + 27$. There will exist a primitive root $r \pmod n$ such that $Ind_r(3) \equiv 1 \pmod 6$. The residues $a_i \pmod n$ such that $Ind_r(a_i) \equiv 0$, 1 or 3 (mod 6) will form a difference set with

$$n = 4t - 1, \quad k = 2t - 1, \quad \lambda = t - 1.$$

Type T (Twin primes). Let n and $n' = n + 2$ be both primes. Then the collection of residues $\{a_1, a_2, \ldots, a_m, 0, n', 2n', \ldots, (n-1)n'\}$ such that $\left(\frac{a_i}{p}\right)_L = \left(\frac{a_i}{q}\right)_L$ $\forall i$ form a difference set (mod nn') with parameters

$$nn' = 4t - 1, \quad k = 2t - 1, \quad \lambda = t - 1.$$

Note that the types, Q, H_6 and T, are *Hadamard* type difference sets.

Type B. Let $n = 4x^2 + 1$, x odd. Then the set of biquadratic (quartic) residues form a difference set with parameters

$$n = 4x^2 + 1, \quad k = x^2, \quad \lambda = \frac{x^2 - 1}{4}.$$

Type B_0. Let $n = 4x^2 + 9$, x odd. Then the set of biquadratic (quartic) residues together with zero form a difference set with parameters

$$n = 4x^2 + 9, \quad k = x^2 + 3, \quad \lambda = \frac{x^2 + 3}{4}.$$

Type O. Let $n = 8x^2 + 1 = 64y^2 + 9$, x and y odd. Then the set of octic residues form a difference set with parameters

$$n = 8x^2 + 1, \quad k = x^2, \quad \lambda = y^2.$$

Type O_0. Let $n = 8x^2 + 49 = 64y^2 + 441$, x odd and y even. Then the set of octic residues together with zero form a difference set with parameters

$$n = 8x^2 + 49, \quad k = x^2 + 6, \quad \lambda = y^2 + 7.$$

Type W_4 (Generalization of type T by Whiteman). Let n and $n' = n + 2$ be both primes such that $(n-1, n'-1) = 4$. Define $d = (n-1)(n'-1)/4$. Let g be a primitive root of both n and n'. Then the collection of residues $\{1, g, g^2, \ldots, g^{d-1}, 0, n', 2n', \ldots, (n-1)n'\}$ form a difference set (mod nn') with parameters

$$nn', \quad k = \frac{nn'-1}{4}, \quad \lambda = \frac{nn'-5}{16}.$$

More difference sets can be generated from a given difference sets. This can be seen from the following theorem.

Theorem 2.8. *A set* $\{n_1, n_2, \ldots, n_k\}$ *is a* (n, k, λ)*-difference set if and only if* $\{n_1 + i, n_2 + i, \ldots, n_k + i\}$ *for every* $i \in \mathbb{Z}_n$ *is a* (n, k, λ)*-difference set.*

Let n be an odd prime and let $\{n_1, n_2, \ldots, n_k\}$ be a (n, k, λ)-difference set. Then $\{\alpha n_1, \alpha n_2, \ldots, \alpha n_k\}$ for every $\mathbb{Z}_n^*$ is also a (n, k, λ)-difference set as α is invertible and $n_i - n_j \equiv a_{ij} \Leftrightarrow \alpha n_i - \alpha n_j \equiv \alpha a_{ij}$.

3. Cyclic subspaces and cyclic frames

Cyclic codes are one of the most useful codes in binary coding.

Definition 3.1. A code $\mathscr{C} \in \mathbb{Z}_2^n$ is cyclic if $(x_{n-1}, x_{n-2}, \ldots, x_1, x_0) \in \mathscr{C}$ implies $(x_{n-2}, x_{n-3}, \ldots, x_0, x_{n-1}) \in \mathscr{C}$.

Thus $\mathscr{C}$ is cyclic if and only if $\mathscr{C} \subseteq \frac{\mathscr{P}(x)}{\langle x^n - 1\rangle}$ is an ideal. These codes are efficient in detecting burst errors. A burst error of size d is an n-tuple whose non-zero entries are in a consecutive span of d coordinates and no fewer.

Cyclic frames are inspired by the cyclic codes. We now look at the construction of cyclic equiangular frames.

Let $\{e_i\}_{i=1}^n$ be the standard orthonormal basis of $\mathbb{C}^n$. Let S be the cyclic shift operator on $\mathbb{C}^n$ such that $Se_i = e_{i+1}$ (mod n) $\forall i = 1, 2, \ldots, n-1$ and $Se_n = e_1$. Then S can be written as

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathscr{M}_n.$$

Note that $SS^* = S^*S = I$.

Definition 3.2. A k-dimensional subspace M of $\mathbb{C}^n$ is called cyclic if M is shift-invariant, i.e. $S(M) \subseteq M$.

Let $w = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$ be an nth root of unity. Define for each $i \in \{0, 1, \ldots, n-1\}$, $v_i \in \mathbb{C}^n$ as

$$v_i = \begin{bmatrix} 1 \\ \overline{w}^i \\ \overline{w}^{2i} \\ \vdots \\ \overline{w}^{(n-1)i} \end{bmatrix}.$$

Note that

$$Sv_i = w^{-i} v_i = \overline{w}^i v_i. \tag{1}$$

It can be shown that $\langle v_i, v_j \rangle = 0 \ \forall i \neq j$, and $\|v_i\| = \sqrt{n}$. Thus $\left\{\frac{1}{\sqrt{n}} v_i\right\}_{i=0}^{n-1}$ is an orthonormal basis for $\mathbb{C}^n$.

Let $\{f_0, f_1, \ldots, f_{n-1}\}$ be a Parseval (n, k)-frame and define

$$V = \begin{bmatrix} f_0^* \\ f_1^* \\ \vdots \\ f_{n-1}^* \end{bmatrix}.$$

Then V is an isometry.

Definition 3.3. The frame $\{f_0, f_1, \ldots, f_{n-1}\}$ is called cyclic if and only if range (V) is shift-invariant.

At the first look, this definition seems different than a, perhaps, more obvious definition of cyclic frames.

Definition 3.4. A frame $\{h_0, h_1, \ldots, h_{n-1}\}$ is called a cyclically permutable frame if there exists a unitary operator U on a Hilbert space $\mathscr{H}$ such that $Uh_i = h_{i+1}$ for all $i \in \{0, 1, \ldots, n-1\}$.

Both definitions depend upon the ordering of the frame. We shall show later in this section that these two definitions are equivalent.

Proposition 3.5. *Let $T \in \mathscr{M}_n(\mathbb{C})$ with n distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ corresponding with the eigenvectors $\{u_1, u_2, \ldots, u_n\}$ such that*

$$Tu_i = \lambda_i u_i \quad \forall i = 1, 2, \ldots, n.$$

If $M \subseteq \mathbb{C}^n$ such that $T(M) \subseteq M$, then $M = \mathrm{span}\{u_i | i \in I\}$ where $I \subseteq \{1, 2, \ldots, n\}$.

Proof. Let $M \subseteq \mathbb{C}^n$ such that $dim(M) = k$. Let $\{u_1', u_2', \ldots, u_k'\}$ be a basis of M. Then we can extend the basis of M to a basis $\{u_1', u_2', \ldots, u_n'\}$ of $\mathbb{C}^n$. Then T can be written as

$$T = \left[\begin{array}{c|c} T_1 & * \\ \hline * & T_2 \end{array}\right],$$

where the block T_1 is corresponding to the subspace M with basis $\{u_1', u_2', \ldots, u_k'\}$.

Let $p_T(z)$ be the characteristic polynomial of T. Then

$$p_T(z) = p_{T_1}(z) p_{T_2}(z).$$

But

$$p_T(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n).$$

Thus there exists $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$ such that

$$p_{T_1}(z) = (z-\lambda_{i_1})(z-\lambda_{i_2})\cdots(z-\lambda_{i_k}),$$

where λ_{i_j}'s are all distinct. Therefore the corresponding eigenvectors belong to M, i.e., $\{u'_{i_1}, u'_{i_2}, \dots, u'_{i_k}\} \subseteq M$.

Since $T_1 u'_{i_j} = \lambda_{i_j} u'_{i_j}$, then $Tu'_{i_j} = \lambda_{i_j} u'_{i_j}$ for all j. Then for every j, there exists a constant α_j such that

$$u_{i_j} = \alpha_j u'_{i_j}.$$

Hence for $I = \{i_1, i_2, \dots, i_k\}$, $M = \operatorname{span}\{u_i | i \in I\}$. $\square$

Now considering the cyclic shift operator S and k-dimensional subspace M, assume $S(M) \subseteq M$. By the above proposition, there exists $I \subseteq \{1, 2, \dots, n\}$ such that $M = \operatorname{span}\{v_i : i \in I\}$.

By using Eq. (1), we can summarize this in the following theorem.

Theorem 3.6. *Let M be a subspace of $\mathbb{C}^n$. Then M is S-invariant if and only if $\exists I \subseteq \{0, 1, 2, \dots, n-1\}$ such that $M = span\{v_i : i \in I\}$.*

Let $\{n_1, n_2, \dots, n_k\} \subseteq \{0, 1, 2, \dots, n-1\}$ and $f_j \in \mathbb{C}^k$ such that for each $j \in \{0, 1, \dots, n-1\}$,

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix},$$

where w is a primitive nth root of unity. These vectors form a (n, k)-frame. In [3], these are called the *harmonic frames*. For this frame,

$$V = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ \overline{w}^{n_1} & \overline{w}^{n_2} & \overline{w}^{n_3} & \cdots & \cdots & \overline{w}^{n_k} \\ \overline{w}^{2n_1} & \overline{w}^{2n_2} & \overline{w}^{2n_3} & \cdots & \cdots & \overline{w}^{2n_k} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \overline{w}^{(n-1)n_1} & \overline{w}^{(n-1)n_2} & \overline{w}^{(n-1)n_3} & \cdots & \cdots & \overline{w}^{(n-1)n_k} \end{bmatrix}$$

and $V : \mathbb{C}^k \to \mathbb{C}^n$ is an isometry. Let M be the range of V.

Note that for any choice of $\{n_1, n_2, \dots, n_k\} \subseteq \{0, 1, \dots, n-1\}$, V is *shift-invariant*. Hence every harmonic frame is *cyclic*.

Note that $\{f_0, f_1, \dots, f_{n-1}\}$ is a *Parseval* frame for $\mathbb{C}^k$. This family of frames was introduced in [3], but we should also notice their cyclic nature.

Let M be an S-invariant subspace of $\mathbb{C}^n$ and let $P_M : \mathbb{C}^n \to M$ be the orthogonal projection. We now prove the following theorem.

Theorem 3.7. *Let M be a subspace of $\mathbb{C}^n$ such that $S(M) \subseteq M$ with orthogonal projection P_M. Then $SP_M = P_M S$.*

Proof. Let $v \in \mathbb{C}^n$ such that

$$v = \sum_{v_i \in M} \alpha_i v_i + \sum_{v_i \notin M} \alpha_i v_i.$$

Then

$$\begin{aligned} SP_M(v) &= S\left(\sum_{v_i \in M} \alpha_i v_i\right) = \sum_{v_i \in M} \alpha_i \overline{w}^i v_i = P_M\left(\sum_{v_i \in M} \alpha_i \overline{w}^i v_i + \sum_{v_i \notin M} \alpha_i v_i\right) \\ &= P_M S(v). \quad \square \end{aligned}$$

Consider $A = (a_{i,j})_{i,j}$ be in the commutant of S. Then

$$\begin{aligned} a_{i,j} &= \langle Ae_j, e_i\rangle \\ &= \langle ASe_{j-1}, Se_{i-1}\rangle \\ &= \langle SAe_{j-1}, Se_{i-1}\rangle \\ &= \langle S^*SAe_{j-1}, e_{i-1}\rangle \\ &= \langle Ae_{j-1}, e_{i-1}\rangle \\ &= a_{i-1,j-1}. \end{aligned}$$

Therefore, every A such that $AS = SA$ is of the form

$$A = a_0 I + a_1 S + a_2 S^2 + \cdots + a_{n-1} S^{n-1}$$

for some constants $a_0, a_1, a_2, \ldots, a_{n-1}$. So P_M can be written as

$$P_M = \begin{bmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & \cdots & a_3 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & \ddots & a_0 \end{bmatrix},$$

where $P_M = VV^* = (\langle f_j, f_i\rangle)_{i,j}$ is *Toeplitz*. Moreover, P_M is *circulant* due to its cyclic nature. Note that this frame forms an ordered collection as changing the order of f_i's disturbs the Toeplitz structure of P_M.

Theorem 3.8. *Let $\{h_1, h_2, \ldots, h_n\}$ be a cyclic (n, k)-frame. Then $\exists$ a unitary U and $\{n_1, n_2, \ldots, n_k\} \subseteq \{0, 1, \ldots, n-1\}$ such that $Uh_i = f_i$ $\forall i$, where*

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}$$

defines a harmonic (n, k)-frame.

Proof. Let $\{h_1, h_2, \ldots, h_n\}$ be a cyclic (n, k)-frame. Then $P_H = \left(\langle h_j, h_i\rangle\right)_{i,j}$ is Toeplitz and can be expressed as a polynomial $p(S)$. We know that the eigenvalues of S are $\{w^l : l = 0, 1, \ldots,$

$n-1\}$ where w is the primitive nth root of unity. Thus, the eigenvalues of P_H are given by $\{p(w^l) : l = 0, 1, \ldots, n-1\}$. Since P_H is also a projection and trace$(P_H) = k$, therefore there are exactly k 1's and $(n-k)$ 0's. Let $p(w^l) = 1$ for $l = n_1, n_2, \ldots, n_k$ and $p(w^l) = 0$ otherwise.

Since the eigenvectors of S are also the eigenvectors of $P_H = p(S)$, therefore for each $j = 1, 2, \ldots, n$ the eigenvector of P_H are given by

$$v_j = \begin{bmatrix} 1 \\ w^{j(n-1)} \\ w^{j(n-2)} \\ \vdots \\ w^j \end{bmatrix}.$$

For each $j = 1, 2, \ldots, n$, define

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Then $\{f_1, f_2, \ldots, f_n\}$ is a cyclic (n, k)-frame. Let $P_F = (\langle f_j, f_i \rangle)_{i,j}$. For some r, consider the ith entry of the vector $P_F(v_r)$, *i.e.*,

$$\sum_{j=1}^{n} \langle f_j, f_i \rangle v_{rj} = \sum_{j=1}^{n} \sum_{t=1}^{k} w^{(j-i)n_t} w^{r(-j+1)} = \sum_{t=1}^{k} w^{(r-in_t)} \sum_{j=1}^{n} w^{(n_t-r)j}.$$

Thus the vector v_r is a zero vector exactly when $r \in \{n_1, n_2, \ldots, n_k\}$. Therefore, both P_F and P_M have exactly same eigenvalues and eigenvectors. Hence $P_H = P_F$. Thus by [2], the cyclic frame $\{h_1, h_2, \ldots, h_n\}$ is unitarily equivalent to the frame $\{f_1, f_2, \ldots, f_n\}$. $\square$

Hence it suffices to only consider the cyclic frames of type $\{f_1, f_2, \ldots, f_n\}$ defined above. We will show that the two definitions of cyclic frames are equivalent.

Proposition 3.9. *Let $\{h_0, h_1, \ldots, h_{n-1}\}$ be a (n, k)-frame. If*

$$V = \begin{bmatrix} h_0^* \\ h_1^* \\ \vdots \\ h_{n-1}^* \end{bmatrix},$$

then range(V) *is shift-invariant if and only if there exists a unitary $U : \mathbb{C}^k \to \mathbb{C}^k$ such that $Uh_i = h_{i+1}$ for all $i \in \{0, 1, \ldots, n-1\}$.*

Proof. Let range (V) be shift-invariant. Thus by Theorem 3.8, $\exists$ a unitary U' and $\{n_1, n_2, \ldots, n_k\} \subseteq \{0, 1, \ldots, n-1\}$ such that $U'h_i = f_i\ \forall i$, where

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Note that $Wf_i = f_{i+1}$ for all i, where $W = \mathrm{diag}\{w^{n_1}, w^{n_2}, \dots, w^{n_k}\}$. Thus

$$h_{i+1} = U'^* f_{i+1} = U'^* W f_i = U'^* W U' h_i.$$

Hence there exists a unitary $U'^* W U'$ such that for all i, $U'^* W U' h_i = h_{i+1}$.

Conversely, let there exist a unitary $U : \mathbb{C}^k \to \mathbb{C}^k$ such that $Uh_i = h_{i+1}$ for all $i \in \{0, 1, \dots, n-1\}$. Let us look at the entries of the correlation matrix $(\langle h_j, h_i \rangle)$. For any $x \in \mathbb{C}^k$, we have

$$(SV(x))_i = \langle x, h_{i-1} \rangle = \langle x, U^* h_i \rangle = \langle Ux, h_i \rangle = (VUx)_i.$$

Hence $SV = VU$ and so $S(\mathrm{range}(V)) = \mathrm{range}(V)$, i.e., $\mathrm{range}(V)$ is shift-invariant. □

Note that to find harmonic frames, we only need to determine the set $\{n_1, n_2, \dots, n_k\}$. Therefore, in order to study a harmonic frame with vectors

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix},$$

we shall only study the set $s_j = \{w^{jn_1}, w^{jn_2}, \dots, w^{jn_k}\}$.

We will now try to find optimal cyclic frames for two-erasures indexed by the subset $I \subseteq \{0, 1, 2, \dots, n-1\}$, where order of I is k.

Since for optimal equiangular cyclic frames we have $|\langle f_j, f_i \rangle| =$ constant and P_M is a circulant matrix, the problem of finding optimal equiangular cyclic frames is now reduced to showing that

$$|\langle f_i, f_0 \rangle| = |\langle f_j, f_0 \rangle| \quad \forall i \neq j,\ i \neq 0,\ j \neq 0.$$

Hence, we need to find a subset $I \subseteq \{0, 1, 2, \dots, n-1\}$ such that for each $j \in \{2, 3, \dots, n-1\}$, the *absolute condition* is satisfied, *i.e.*,

$$\left| \sum_{i=1}^{k} w^{n_i} \right| = \left| \sum_{i=1}^{k} w^{jn_i} \right|.$$

In the rest of this paper, we will establish some conditions on n and k to show the existence of equiangular cyclic (n, k)-frames defined as above.

4. Equiangular cyclic frames

We now study the possible selections of the set $I \subseteq \mathbb{Z}_n$ in order to generate equiangular cyclic (n, k)-frames. We note that in order to determine an equiangular cyclic (n, k)-frame as above, we need to determine the frame vector f_1 only. Therefore, we will call the vector f_1 as the *generator* of the frame. In [4], complex equiangular frames were conjectured to exist by numerical experimentation for many pairs (n, k). Most of these can be shown to exist because of the existence of difference sets of the appropriate sizes. Many of the following results were discovered while attempting to construct these frames.

Since the absolute value of a sum does not change when the entries are permuted, thus we will mainly consider the set $s_j = \{w^{jn_1}, w^{jn_2}, \dots, w^{jn_k}\}$ of the entries in vector

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Therefore although we will state the results for the vectors $\{f_j\}_{j\in J}$, it would suffice to prove it for the sets $\{s_j\}_{j\in J}$.

These frames were studied in [8] where the following result is proved for complex MWBE (Maximum Welch Bound Equality) codebooks. The construction of these codebooks show that they are the same as equiangular cyclic frames and therefore, we state the theorem for equiangular cyclic frame and provide a slightly different proof.

Theorem 4.1. *The collection* $\{f_0, f_1, f_2, \dots, f_{n-1}\}$ *is an equiangular cyclic* (n, k)*-frame if and only if the set* $\{n_1, n_2, \dots, n_k\}$ *is a* (n, k, λ)*-difference set, where*

$$f_j = \frac{1}{\sqrt{n}}\begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Proof. Let the collection $\{f_0, f_1, f_2, \dots, f_{n-1}\}$ be an equiangular cyclic (n, k)-frame. Then by [2], we know that for every $l \neq 0$, we have

$$|\langle f_0, f_l\rangle|^2 = \frac{1}{n^2}\sum_{i,j=1}^{k} w^{l(n_i-n_j)} = c^2,$$

where $c = \sqrt{\frac{k(n-k)}{n^2(n-1)}}$.

Let a_r be the order of the set $\{(i, j) \mid n_i - n_j \equiv r(\mathrm{mod}\, n\}$. Then $a_0 = k$. Clearly,

$$\frac{1}{n^2}\sum_{i,j=1}^{k} w^{l(n_i-n_j)} = \frac{1}{n^2}\sum_{t=0}^{n-1} a_t w^{lt} = c^2.$$

Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}$. Then $p(w^l) = n^2c^2\ \forall l \neq 0$ and $p(1) = a_0 + a_1 + \cdots + a_{n-1}$.

Consider the $n \times n$ matrix $U = (w^{ij})_{i,j}$. Then $U^* = (w^{-ij})_{i,j}$ and

$$U^*U = \left(\sum_{t=0}^{n-1} w^{-it}w^{tj}\right)_{i,j} = \left(\sum_{t=0}^{n-1} w^{(j-i)t}\right)_{i,j} = nI.$$

Therefore,

$$U\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + \cdots + a_{n-1} \\ p(w) \\ p(w^2) \\ \vdots \\ p(w^{n-1)} \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 + \cdots + a_{n-1} \\ n^2c^2 \\ n^2c^2 \\ \vdots \\ n^2c^2 \end{bmatrix}.$$

So, we get,

$$n\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = U^*U\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = U^*\begin{bmatrix} a_0 + a_1 + a_2 + \cdots + a_{n-1} \\ n^2c^2 \\ n^2c^2 \\ \vdots \\ n^2c^2 \end{bmatrix}.$$

Thus,

$$n\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 + \cdots + a_{n-1} + (n-1)n^2c^2 \\ a_0 + a_1 + a_2 + \cdots + a_{n-1} - n^2c^2 \\ a_0 + a_1 + a_2 + \cdots + a_{n-1} - n^2c^2 \\ \vdots \\ a_0 + a_1 + a_2 + \cdots + a_{n-1} - n^2c^2 \end{bmatrix}.$$

Hence, $na_r = a_0 + a_1 + a_2 + \cdots + a_{n-1} - n^2c^2$, which is independent of r. Therefore,

$$a_1 = a_2 = \cdots = a_{n-1} = \lambda$$

for some constant λ. So, $n\lambda = k + (n-1)\lambda - n^2c^2$. Thus,

$$\lambda = k - n^2c^2 = k - \frac{k(n-k)}{n-1} = \frac{k(k-1)}{n-1}.$$

Hence, $\{n_1, n_2, \ldots, n_k\}$ forms a difference set.

Conversely, let $\{n_1, n_2, \ldots, n_k\}$ be a difference set. For each non-zero j, define

$$f_j = \frac{1}{\sqrt{n}}\begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Then [2] showed that the collection $\{f_0, f_1, f_2, \ldots, f_{n-1}\}$ is equiangular if and only if

$$|\langle f_i, f_j\rangle| = c$$

for all $i \neq j$. Consider

$$\begin{aligned} |\langle f_i, f_j\rangle|^2 &= \langle f_i, f_j\rangle\overline{\langle f_i, f_j\rangle} \\ &= \frac{1}{n^2}\left(\sum_{l=1}^{k} w^{in_l - jn_l}\right)\left(\sum_{m=1}^{k} w^{jn_m - in_m}\right) \\ &= \frac{1}{n^2}\left(\sum_{l,m=1}^{k} w^{(i-j)(n_l - n_m)}\right) \\ &= \frac{1}{n^2}\left(k + \sum_{l \neq m} w^{(i-j)(n_l - n_m)}\right) \\ &= \frac{1}{n^2}\left(k + \sum_{r=1}^{n-1} a_r w^{(i-j)r}\right) \\ &= \frac{1}{n^2}\left(k + \sum_{r=1}^{n-1} \lambda w^{(i-j)r}\right) \\ &= \frac{1}{n^2}\left(k + \lambda\sum_{r=1}^{n-1} w^{(i-j)r}\right) \\ &= \frac{1}{n^2}\left(k + \lambda(-1)\right) = \frac{1}{n^2}\left(k - \lambda\right) = c^2. \end{aligned}$$

Hence, as the absolute condition is satisfied, therefore the collection $\{f_i\}_{i=0}^{n-1}$ is an equiangular cyclic (n, k)-frame. $\square$

If $\{n_1, n_2, \dots, n_k\}$ is a (n, k, λ)-difference set, then we can generate another (n, k, λ)-difference set by translation, i.e., $\{n_1 + a, n_2 + a, \dots, n_k + a\}$ for some a (mod n). The equiangular cyclic frame generated by the translated difference set is *type III* equivalent to the one generated by original difference set. If the elements of a difference set are multiplied by a number j (mod n), then the equiangular cyclic frame generated by new difference set is *type II* equivalent to the equiangular cyclic frame generated by the original set. However, there exist multiple (n, k, λ)-difference sets for some pair (n, k) which generate inequivalent cyclic frames. We shall see more on the difference sets that are inequivalent in this sense, later in this article.

Since $\lambda = \frac{k(k-1)}{n-1}$ must be an integer, we have the following corollary which gives a necessary condition for the existence of a (n, k, λ)-difference set.

Corollary 4.2. *If there exist a (n, k, λ)-difference set, then $n - 1$ must divide $k(k - 1)$.*

We now look at the following results which are obtained independently of [8]. These examine equiangular cyclic frames without involving difference sets. A close observation of the results from [8] with the following results reveal some very interesting properties of difference sets which might not be that obvious by their definition.

We start with an example of an equiangular cyclic $(7, 3)$-frame and depict the use of the *absolute condition*.

For $n = 7$ and $k = 3$, let us choose

$$f_1 = \frac{1}{\sqrt{7}} \begin{bmatrix} w \\ w^2 \\ w^3 \end{bmatrix}.$$

So $s_1 = \{w, w^2, w^3\}$. We can check that the absolute condition fails as

$$|\langle f_1, f_0\rangle| = |w + w^2 + w^3| \neq |w^2 + w^4 + w^6| = |\langle f_2, f_0\rangle|.$$

Thus, the above chosen f_1 does not generate an equiangular cyclic $(7, 3)$-frame.

However, let us now choose

$$f_j = \frac{1}{\sqrt{7}} \begin{bmatrix} w \\ w^2 \\ w^4 \end{bmatrix}.$$

So $s_1 = \{w, w^2, w^4\}$. Then we can check that $s_i = \{w, w^2, w^4\}$ for $i = 1, 2, 4$ and $s_i = \{w^3, w^5, w^6\}$ for $i = 3, 5, 6$, and

$$|w + w^2 + w^4| = |w^3 + w^5 + w^6|.$$

Thus the absolute condition is satisfied and hence f_1 generates such an equiangular cyclic $(7, 3)$-frame.

The following is our first theorem which demonstrates the relation between n, k and the roots of unity required to generate an equiangular cyclic (n, k)-frame.

Theorem 4.3. *Let $\{n_1, n_2, \dots, n_k\} \subseteq \mathbb{Z}_n$ such that $\forall j \in \{2, 3, \dots, n - 1\}$, $\exists l_j \in \mathbb{Z}_n$ and a permutation π_j of $\mathbb{Z}_k$ such that either*

(i) $jn_i - n_{\pi_j(i)} \equiv l_j \pmod n \quad \forall i = 1, 2, \ldots, k$

or,

(ii) $jn_i + n_{\pi_j(i)} \equiv l_j \pmod n \quad \forall i = 1, 2, \ldots, k.$

Then the collection $\{f_0, f_1, f_2, \ldots, f_{n-1}\}$ *is an equiangular cyclic* (n, k)*-frame with*

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Proof. Let $\{n_1, n_2, \ldots, n_k\} \subseteq \mathbb{Z}_n$ be chosen as above. Then $\{f_0, f_1, f_2, \ldots, f_{n-1}\}$ will be an equiangular cyclic (n, k)-frame if and only if the absolute condition is satisfied, *i.e.* for each $j \in \{2, 3, \ldots, n-1\}$

$$\left|\sum_{i=1}^{k} w^{n_i}\right| = \left|\sum_{i=1}^{k} w^{jn_i}\right|.$$

Firstly, choose j such that (i) is satisfied. Then for $j \in \{2, 3, \ldots, n-1\}$,

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{jn_1} \\ w^{jn_2} \\ \vdots \\ w^{jn_k} \end{bmatrix}.$$

Then $s_j = \{w^{jn_1}, w^{jn_2}, \ldots, w^{jn_k}\}$.

Then,

$$w^{jn_m} = w^{l_j + n_{\pi_j(m)}} = w^{l_j} w^{n_{\pi_j(m)}}.$$

Therefore,

$$\left|\sum_{i=1}^{k} w^{jn_i}\right| = \left|\sum_{i=1}^{k} w^{l_j} w^{n_{\pi_j(i)}}\right| = \left|w^{l_j} \sum_{i=1}^{k} w^{n_{\pi_j(i)}}\right| = |w^{l_j}| \left|\sum_{m=1}^{k} w^{n_m}\right| = \left|\sum_{m=1}^{k} w^{n_m}\right|.$$

Now, choose j such that (ii) is satisfied. Then,

$$w^{jn_m} = w^{l_j - n_{\pi_j(m)}} = w^{l_j} \overline{w^{n_{\pi_j(m)}}}.$$

Therefore,

$$\left|\sum_{i=1}^{k} w^{jn_i}\right| = \left|\sum_{i=1}^{k} w^{l_j} \overline{w^{n_{\pi_j(i)}}}\right| = \left|w^{l_j} \sum_{i=1}^{k} \overline{w^{n_{\pi_j(i)}}}\right| = |w^{l_j}| \left|\overline{\sum_{m=1}^{k} w^{n_m}}\right| = \left|\sum_{m=1}^{k} w^{n_m}\right|.$$

Thus the absolute condition is satisfied for all j. Hence, $\{f_0, f_1, f_2, \ldots, f_{n-1}\}$ is an equiangular cyclic (n, k)-frame. □

Since a set $\{n_1, n_2, \ldots, n_k\} \subseteq \mathbb{Z}_n$ generates an equiangular cyclic (n, k)-frame if and only if the set is a difference set, we have the following corollary.

Corollary 4.4. *Any set* $\{n_1, n_2, \ldots, n_k\} \subseteq \mathbb{Z}_n$ *satisfying the properties of Theorem* 4.2 *must be a difference set.*

More frames can be developed from a given frame. We use the fact that the sum of all roots of unity is zero to prove the following theorem.

Theorem 4.5. *Let w be a primitive nth root of unity. Let $n_1, n_2, \ldots, n_k \in \mathbb{Z}_n$ such that*

$$f_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{n_1} \\ w^{n_2} \\ \vdots \\ w^{n_k} \end{bmatrix}$$

generates an equiangular cyclic (n, k)-frame. Then the remaining roots of unity generate an equiangular cyclic $(n, n-k)$-frame generated by

$$h_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{n_{k+1}} \\ w^{n_{k+2}} \\ \vdots \\ w^{n_n} \end{bmatrix}.$$

Proof. Let $\{n_{k+1}, n_{k+2}, \ldots, n_n\} \subseteq \mathbb{Z}_n$ such that $\{w^{n_{k+1}}, w^{n_{k+2}}, \ldots, w^{n_n}\}$ forms the set of remaining nth roots of unity. Since

$$w^{n_1} + w^{n_2} + \cdots + w^{n_n} = 0,$$

therefore,

$$w^{jn_1} + w^{jn_2} + \cdots + w^{jn_n} = 0.$$

As n is prime, then

$$w^{jn_r} = w^{jn_s} \Leftrightarrow n_r \equiv n_s \pmod{n}.$$

So,

$$w^{jn_1} + w^{jn_2} + \cdots + w^{jn_k} = -w^{jn_{k+1}} - w^{jn_{k+2}} - \cdots + w^{jn_n} \quad \forall 1 \leqslant j \leqslant n-1.$$

Then,

$$|w^{n_1} + w^{n_2} + \cdots + w^{n_k}| = |w^{n_{k+1}} + w^{n_{k+2}} + \cdots + w^{n_n}|$$

and,

$$|w^{jn_1} + w^{jn_2} + \cdots + w^{jn_k}| = |w^{jn_{k+1}} + w^{jn_{k+2}} + \cdots + w^{jn_n}|.$$

But $\{f_i\}$ is an equiangular cyclic (n, k)-frame, therefore,

$$|w^{n_1} + w^{n_2} + \cdots + w^{n_k}| = |w^{jn_1} + w^{jn_2} + \cdots + w^{jn_k}|.$$

Therefore the absolute condition is satisfied, i.e.,

$$|w^{n_{k+1}} + w^{n_{k+2}} + \cdots + w^{n_n}| = |w^{jn_{k+1}} + w^{jn_{k+2}} + \cdots + w^{jn_n}|.$$

So $\{g_0, g_1, \ldots, g_{n-1}\}$ generated by set $t_1 = \{w^{n_{k+1}}, w^{n_{k+2}}, \ldots, w^{n_n}\}$ is an equiangular cyclic $(n, k+1)$-frame. $\square$

Then by Theorem 3.4 we get that the frame generated by $g_1' = (1, w^3, w^5, w^6)$ is an equiangular cyclic $(7, 4)$-frame.

In general, if $\{f_i\}_{i=1}^M$ is a uniform Parseval frame for l_2^N then without loss of generality we may assume $f_i \in l_2^M$ and the orthogonal projection P of l_2^M onto the span of the $\{f_i\}$ satisfies $Pe_i = f_i$ for $i = 1, 2, \ldots, M$ where $\{e_i\}_{i=1}^M$ is the orthonormal basis of l_2^M. Now, $\{(I - P)e_i\}_{i=1}^M$ is also an uniform Parseval frame for l_2^{M-N}.

Corollary 4.6. *If* $\{n_1, n_2, \ldots, n_k\} \subseteq \mathbb{Z}_n$ *is a* (n, k, λ)*-difference set, then the complement must also be a* $(n, n-k, \bar{\lambda})$*-difference set, where*

$$\bar{\lambda} = \frac{(n-k)(n-k-1)}{n-1}.$$

Since $\bar{\lambda} = \frac{(n-k)(n-k-1)}{n-1}$ must be an integer, $n - 1$ must divide the two quantities $k(k-1)$ and $(n-k)(n-k-1)$. Note that

$$(n-k)(n-k-1) \pmod n \equiv k^2 + k \pmod n \equiv k(k-1) \pmod n.$$

Therefore it suffices to state that $n - 1$ must divide $k(k-1)$.

Note that the assumption of Corollary 4.6 hold for any equiangular cyclic (n, k)-frame which is generated by nth roots of unity as above. It can also be deduced from above that we always have an equiangular cyclic $(n, 1)$-frame, and hence, an equiangular cyclic $(n, n-1)$-frame.

Look at the example of an equiangular cyclic $(7, 3)$-frame. We note that the set $\{1, 2, 4\}$ is the set of quadratic residues of 7. Also, these are the powers of w which generate an equiangular cyclic $(7, 3)$-frame.

We now generalize the above observation for the case of an equiangular cyclic $(7, 3)$-frame.

Theorem 4.7. *Let* n *be a prime integer such that* $n = 2k + 1$, *where* k *is odd. For each* $j \in \{0, 1, \ldots, n-1\}$, *define*

$$f_j = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{j1^2} \\ w^{j2^2} \\ \vdots \\ w^{jk^2} \end{bmatrix}.$$

Then the collection $\{f_0, f_1, \ldots, f_{n-1}\}$ *is an equiangular cyclic* (n, k)*-frame.*

Proof. Let m be in the reduced residue system of n. Since in a reduced residue system, the product of two quadratic residues (non-residues) is a quadratic residue and product of a quadratic non-residue with a quadratic residue is a quadratic non-residue, thus jm^2 is a quadratic residue (non-residue) if and only if j is a quadratic residue (non-residue). Let $\{-1^2, -2^2, \ldots, -k^2\}$ be the set of quadratic non-residues of n. Since the set of all quadratic residues is closed with respect to multiplication, it follows that $s_j = \{w^{1^2}, w^{2^2}, \ldots, w^{k^2}\}$ when j is a quadratic residue, and, $s_j = \{w^{-1^2}, w^{-2^2}, \ldots, w^{-k^2}\}$ when j is a quadratic non-residue. Since a is a quadratic residue (non-residue) of n if and only if $(n - a)$ is a quadratic non-residue (residue) of n, we have

$$\begin{aligned} |w^{1^2} + w^{2^2} + \cdots + w^{k^2}| &= |\overline{w^{1^2} + w^{2^2} + \cdots + w^{k^2}}| = |\overline{w^{1^2}} + \overline{w^{2^2}} + \cdots + \overline{w^{k^2}}| \\ &= |\overline{w}^{1^2} + \overline{w}^{2^2} + \cdots + \overline{w}^{k^2}| = |w^{n-1^2} + w^{n-2^2} + \cdots + w^{n-k^2}| \\ &= |w^{-1^2} + w^{-2^2} + \cdots + w^{-k^2}|. \end{aligned}$$

Since the absolute condition is satisfied, $\{f_0, f_1, \ldots, f_{n-1}\}$ is an equiangular cyclic (n, k)-frame. □

We now look at another way of generating frames from a given equiangular cyclic (n, k)-frame.

Corollary 4.8. *Let n be a prime integer such that* $n = 2k + 1$, *where k is odd. Let*

$$f_j = \frac{1}{\sqrt{n}}\begin{bmatrix} w^{j1^2} \\ w^{j2^2} \\ \vdots \\ w^{jk^2} \end{bmatrix}$$

for each $j \in \{0, 1, \ldots, n-1\}$ *be the equiangular cyclic* (n, k)*-frame constructed as above. Then*

$$Re(w^{1^2} + w^{2^2} + \cdots + w^{k^2}) = -\frac{1}{2}.$$

Consequently,

$$g_1 = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 \\ w^{1^2} \\ w^{2^2} \\ \vdots \\ w^{k^2} \end{bmatrix}$$

generates an equiangular cyclic $(n, k+1)$*-frame.*

Proof. Since all nth roots of unity sum up to zero,

$$1 + \sum_{i=1}^{k} w^{i^2} = -\sum_{i=1}^{k} w^{-i^2}.$$

So,

$$\left|1 + \sum_{i=1}^{k} w^{i^2}\right| = \left|\sum_{i=1}^{k} w^{-i^2}\right| = \left|\sum_{i=1}^{k} w^{i^2}\right|.$$

Also, since n is prime, therefore, for each non-zero $j \in \mathbb{Z}_n$

$$1 + \sum_{i=1}^{k} w^{ji^2} = -\sum_{i=1}^{k} w^{-ji^2}.$$

Therefore,

$$\left|1 + \sum_{i=1}^{k} w^{ji^2}\right| = \left|\sum_{i=1}^{k} w^{-ji^2}\right| = \left|\sum_{i=1}^{k} w^{ji^2}\right|.$$

Since $|1 + z| = |z|$ if and only if $\mathrm{Re}(z) = -\frac{1}{2}$, we get

$$\mathrm{Re}(w^{1^2} + w^{2^2} + \cdots + w^{k^2}) = -\frac{1}{2}.$$

Then, $\mathrm{Re}(1 + w^{1^2} + w^{2^2} + \cdots + w^{k^2}) = \frac{1}{2}$, such that

$$\left|1 + \sum_{i=1}^{k} w^{i^2}\right| = \left|\sum_{i=1}^{k} w^{i^2}\right|.$$

As $\{f_0, f_1, \ldots, f_{n-1}\}$ is an equiangular cyclic (n, k)-frame, for each non-zero $j \in \mathbb{Z}_n$,

$$\left|\sum_{i=1}^{k} w^{i^2}\right| = \left|\sum_{i=1}^{k} w^{ji^2}\right|.$$

Thus for each non-zero $j \in \mathbb{Z}_n$,

$$\left|1 + \sum_{i=1}^{k} w^{i^2}\right| = \left|1 + \sum_{i=1}^{k} w^{ji^2}\right|.$$

Hence

$$g_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ w^{1^2} \\ w^{2^2} \\ \vdots \\ w^{k^2} \end{bmatrix}$$

generates an equiangular cyclic $(n, k + 1)$-frame. □

Corollary 4.9. *Let n be a prime integer such that $n = 2k + 1$ where k is odd. Then the set of quadratic residues form a difference set. Moreover, the set of residues together with $\{0\}$ also form a difference set.*

Let n be a prime integer such that $n = 2k + 1$ where k is odd. We have now seen two different ways of generating equiangular cyclic $(n, k + 1)$-frames. Let us now look more closely at these two frames.

Let G be the frame generated by

$$g_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ w^{1^2} \\ w^{2^2} \\ \vdots \\ w^{k^2} \end{bmatrix}.$$

And, let H be the frame generated by

$$h_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ w^{-1^2} \\ w^{-2^2} \\ \vdots \\ w^{-k^2} \end{bmatrix}.$$

Note that

$$w^{(n-j)i^2} = w^{-ji^2} = w^{j(n-i^2)}.$$

Thus $g_j = h_{n-j}$ for every j. Therefore, the two frames G and H are just permutations of each other, and hence, are *type I* equivalent.

5. Gauss sums and equiangular cyclic frames

We shall now look at various properties of equiangular cyclic frames that are derived from the concept of Gauss sums. We first start with the definition of a Gauss sum (Gauss period). This is also useful in gaining some specific information on the projection matrices of the above constructed equiangular cyclic frames.

Definition 5.1. A Gauss sum is a sum of roots of unity written as

$$\varphi(a, n) = \sum_{r \in \mathbb{Z}_n} \mathrm{e}^{\frac{-\mathrm{i}\pi r^2 a}{n}},$$

where a and n are relatively prime integers.

We look at the case $a = -2$ and odd prime $n \equiv 3 \pmod 4$. These are also called *quadratic Gauss sums*. Then,

$$\varphi(-2, n) = \sum_{r \in \mathbb{Z}_n} w^{r^2}.$$

Define R, T and N as

$$R = \sum_{r=1}^{k} w^{r^2}, \quad T = \sum_{r=1}^{n-1} \left(\frac{r}{n}\right)_L w^r \quad \text{and} \quad N = \sum_{r=1}^{k} w^{-r^2}.$$

Clearly, $N = \overline{R}$ and $\varphi(-2, n) = 1 + 2R$.

Also, $T = R - N$ and $1 + R + N = 0$.

Therefore, $T = N + R = 1 + 2R = \varphi(-2, n)$.

Then Gauss showed that $\varphi(-2, n) = \sqrt{n}\mathrm{i}$. Let $R = a + b\mathrm{i}$. So,

$$\sqrt{n}\mathrm{i} = \varphi(-2, n) = 1 + 2R = (1 + 2a) + 2b\mathrm{i}.$$

Let us now compare real and imaginary parts. We get $a = -\frac{1}{2}$ as shown earlier and $b = \frac{\sqrt{n}}{2}$. Hence,

$$R = -\frac{1}{2} + \frac{\sqrt{n}}{2}\mathrm{i}, \quad N = \overline{R} = -\frac{1}{2} - \frac{\sqrt{n}}{2}\mathrm{i}$$

and,

$$|R| = |N| = \frac{\sqrt{n+1}}{2}.$$

Using Gauss sums we were able to identify the value of R which is used in the following results.

Proposition 5.2. *Let n be a prime integer such that $n = 2k + 1$ where k is odd. Consider the equiangular cyclic (n, k)-frame generated by*

$$f_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{1^2} \\ w^{2^2} \\ \vdots \\ w^{k^2} \end{bmatrix}.$$

Then,

$$\langle f_j, f_i \rangle = \begin{cases} \frac{1}{n} R & (j-i) \textit{ is a quadratic residue}, \\ \frac{1}{n} \overline{R} & (j-i) \textit{ is a quadratic non-residue}. \end{cases}$$

Proof. Let us consider the inner-product $\langle f_j, f_i \rangle$ as follows:

$$\left\langle \frac{1}{\sqrt{n}} \begin{bmatrix} w^{j1^2} \\ w^{j2^2} \\ \vdots \\ w^{jk^2} \end{bmatrix}, \frac{1}{\sqrt{n}} \begin{bmatrix} w^{i1^2} \\ w^{i2^2} \\ \vdots \\ w^{ik^2} \end{bmatrix} \right\rangle = \frac{1}{n} \sum_{r=0}^{k} w^{(j-i)r^2}.$$

Hence, we get

$$\langle f_j, f_i \rangle = \begin{cases} \frac{1}{n} \sum_{r=0}^{k} w^{r^2} & (j-i) \text{ is a quadratic residue}, \\ \frac{1}{n} \overline{\sum_{r=0}^{k} w^{r^2}} & (j-i) \text{ is a quadratic non-residue}. \end{cases}$$

Therefore,

$$\langle f_j, f_i \rangle = \begin{cases} \frac{1}{n} R & (j-i) \text{ is a quadratic residue}, \\ \frac{1}{n} \overline{R} & (j-i) \text{ is a quadratic non-residue}. \end{cases} \qquad \square$$

As these frames are equiangular, we know that $|\langle f_j, f_i \rangle| = c$ for all $i \neq j$, where c is some constant. By [2], we know that for equiangular cyclic (n, k)-frames, the value of this constant is given by

$$c = \sqrt{\frac{k(n-k)}{n^2(n-1)}}.$$

Let us first consider the case when $(j-i)$ is a quadratic residue and let $\langle f_j, f_i \rangle = c\lambda_{ij}$ such that $|\lambda_{ij}| = 1$. Therefore, by the above proposition, we get

$$\sqrt{\frac{k(n-k)}{n^2(n-1)}} \lambda_{ij} = \frac{R}{n} = -\frac{1}{2n} + \frac{\sqrt{n}}{2n} i.$$

Now by comparing real and imaginary parts, we can show that

$$\lambda_{ij} = \frac{-1}{\sqrt{n+1}} + \sqrt{\frac{n}{n+1}} i.$$

Then we can see that

$$\lambda_{ij} = \begin{cases} \frac{-1}{\sqrt{n+1}} + \sqrt{\frac{n}{n+1}} i & (j-i) \text{ is a quadratic residue}, \\ \frac{-1}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}} i & (j-i) \text{ is a quadratic non-residue}. \end{cases}$$

Now, we use Gauss sums for prime $n \equiv 1 \pmod 4$ such that $n = 4k^2 + 1$, where k is an odd integer. We know that for such prime numbers n, the set of quartic residues forms a difference set and hence, generates an equiangular cyclic (n, k^2)-frame.

Define

$$Q_S = \sum_{r \in S} w^r$$

for some set S.

As shown before, the four orbits of H_4 form a partition of $\mathbb{Z}_n$. Let us denote the orbits as $\{H_4, \overline{H}_4, N_4, \overline{N}_4\}$, where N_4 is the orbit aH_4 for some quadratic non-residue a. As H_4 generates an equiangular cyclic frame, therefore,

$$|Q_{H_4}| = |Q_{\overline{H}_4}| = |Q_{N_4}| = |Q_{\overline{N}_4}|.$$

Also we know that $Q_{H_4} = \overline{Q_{\overline{H}_4}}$ and $Q_{N_4} = \overline{Q_{\overline{N}_4}}$.

Let $Q_{H_4} = a + b\mathrm{i}$ and $Q_{N_4} = c + d\mathrm{i}$. Then, Gauss showed $1 + 4Q_{H_4} = \sqrt{n} + \sqrt{2n + 2\sqrt{n}}$.

Now comparing the real and imaginary parts we get, $a = \frac{\sqrt{n}-1}{4}$ and $b = \frac{\sqrt{2n+2\sqrt{n}}}{4}$.

As shown above, $T = Q_{H_2} - Q_{\overline{H}_2}$. Then, $T = \sqrt{n}$. Since $Q_{H_2} = 2a$ and $Q_{\overline{H}_2} = 2c$, therefore, $c = \frac{-\sqrt{n}-1}{4}$ and $d = \frac{\sqrt{2n-2\sqrt{n}}}{4}$.

By giving similar arguments as above, we can prove the following proposition.

Proposition 5.3. *Let n be a prime integer such that $n = 4k^2 + 1$ for some odd integer k. Consider the equiangular cyclic (n, k^2)-frame generated by*

$$f_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} w^{n_1} \\ w^{n_2} \\ \vdots \\ w^{n_{k^2}} \end{bmatrix},$$

where $\{n_1, n_2, \ldots, n_{k^2}\}$ is the set of all quartic residues of n. Then,

$$\langle f_j, f_i \rangle = \begin{cases} \frac{1}{n} Q_{H_4} & \text{if } (j-i) \in H_4, \\ \frac{1}{n} \overline{Q_{H_4}} & \text{if } (j-i) \in \overline{H}_4, \\ \frac{1}{n} Q_{N_4} & \text{if } (j-i) \in N_4, \\ \frac{1}{n} \overline{Q_{N_4}} & \text{if } (j-i) \in \overline{N}_4. \end{cases}$$

As these frames are also equiangular, therefore we know that $\langle f_j, f_i \rangle = c$ for all $i \neq j$, where c is some constant. In this case,

$$c = \sqrt{\frac{k^2(n-k^2)}{n^2(n-1)}}.$$

Let us first consider the case when $(j - i)$ is in H_4 and let $\langle f_j, f_i \rangle = c\lambda_{ij}$ such that $|\lambda_{ij}| = 1$. Therefore, by proposition above, we get

$$\sqrt{\frac{k^2(n-k^2)}{n^2(n-1)}} \lambda_{ij} = \frac{R}{n} = \frac{\sqrt{n}-1}{4n} + \frac{\sqrt{2n+2\sqrt{n}}}{4n} i.$$

Now by comparing real and imaginary parts, we can show that

$$\lambda_{ij} = \frac{\sqrt{n}-1}{\sqrt{3n+1}} + \sqrt{\frac{2n+2\sqrt{n}}{3n+1}}i.$$

Similarly, it can be shown that

$$\lambda_{ij} = \begin{cases} \frac{\sqrt{n}-1}{\sqrt{3n+1}} + \sqrt{\frac{2n+2\sqrt{n}}{3n+1}}i & \text{if } (j-i) \in H_4, \\ \frac{\sqrt{n}-1}{\sqrt{3n+1}} - \sqrt{\frac{2n+2\sqrt{n}}{3n+1}}i & \text{if } (j-i) \in \overline{H}_4, \\ \frac{-\sqrt{n}-1}{\sqrt{3n+1}} + \sqrt{\frac{2n-2\sqrt{n}}{3n+1}}i & \text{if } (j-i) \in N_4, \\ \frac{-\sqrt{n}-1}{\sqrt{3n+1}} - \sqrt{\frac{2n-2\sqrt{n}}{3n+1}}i & \text{if } (j-i) \in \overline{N}_4. \end{cases}$$

We shall use this information in the following section.

6. Random and burst errors

Recall from [2] that the operator norm of the $m \times m$ correlation matrix $(\langle f_{i_k}, f_{i_l}\rangle)_{k,l=1}^m$ gives the error of m-erasures occurring in locations $\{i_1, \ldots, i_m\}$.

Definition 6.1. A set of m-erasures is called a burst error if $\{i_1, i_2, \ldots, i_m\}$ are consecutive integers. For any arbitrary collection $\{i_1, \ldots, i_m\}$, the set of m-erasures is called a random error.

We shall now look at the characteristic properties of the correlation matrix $(\langle f_j, f_i\rangle)_{ij}$. In case of j erasures, we consider the corresponding adjacent submatrix of order j. Recall from Section 3 that the correlation matrix of an equiangular cyclic (n, k)-frame is *Toeplitz*. Hence all principle submatrices with consecutive rows and columns are same. Therefore we get the following result.

Proposition 6.2. *For every m, the norm of the burst error for m-erasures is constant.*

Since burst errors are a particular type of random errors, it is natural to assume that the minimum random error would be very small as compared to the burst error. However, it need not be true. As can be seen by numerical computation, for the case of equiangular cyclic $(11, 5)$-frame generated by quadratic residues, the minimum random error 0.7106 is less than the burst error 0.7611. However for the case of equiangular cyclic $(7, 3)$-frame generated by quadratic residues, the minimum random error 0.7517 is same as the burst error.

By [2], we know that the equiangular cyclic frames are all optimal for 2-erasures. Let us now consider the case of 3-erasures.

First consider the case when n is prime such that $n = 2k + 1$, where k is odd. Let us consider the case when the $\{i, j, l\}$ coefficients are lost. Then the 3×3 correlation matrix is given by

$$C_{i,j,l} = \begin{bmatrix} \langle f_i, f_i\rangle & \langle f_j, f_i\rangle & \langle f_l, f_i\rangle \\ \langle f_i, f_j\rangle & \langle f_j, f_j\rangle & \langle f_l, f_j\rangle \\ \langle f_i, f_l\rangle & \langle f_j, f_l\rangle & \langle f_l, f_l\rangle \end{bmatrix}.$$

We shall now try to get some information about the norm of this correlation submatrix. We know that $\langle f_a, f_b\rangle = \sum_{t=1}^k w^{(a-b)t^2}$. Therefore

$$C_{i,j,l} = \begin{bmatrix} \frac{k}{n} & \sum_{t=1}^{k} w^{(j-i)t^2} & \sum_{t=1}^{k} w^{(l-i)t^2} \\ \sum_{t=1}^{k} w^{(i-j)t^2} & \frac{k}{n} & \sum_{t=1}^{k} w^{(l-j)t^2} \\ \sum_{t=1}^{k} w^{(i-l)t^2} & \sum_{t=1}^{k} w^{(j-l)t^2} & \frac{k}{n} \end{bmatrix}.$$

It is shown in the above theorem that there are only two possible off-diagonal entries in the error matrix. Hence,

$$C_{i,j,l} = \begin{bmatrix} \frac{k}{n} & \lambda_{ij}c & \lambda_{il}c \\ \bar{\lambda}_{ij}c & \frac{k}{n} & \lambda_{jl}c \\ \bar{\lambda}_{il}c & \bar{\lambda}_{jl}c & \frac{k}{n} \end{bmatrix}.$$

Then we can write $C_{i,j,l} = \frac{k}{n}I + cJ$, where J is given by

$$J = \begin{bmatrix} 0 & \lambda_{ij} & \lambda_{il} \\ \bar{\lambda}_{ij} & 0 & \lambda_{jl} \\ \bar{\lambda}_{il} & \bar{\lambda}_{jl} & 0 \end{bmatrix},$$

where the only possible values for $\lambda_{ij}, \lambda_{il}, \lambda_{jl}$ are λ or $\bar{\lambda}$ depending on whether $(i-j)$, $(i-l)$, $(j-l)$ are quadratic residues or non-residues respectively. Note that the eigenvalue of $C_{i,j,l}$ is given by $\frac{k}{n} + c\alpha$ where α is an eigenvalue of J. Computing the characteristic polynomial of J we get

$$-x^3 + 3x + 2\mathrm{Re}(\lambda_{ij}\bar{\lambda}_{il}\lambda_{jl}) = 0.$$

It is easy to check that

$$\mathrm{Re}(\lambda_{ij}\bar{\lambda}_{il}\lambda_{jl}) = \begin{cases} \mathrm{Re}(\lambda^3) & \lambda_{ij} = \lambda_{jl} \neq \lambda_{il}, \\ \mathrm{Re}\,\lambda & \text{otherwise.} \end{cases}$$

Therefore, the matrix $C_{i,j,l}$ can have at most two distinct possible norms in this case. Also we get that following are the only inequivalent possible forms of matrices for 3-erasures. These are obtained as $DC_{i,j,l}D^{-1}$, where D is a diagonal matrix chosen to make the off-diagonal entries of first row as 1.

$$\text{(i)} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \lambda \\ 1 & \bar{\lambda} & 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \lambda^3 \\ 1 & \bar{\lambda}^3 & 0 \end{bmatrix},$$

$$\text{(iii)} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \bar{\lambda} \\ 1 & \lambda & 0 \end{bmatrix}, \quad \text{(iv)} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \bar{\lambda}^3 \\ 1 & \lambda^3 & 0 \end{bmatrix}.$$

Note that to obtain two different sets of eigenvalues, we only need to consider forms (i) and (ii).

As mentioned above, the only possible values for $\lambda_{ij}, \lambda_{il}, \lambda_{jl}$ are λ or $\bar{\lambda}$ depending on whether $(i-j)$, $(i-l)$, $(j-l)$ are quadratic residues or non-residues respectively. We also know that for a burst error, the corresponding adjacent submatrix is *Toeplitz*. Let $j = i+1$ and $l = i+2$. Then we know that $\lambda_{ij} = \lambda_{jl}$. As shown above, we get the following equivalent classes of J in case of burst errors.

$$\text{(a)} \begin{bmatrix} 0 & \lambda & \lambda \\ \bar{\lambda} & 0 & \lambda \\ \bar{\lambda} & \bar{\lambda} & 0 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 0 & \lambda & \bar{\lambda} \\ \bar{\lambda} & 0 & \lambda \\ \lambda & \bar{\lambda} & 0 \end{bmatrix}.$$

Observe that to obtain form (a) or (b), we must have 2 as a quadratic residue or non-residue respectively. Now by combining Theorem 2.5 with this observation, we get the following result about the matrix J defined above.

Proposition 6.3. *The matrix J is of the form* (a) *or* (b) *whenever* $n \equiv \pm 1 \pmod 8$ *or* $n \equiv \pm 3 \pmod 8$ *respectively*.

Similarly the inequivalent forms of $E_{i_1,\ldots,i_m}$ can be obtained for $m > 3$ and similar observations can be made.

Also, we can follow the same procedure to find such forms for $n \equiv 1 \pmod 4$ such that $n = 4k^2 + 1$, k is odd.

7. Inequivalent frames

From Section 2, we know that there exist multiple types of difference sets. Thus, it is natural to find out whether different types of difference sets generate equivalent or inequivalent equiangular cyclic (n, k)-frames.

From [2], we know that the errors for any m-erasures are the same for equivalent frames. Therefore, one way to find out is to check for the error of 3-erasures in case more than one difference set exists for (n, k).

There exist two distinct types of difference sets for $n = 31$, $k = 15$ and $n = 43$, $k = 21$ as seen in [9]. On numerically computing the maximum of norms of all 3×3 correlation submatrices for equiangular cyclic (31, 15)-frames, we see that the frames generated by type H_6 gives 0.6663 and the one generated by type Q gives 0.6555. In case of $n = 43$, $k = 21$, the maximum of norms of all 3×3 correlation submatrices generated by type H_6 is 0.6426 whereas for the frame generated by type Q is 0.6321.

As can be seen, in these cases, the frames generated by the set of quadratic residues is optimal for 3 erasures. The computations also show that the two frames generated by distinct difference sets need not be equivalent. Following is a list of cases from [9], where multiple difference sets exist.

n	k	*Type*	*Difference set*
31	15	H_6	1, 2, 3, 4, 6, 8, 12, 15, 16, 17, 23, 24, 27, 29, 30
31	15	Q	1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28
43	21	H_6	1, 2, 3, 4, 5, 8, 11, 12, 16, 19, 20, 21, 22, 27, 32, 33, 35 37, 39, 41, 42
43	21	Q	1, 4, 6, 9, 10, 11, 13, 14, 15, 16, 17, 21, 23, 24, 25, 31, 35 36, 38, 40, 41

8. Spherical 1-design

A characterization of spherical 1-design and 2-designs for real uniform frames is given in [2]. In the following theorem, we show that the same characterization holds for spherical 1-designs for the case of complex uniform frames. Let us begin by the definition of a spherical 1-design.

Definition 8.1. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ forms a spherical 1-design if and only if $\{v_1, v_2, \ldots, v_n\}$ is in the sphere of $\mathbb{C}^k$ and

$$\int f \, \mathrm{d}S = \frac{1}{n} \sum_{i=1}^{n} f(v_i)$$

for all polynomials f of degree 1.

The following gives a characterization of the spherical 1-designs for complex frames.

Theorem 8.2. *Let* $\{v_1, v_2, \ldots, v_n\}$ *be a set of vectors in* $\mathbb{C}^k$. *Then the set of vectors* $\{v_1, v_2, \ldots, v_n\}$ *forms a* 1-*design if and only if*

$$\sum_{i=1}^{n} v_i = 0.$$

Proof. Let

$$f(z) = a_0 + a_1 z + \cdots + a_k z_k + b_1 \bar{z}_1 + b_2 \bar{z}_2 + \cdots + b_k \bar{z}_k$$

be a polynomial of degree 1. Let $z_i = x_i + y_i$.
Then $\bar{z}_i = x_i - y_i$. So,

$$f(z) = a_0 + (a_1 + b_1)x_1 + \cdots + (a_k + b_k)x_k + (a_1 - b_1)y_1 i + \cdots + (a_k - b_k)y_k i.$$

Therefore,

$$\int f \, \mathrm{d}S = a_0.$$

Let $a = (a_1 + b_1, (a_1 - b_1)i, \ldots, a_k + b_k, (a_k - b_k)i)$. Let

$$v_j = \begin{bmatrix} c_{j1} + d_{j1} i \\ c_{j2} + d_{j2} i \\ \vdots \\ c_{jk} + d_{jk} i \end{bmatrix}.$$

Then consider $v_j \to w_j$ such that

$$w_j = \begin{bmatrix} c_{j1} \\ d_{j1} \\ c_{j2} \\ d_{j2} \\ \vdots \\ c_{jk} \\ d_{jk} \end{bmatrix}.$$

Then $f(v_i) = a_0 + a \cdot w_i$. Therefore,

$$\frac{1}{n} \sum_{i=1}^{n} f(v_i) = \frac{1}{n} \sum_{i=1}^{n} (a_0 + a \cdot w_i) = a_0 + \frac{1}{n} a . \sum_{i=1}^{n} w_i.$$

Hence,

$$\int f \, \mathrm{d}S = a_0 = \frac{1}{n} \sum_{i=1}^{n} f(v_i)$$

if and only if

$$\sum_{i=1}^{n} v_i = 0. \qquad \square$$

Note that for a prime n, every equiangular cyclic (n, k)-frames form a spherical 1-design since

$$\sum_{i=0}^{n} f_i = 0.$$

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