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Existence of mild solutions for fractional neutral evolution equations☆

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ABSTRACT

In this paper, by using the fractional power of operators and some fixed point theorems, we discuss a class of fractional neutral evolution equations with nonlocal conditions and obtain various criteria on the existence and uniqueness of mild solutions. In the end, we give an example to illustrate the applications of the abstract results.

1. Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [1–6]). There has been a significant development in fractional differential equations in recent years, see the monographs of Kilbas et al. [7], Miller and Ross [8], Podlubny [9], Lakshmikantham et al. [10], and the papers [11–27] and the references therein.

In this paper, we assume that E is a Banach space with the norm $|\cdot|$. Let $J \subset R$. Denote $C(J, E)$ to be the Banach space of continuous functions from J into E with the norm $\|x\| = \sup_{t\in J} |x(t)|$, where $x \in C(J, E)$.

Let $r > 0$ and $\mathcal{C} = C([-r, 0], E)$ be the space of continuous functions from $[-r, 0]$ into E. For any element $z \in \mathcal{C}$, define the norm $\|z\|_* = \sup_{\vartheta\in[-r,0]} |z(\vartheta)|$.

Consider the nonlocal Cauchy problem of the following form

$$\begin{cases} {}^cD^q[x(t) - h(t, x_t)] + Ax(t) = f(t, x_t) & t \in (0, a], \\ x_0(\vartheta) + (g(x_{t_1}, \dots, x_{t_n}))(\vartheta) = \varphi(\vartheta), & \vartheta \in [-r, 0], \end{cases} \tag{1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $0 < q < 1$, $0 < t_1 < \cdots < t_n \le a$, $a > 0$, $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$ of operators on E, f, h: $[0, \infty) \times \mathcal{C} \to E$ and $g : \mathcal{C}^n \to \mathcal{C}$ are given functions satisfying some assumptions, $\varphi \in \mathcal{C}$ and define x_t by $x_t(\vartheta) = x(t + \vartheta)$, for $\vartheta \in [-r, 0]$.

A strong motivation for investigating the nonlocal Cauchy problem (1) comes from physics. For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. They are useful to model anomalous diffusion, where a plume of particles spreads in a different manner than the classical diffusion equation predicts. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 1)$, namely

$$\partial_t^\alpha u(z, t) = Au(z, t), \quad t \ge 0, \ z \in R.$$

We can take $A = \partial_z^{\beta_1}$, for $\beta_1 \in (0, 1]$, or $A = \partial_z + \partial_z^{\beta_2}$ for $\beta_2 \in (1, 2]$, where ∂_t^α, $\partial_z^{\beta_1}$, $\partial_z^{\beta_2}$ are the fractional derivatives of order α, β_1, β_2 respectively. We refer the interested reader to [11,22,24,25] and the references therein for more details.

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The nonlocal condition can be applied in physics with a better effect than the classical initial condition $x_0(\vartheta) = \varphi(\vartheta),\ \vartheta \in [-r, 0]$. For example, $g(x_{t_1}, \ldots, x_{t_n})$ can be written as

$$(g(x_{t_1}, \ldots, x_{t_n}))(\vartheta) = \sum_{i=1}^{m} c_i x_{t_i}(\vartheta),$$

where $c_i (i = 1, 2, \ldots, n)$ are given constants and $0 < t_1 < \cdots < t_n \leq a$. Nonlocal conditions were initiated by Byszewski [28] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski and Lakshmikantham [29], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

In [15], El-Sayed discussed fractional order diffusion-wave equations. Recently, Meerschaert et al. [22] studied a stochastic solution of space-time fractional diffusion equations. Eidelman and Kochubei [12] investigated the Cauchy problem for fractional diffusion equations. El-Borai [13,14] studied a fundamental solution of fractional evolution equations in a Banach space. Baeumer et al. [11] gave the existence of solutions of inhomogeneous fractional diffusion equations with a forcing function. Jardat, Al-Omrai and Momani [18] considered the existence and uniqueness of mild solution for the semilinear initial Value problem of non-integer order. Muslim [23] investigated the existence and approximation of solutions to fractional evolution equation in a Banach space. In addition, regarding works on the existence and uniqueness of different types of solutions to integer-order evolution equations, we refer to [30,28,29,31] and the references therein.

In the next section, we will introduce some useful preliminaries. In Section 3, we establish criteria on the existence and uniqueness of mild solutions for nonlocal Cauchy problem (1) by considering a integral equation which is given in terms of probability density and semigroup. The methods of the functional analysis concerning an analytic semigroup of operators and some fixed point theorems are applied effectively. In Section 4, we also give an example to illustrate the applications of the abstract results.

2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Throughout this paper, let $-A$ be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on E. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for $0 < \eta \leq 1$, it is possible to define the fractional power A^η as a closed linear operator on its domain $D(A^\eta)$. For analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used.

(i) There is a $M \geq 1$ such that

$$M := \sup_{t\in[0,+\infty)} |T(t)| < \infty, \tag{2}$$

(ii) for any $\eta \in (0, 1]$, there exists a positive constant C_η such that

$$|A^\eta T(t)| \leq \frac{C_\eta}{t^\eta}, \quad 0 < t \leq a. \tag{3}$$

For more details about the above preliminaries, we refer to [30].

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1 (*[9]*). The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t > 0,\ \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2 (*[9]*). The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} \mathrm{d}s = I^{n-\alpha} f^{(n)}(t), \quad t > 0,\ 0 \leq n-1 < \alpha < n.$$

If f is an abstract function with values in E, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Assume that $J \subset R$, and $1 \leq p \leq \infty$. For measurable functions $m : J \to R$, define the norm

$$\|m\|_{L^pJ} = \begin{cases} \left(\int_J |m(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \{\sup\limits_{t\in J-\bar{J}} |m(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J, R)$ be the Banach space of all Lebesgue measurable functions $m : J \to R$ with $\|m\|_{L^pJ} < \infty$.

Lemma 2.1 (*Hölder Inequality*). *Assume that $\sigma, p \geq 1$, and $\frac{1}{\sigma} + \frac{1}{p} = 1$. If $l \in L^{\sigma}(J, R)$, $m \in L^{p}(J, R)$, then for $1 \leq p \leq \infty$, $lm \in L^1(J, R)$ and*

$$\|lm\|_{L^1 J} \leq \|l\|_{L^{\sigma} J} \|m\|_{L^p J}.$$

Lemma 2.2 (*Bochner's Theorem*). *A measurable function $Q : [0, a] \to E$ is Bochner integrable if $|Q|$ is Lebesgue integrable.*

Lemma 2.3 (*Krasnoselskii's Fixed Point Theorem*). *Let E be a Banach space, let B be a bounded closed and convex subset of E and let F_1, F_2 be maps of B into E such that $F_1 x + F_2 y \in B$ for every pair $x, y \in B$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1 x + F_2 x = x$ has a solution on B.*

Lemmas 2.1–2.3 are classical, which can be found in many books.

3. Existence and uniqueness of mild solutions

According to Definitions 2.1 and 2.2, it is suitable to rewrite the nonlocal Cauchy problem (1) in the equivalent integral equation

$$\begin{cases} x(t) = \varphi(0) - (g(x_{t_1}, \ldots, x_{t_n}))(0) - h(0, x_0) + h(t, x_t) \\ \qquad + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} [-Ax(s) + f(s, x_s)] \mathrm{d}s, \quad t \in [0, a], \\ x_0(\vartheta) + (g(x_{t_1}, \ldots, x_{t_n}))(\vartheta) = \varphi(\vartheta), \quad \vartheta \in [-r, 0], \end{cases} \tag{4}$$

provided that the integral in (4) exists.

Before giving the definition of mild solution of (1), we first prove the following lemma.

Lemma 3.1. *If (4) holds, then we have*

$$\begin{cases} x(t) = \displaystyle\int_0^{\infty} \phi_q(\theta) T(t^q \theta) [\varphi(0) - (g(x_{t_1}, \ldots, x_{t_n}))(0) - h(0, x_0)] \mathrm{d}\theta + h(t, x_t) \\ \quad + q \displaystyle\int_0^t \int_0^{\infty} \theta (t-s)^{q-1} \phi_q(\theta) A T((t-s)^q \theta) h(s, x_s) \mathrm{d}\theta \mathrm{d}s \\ \quad + q \displaystyle\int_0^t \int_0^{\infty} \theta (t-s)^{q-1} \phi_q(\theta) T((t-s)^q \theta) f(s, x_s) \mathrm{d}\theta \mathrm{d}s, \quad t \in [0, a], \\ x_0(\vartheta) + (g(x_{t_1}, \ldots, x_{t_n}))(\vartheta) = \varphi(\vartheta), \quad \vartheta \in [-r, 0], \end{cases} \tag{5}$$

where ϕ_q is a probability density function defined on $(0, \infty)$, that is

$$\phi_q(\theta) \geq 0, \quad \theta \in (0, \infty) \quad \text{and} \quad \int_0^{\infty} \phi_q(\theta) \mathrm{d}\theta = 1.$$

Proof. Let $\lambda > 0$. Applying Laplace transforms

$$\nu(\lambda) = \int_0^{\infty} \mathrm{e}^{-\lambda s} x(s) \mathrm{d}s, \qquad \chi(\lambda) = \int_0^{\infty} \mathrm{e}^{-\lambda s} h(s, x_s) \mathrm{d}s,$$

and

$$\omega(\lambda) = \int_0^{\infty} \mathrm{e}^{-\lambda s} f(s, x_s) \mathrm{d}s$$

to (4), we have

$$\begin{aligned} \nu(\lambda) &= \frac{1}{\lambda} [\varphi(0) - (g(x_{t_1}, \ldots, x_{t_n}))(0) - h(0, x_0)] + \chi(\lambda) - \frac{1}{\lambda^q} A \nu(\lambda) + \frac{1}{\lambda^q} \omega(\lambda) \\ &= \lambda^{q-1} (\lambda^q I + A)^{-1} [\varphi(0) - (g(x_{t_1}, \ldots, x_{t_n}))(0) - h(0, x_0)] \\ &\quad + \lambda^q (\lambda^q I + A)^{-1} \chi(\lambda) + (\lambda^q I + A)^{-1} \omega(\lambda) \\ &= \lambda^{q-1} \int_0^{\infty} \mathrm{e}^{-\lambda^q s} T(s) [\varphi(0) - (g(x_{t_1}, \ldots, x_{t_n}))(0) - h(0, x_0)] \mathrm{d}s \\ &\quad + \lambda^q \int_0^{\infty} \mathrm{e}^{-\lambda^q s} T(s) \chi(\lambda) \mathrm{d}s + \int_0^{\infty} \mathrm{e}^{-\lambda^q s} T(s) \omega(\lambda) \mathrm{d}s, \end{aligned} \tag{6}$$

where I is the identity operator defined on E.

Consider the one-sided stable probability density [21]

$$\psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

whose Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\lambda\theta}\psi_q(\theta)\mathrm{d}\theta = \mathrm{e}^{-\lambda^q}, \quad \text{where } q \in (0,1). \tag{7}$$

Using (7), we get

$$\begin{aligned}
&\lambda^{q-1}\int_0^\infty \mathrm{e}^{-\lambda^q s}T(s)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}s\\
&= \int_0^\infty q(\lambda t)^{q-1}\mathrm{e}^{-(\lambda t)^q}T(t^q)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}t\\
&= \int_0^\infty -\frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-(\lambda t)^q}]T(t^q)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}t\\
&= \int_0^\infty\int_0^\infty \theta\psi_q(\theta)\mathrm{e}^{-\lambda t\theta}T(t^q)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}\theta\mathrm{d}t\\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\left[\int_0^\infty \psi_q(\theta)T\left(\frac{t^q}{\theta^q}\right)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}\theta\right]\mathrm{d}t,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\lambda^q s}T(s)\omega(\lambda)\mathrm{d}s &= \int_0^\infty\int_0^\infty qt^{q-1}\mathrm{e}^{-(\lambda t)^q}T(t^q)\mathrm{e}^{-\lambda s}f(s,x_s)\mathrm{d}s\mathrm{d}t\\
&= \int_0^\infty\int_0^\infty\int_0^\infty q\psi_q(\theta)\mathrm{e}^{-(\lambda t\theta)}T(t^q)\mathrm{e}^{-\lambda s}t^{q-1}f(s,x_s)\mathrm{d}\theta\mathrm{d}s\mathrm{d}t\\
&= \int_0^\infty\int_0^\infty\int_0^\infty q\psi_q(\theta)\mathrm{e}^{-\lambda(t+s)}T\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q}f(s,x_s)\mathrm{d}\theta\mathrm{d}s\mathrm{d}t\\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\left[q\int_0^t\int_0^\infty \psi_q(\theta)T\left(\frac{(t-s)^q}{\theta^q}\right)f(s,x_s)\frac{(t-s)^q}{\theta^q}\mathrm{d}\theta\mathrm{d}s\right]\mathrm{d}t,
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\lambda^q\int_0^\infty \mathrm{e}^{-\lambda^q s}T(s)\chi(\lambda)\mathrm{d}s &= \int_0^\infty\int_0^\infty q\lambda^q t^{q-1}\mathrm{e}^{-(\lambda t)^q}T(t^q)\mathrm{e}^{-\lambda s}h(s,x_s)\mathrm{d}s\mathrm{d}t\\
&= \int_0^\infty\left[\int_0^\infty -T(t^q)\mathrm{e}^{-\lambda s}h(s,x_s)\mathrm{d}s\right]\mathrm{d}\mathrm{e}^{-(\lambda t)^q}\\
&= \left(\mathrm{e}^{-(\lambda t)^q}\int_0^\infty -T(t^q)\mathrm{e}^{-\lambda s}h(s,x_s)\mathrm{d}s\right)\Bigg|_{t=0}^{\infty}\\
&\quad + \int_0^\infty\int_0^\infty qt^{q-1}\mathrm{e}^{-(\lambda t)^q}AT(t^q)\mathrm{e}^{-\lambda s}h(s,x_s)\mathrm{d}s\mathrm{d}t\\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\left[h(t,x_t) + q\int_0^t\int_0^\infty \psi_q(\theta)AT\left(\frac{(t-s)^q}{\theta^q}\right)h(s,x_s)\frac{(t-s)^q}{\theta^q}\mathrm{d}\theta\mathrm{d}s\right]\mathrm{d}t.
\end{aligned} \tag{10}$$

According to (6) and (8)–(10), we have

$$\begin{aligned}
\nu(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t}&\left[\int_0^\infty \psi_q(\theta)T\left(\frac{t^q}{\theta^q}\right)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}\theta\right.\\
&+ h(t,x_t) + q\int_0^t\int_0^\infty \psi_q(\theta)AT\left(\frac{(t-s)^q}{\theta^q}\right)h(s,x_s)\frac{(t-s)^q}{\theta^q}\mathrm{d}\theta\mathrm{d}s\\
&\left.+ q\int_0^t\int_0^\infty \psi_q(\theta)T\left(\frac{(t-s)^q}{\theta^q}\right)f(s,x_s)\frac{(t-s)^q}{\theta^q}\mathrm{d}\theta\mathrm{d}s\right]\mathrm{d}t.
\end{aligned}$$

Now we can invert the last Laplace transform to get

$$x(t) = \int_0^\infty \psi_q(\theta)T\left(\frac{t^q}{\theta^q}\right)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]\mathrm{d}\theta + h(t,x_t)$$

$$
\begin{aligned}
&+q\int_0^t\int_0^\infty \psi_q(\theta)AT\left(\frac{(t-s)^q}{\theta^q}\right)h(s,x_s)\frac{(t-s)^q}{\theta^q}\mathrm{d}\theta\mathrm{d}s\\
&+q\int_0^t\int_0^\infty \psi_q(\theta)T\left(\frac{(t-s)^q}{\theta^q}\right)f(s,x_s)\frac{(t-s)^q}{\theta^q}\mathrm{d}\theta\mathrm{d}s\\
=&\int_0^\infty \phi_q(\theta)T(t^q\theta)[\varphi(0)-(g(x_{t_1},\ldots,x_{t_n}))(0)-h(0,x_0)]\mathrm{d}\theta+h(t,x_t)\\
&+q\int_0^t\int_0^\infty \theta(t-s)^{q-1}\phi_q(\theta)AT((t-s)^q\theta)h(s,x_s)\mathrm{d}\theta\mathrm{d}s\\
&+q\int_0^t\int_0^\infty \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s,
\end{aligned}
$$

where $\phi_q(\theta)=\frac{1}{q}\theta^{-1-1/q}\psi_q(\theta^{-1/q})$ is the probability density function defined on $(0,\infty)$. This completes the proof. $\square$

For any $x\in E$, Define operators $\{S_q(t)\}_{t\geq 0}$ and $\{T_q(t)\}_{t\geq 0}$ by

$$
S_q(t)x=\int_0^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta \quad\text{and}\quad T_q(t)x=q\int_0^\infty \theta\phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta.
$$

Due to Lemma 3.1, we give the following definition of the mild solution of (1).

Definition 3.1. By the mild solution of the nonlocal Cauchy problem (1), we mean that the function $x\in C([-r,a],E)$ which satisfies

$$
\begin{cases}
x(t)=S_q(t)[\varphi(0)-(g(x_{t_1},\ldots,x_{t_n}))(0)-h(0,x_0)]+h(t,x_t)+\displaystyle\int_0^t (t-s)^{q-1}AT_q(t-s)h(s,x_s)\mathrm{d}s\\
\qquad+\displaystyle\int_0^t (t-s)^{q-1}T_q(t-s)f(s,x_s)\mathrm{d}s,\quad t\in[0,a],\\
x_0(\vartheta)+(g(x_{t_1},\ldots,x_{t_n}))(\vartheta)=\varphi(\vartheta),\quad \vartheta\in[-r,0].
\end{cases}
$$

Before stating and proving the main results, we introduce the following hypotheses.

(H_1) $T(t)$ is a compact operator for every $t>0$,

(H_2) for almost all $t\in[0,a]$, the function $f(t,\cdot):\mathcal{C}\to E$ is continuous and for each $z\in\mathcal{C}$, the function $f(\cdot,z):[0,a]\to E$ is strongly measurable,

(H_3) there exists a constant $q_1\in[0,q)$ and $m\in L^{\frac{1}{q_1}}([0,a],R^+)$ such that $|f(t,z)|\leq m(t)$ for all $z\in\mathcal{C}$ and almost all $t\in[0,a]$,

(H_4) there exists a constant $L>0$ such that $\|g(x_{t_1},\ldots,x_{t_n})-g(y_{t_1},\ldots,y_{t_n})\|_*\leq L\|x-y\|$, for $x,y\in C([-r,a],E)$,

(H_5) $h:[0,a]\times\mathcal{C}\to E$ is continuous function and there exists a constant $\beta\in(0,1)$ and $H,H_1>0$ such that $h\in D(A^\beta)$ and for any $z,y\in\mathcal{C}$, $t\in[0,a]$, the function $A^\beta h(\cdot,z)$ is strongly measurable and $A^\beta h(t,\cdot)$ satisfies the Lipschitz condition

$$
|A^\beta h(t,z)-A^\beta h(t,y)|\leq H\|z-y\|_* \tag{11}
$$

and the inequality

$$
|A^\beta h(t,z)|\leq H_1(\|z\|_*+1). \tag{12}
$$

Remark 3.1. The condition (H_3) can be replaced by the condition

(H_3)$'$ there exists a constant $q_1\in[0,q)$ and $m_k\in L^{\frac{1}{q_1}}([0,a],R^+)$ such that $|f(t,z)|\leq m_k(t)$ for all $z\in\mathcal{C}$, $\|z\|_*\leq k$ and almost all $t\in[0,a]$, where k is any positive constant.

Since (H_3)$'$ would not be an essential generalization, we only consider (H_3) throughout the following text.

We prove the following lemmas relative to operators $\{S_q(t)\}_{t\geq 0}$ and $\{T_q(t)\}_{t\geq 0}$ before we proceed further.

Lemma 3.2. *For any fixed $t\geq 0$, $S_q(t)$ and $T_q(t)$ are linear and bounded operators.*

Proof. For any fixed $t\geq 0$, since $T(t)$ is linear operator, it is easy to see that $S_q(t)$ and $T_q(t)$ are also linear operators. For $\xi\in[0,1]$, according to [21], direct calculation gives that

$$
\int_0^\infty \frac{1}{\theta^\xi}\psi_q(\theta)\mathrm{d}\theta=\frac{\Gamma(1+\frac{\xi}{q})}{\Gamma(1+\xi)}.
$$

Then we have

$$
\int_0^\infty \theta^\xi\phi_q(\theta)\mathrm{d}\theta=\int_0^\infty \frac{1}{\theta^{q\xi}}\psi_q(\theta)\mathrm{d}\theta=\frac{\Gamma(1+\xi)}{\Gamma(1+q\xi)}. \tag{13}
$$

In the case $\xi = 1$, we have

$$\int_0^\infty \theta\phi_q(\theta)\mathrm{d}\theta = \int_0^\infty \frac{1}{\theta^q}\psi_q(\theta)\mathrm{d}\theta = \frac{1}{\Gamma(1+q)}.$$

For any $x \in E$, according to (2) and (13), we have

$$|S_q(t)x| = \left|\int_0^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta\right| \le M|x|$$

and

$$|T_q(t)x| = \left|q\int_0^\infty \theta\phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta\right| \le \frac{qM}{\Gamma(1+q)}|x|.$$

This completes the proof. □

Lemma 3.3. *Operators $\{S_q(t)\}_{t\ge 0}$ and $\{T_q(t)\}_{t\ge 0}$ are strongly continuous, which means that for $\forall x \in E$ and $0 \le t' < t'' \le a$, we have*

$$|S_q(t'')x - S_q(t')x| \to 0 \quad \textit{and} \quad |T_q(t'')x - T_q(t')x| \to 0 \quad \textit{as } t' \to t''.$$

Proof. For any $x \in E$ and $0 \le t' < t'' \le a$, we get that

$$\begin{aligned}|T_q(t'')x - T_q(t')x| &= \left|q\int_0^\infty \theta\phi_q(\theta)[T((t'')^q\theta) - T((t')^q\theta)]x\mathrm{d}\theta\right| \\ &\le qM\int_0^\infty \theta\phi_q(\theta)|[T((t'')^q\theta - (t')^q\theta) - I]x|\mathrm{d}\theta.\end{aligned}$$

According to the strongly continuity of $\{T(t)\}_{t\ge 0}$ and (13), we know that $|T_q(t'')x - T_q(t')x|$ tends to zero as $t'' - t' \to 0$, which means that $\{T_q(t)\}_{t\ge 0}$ is strongly continuous. Using a similar method, we can also obtain that $\{S_q(t)\}_{t\ge 0}$ is also strongly continuous, and this completes the proof. □

Lemma 3.4. *If the assumption* (H_1) *is satisfied, then $S_q(t)$ and $T_q(t)$ are also compact operators for every $t > 0$.*

Proof. For each positive constant k, set $Y_k = \{x \in E : |x| \le k\}$. Then Y_k is clearly a bounded subset in E.

We only need prove that for any positive constant k and $t > 0$, the sets

$$V_1(t) = \left\{\int_0^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta,\ x \in Y_k\right\} \quad \text{and} \quad V_2(t) = \left\{q\int_0^\infty \theta\phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta,\ x \in Y_k\right\}$$

are relatively compact in E.

Let $t > 0$ be fixed. For $\forall\ \delta > 0$, define the subset in E by

$$V_\delta(t) = \left\{\int_\delta^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta,\ x \in Y_k\right\}.$$

Then for any $x \in Y_k$, we have

$$\int_\delta^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta = T(t^q\delta)\int_\delta^\infty \phi_q(\theta)T(t^q\theta - t^q\delta)x\mathrm{d}\theta.$$

From the compactness of $T(t^q\delta)(t^q\delta > 0)$, we obtain that the set $V_\delta(t)$ is relatively compact in E for $\forall\ \delta > 0$. Moreover, for every $x \in Y_k$, we have

$$\left|\int_0^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta - \int_\delta^\infty \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta\right| = \left|\int_0^\delta \phi_q(\theta)T(t^q\theta)x\mathrm{d}\theta\right| \le Mk\int_0^\delta \phi_q(\theta)\mathrm{d}\theta.$$

Therefore, there are relatively compact sets arbitrarily close to the set $V_1(t),\ t > 0$. Hence the set $V_1(t),\ t > 0$ is also relatively compact in E. Similarly, we can conclude that the set $V_2(t),\ t > 0$ is relatively compact in E. The proof is complete. □

Lemma 3.5. *For any $x \in E,\ \beta \in (0,1)$ and $\eta \in (0,1]$, we have*

$$AT_q(t)x = A^{1-\beta}T_q(t)A^\beta x, \quad 0 \le t \le a,$$

and

$$|A^\eta T_q(t)| \le \frac{qC_\eta}{t^{q\eta}}\frac{\Gamma(2-\eta)}{\Gamma(1+q(1-\eta))}, \quad 0 < t \le a.$$

Proof. For any $x \in E$, $\beta \in (0,1)$ and $\eta \in (0,1]$, we have

$$\begin{aligned} AT_q(t)x &= q\int_0^\infty \theta\phi_q(\theta)AT(t^q\theta)x\mathrm{d}\theta \\ &= q\int_0^\infty \theta\phi_q(\theta)A^{1-\beta}T(t^q\theta)A^\beta x\mathrm{d}\theta \\ &= A^{1-\beta}T_q(t)A^\beta x. \end{aligned}$$

By (3) and (13), we get

$$\begin{aligned} |A^\eta T_q(t)x| &= \left|q\int_0^\infty \theta\phi_q(\theta)A^\eta T(t^q\theta)x\mathrm{d}\theta\right| \le \int_0^\infty \theta\phi_q(\theta)\frac{C_\eta}{(t^q\theta)^\eta}|x|\mathrm{d}\theta \\ &= \frac{qC_\eta|x|}{t^{q\eta}}\int_0^\infty \theta^{1-\eta}\phi_q(\theta)\mathrm{d}\theta \\ &= \frac{qC_\eta}{t^{q\eta}}\frac{\Gamma(2-\eta)}{\Gamma(1+q(1-\eta))}|x|, \quad 0<t\le a. \end{aligned}$$

The proof is complete. □

For each positive constant k, let $B_k = \{x \in C([-r,a],E) : \|x\| \le k\}$. Then B_k is clearly a bounded closed and convex subset in $C([-r,a],E)$.

The following existence results for the nonlocal Cauchy problem (1) is based on Krasnoselskii's fixed point theorem.

Theorem 3.1. *If* (H_1)–(H_5) *are satisfied, then the nonlocal Cauchy problem* (1) *has a mild solution provided that*

(i) $ML + (M+1)|A^{-\beta}|H_1 + \frac{\Gamma(1+\beta)C_{1-\beta}H_1}{\beta\Gamma(1+q\beta)}a^{q\beta} < 1$,

(ii) $ML + (M+1)|A^{-\beta}|H + \frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}a^{q\beta} < 1$.

Proof. Define the function $\upsilon \in C([-r,a],E)$ such that $|\upsilon(t)| \equiv 0$, $t \in [-r,a]$. For any positive constant k and $x \in B_k$, in view of (12), Lemma 3.2 and the condition (H_4), for $t \in [0,a]$, it follows that

$$\begin{aligned} |S_q(t)[\varphi(0) - (g(x_{t_1},\ldots,x_{t_n}))(0) - h(0,x_0)]| &\le M(|\varphi(0)| + L\|x-\upsilon\| + |g(\upsilon_{t_1},\ldots,\upsilon_{t_n})(0)| + |A^{-\beta}A^\beta h(0,x_0)|) \\ &\le M[\|\varphi\|_* + Lk + \|g(\upsilon_{t_1},\ldots,\upsilon_{t_n})\|_* + |A^{-\beta}|H_1(k+1)]. \end{aligned} \tag{14}$$

For any positive constant k and $x \in B_k$, since x_t is continuous in t, according to (H_2), $f(t,x_t)$ is a measurable function on $[0,a]$. Direct calculation gives that $(t-s)^{q-1} \in L^{\frac{1}{1-q_1}}[0,t]$ for $t \in [0,a]$ and $q_1 \in [0,q)$. Let

$$b = \frac{q-1}{1-q_1} \in (-1,0), \qquad M_1 = \|m\|_{L^{\frac{1}{q_1}}[0,a]}.$$

By using Lemma 2.1 (Hölder inequality) and (H_3), for $t \in [0,a]$, we obtain

$$\begin{aligned} \int_0^t |(t-s)^{q-1}f(s,x_s)|\mathrm{d}s &\le \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}\mathrm{d}s\right)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}[0,t]} \\ &\le \frac{M_1}{(1+b)^{1-q_1}}a^{(1+b)(1-q_1)}. \end{aligned} \tag{15}$$

In view of Lemma 3.2 and (15), we get

$$\begin{aligned} \int_0^t |(t-s)^{q-1}T_q(t-s)f(s,x_s)|\mathrm{d}s &\le \frac{qM}{\Gamma(1+q)}\int_0^t |(t-s)^{q-1}f(s,x_s)|\mathrm{d}s, \\ &\le \frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}a^{(1+b)(1-q_1)}, \quad \text{for } t \in [0,a]. \end{aligned} \tag{16}$$

Thus, $|(t-s)^{q-1}T_q(t-s)f(s,x_s)|$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in [0,a]$. From Lemma 2.2 (Bochner's theorem), it follows that $(t-s)^{q-1}T_q(t-s)f(s,x_s)$ is Bochner integrable with respect to $s \in [0,t]$ for all $t \in [0,a]$.

Since $h \in D(A^\beta)$, the function $A^\beta h(\cdot,z)$ is strongly measurable for any $z \in \mathcal{C}$, $t \in [0,a]$, and $A^\beta h(t,\cdot)$ satisfies the Lipschitz condition, then $A^\beta h(s,x_s)$ is strongly measurable on $[0,a]$. In addition, in view of the fact that $\{T(t)\}_{t\ge0}$ is an analytic semigroup, then for $t \in (0,a]$ and $\theta \in (0,\infty)$, the operator function $s \to (t-s)^{q-1}AT((t-s)^q\theta)$ is continuous in the uniform operator topology in $[0,t)$ and thus $(t-s)^{q-1}AT_q(t-s)h(s,x_s)$ is continuous in $[0,t)$. Applying (12) and

Lemma 3.5, for any $x \in B_k,\ t \in [0, a]$, the following relation holds

$$\begin{aligned}\int_0^t |(t-s)^{q-1}AT_q(t-s)h(s,x_s)|\mathrm{d}s &= \int_0^t |(t-s)^{q-1}A^{1-\beta}T_q(t-s)A^{\beta}h(s,x_s)|\mathrm{d}s \\ &\le \int_0^t (t-s)^{q-1}\frac{q\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+q\beta)(t-s)^{q(1-\beta)}}H_1(k+1)\mathrm{d}s \\ &= \frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}C_{1-\beta}H_1(k+1)\int_0^t (t-s)^{q\beta-1}\mathrm{d}s \\ &= \frac{\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}C_{1-\beta}H_1(k+1)a^{q\beta}. \end{aligned} \tag{17}$$

Thus, $|(t-s)^{q-1}AT_q(t-s)h(s,x_s)|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in [0, a]$. From Lemma 2.2 (Bochner's theorem), it follows that $(t-s)^{q-1}AT_q(t-s)h(s,x_s)$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in [0, a]$.

For each positive k, define two operators F_1 and F_2 on B_k as follows

$$\begin{cases} (F_1x)(t) = S_q(t)[\varphi(0) - (g(x_{t_1},\dots,x_{t_n}))(0) - h(0,x_0)] + h(t,x_t) + \displaystyle\int_0^t (t-s)^{q-1}AT_q(t-s)h(s,x_s)\mathrm{d}s \quad t \in [0,a], \\ (F_1x)(\vartheta) = \varphi(\vartheta) - (g(x_{t_1},\dots,x_{t_n}))(\vartheta), \quad \vartheta \in [-r,0], \end{cases}$$

and

$$\begin{cases} (F_2x)(t) = \displaystyle\int_0^t (t-s)^{q-1}T_q(t-s)f(s,x_s)\mathrm{d}s, & t \in [0,a], \\ (F_2x)(\vartheta) = 0, & \vartheta \in [-r,0], \end{cases}$$

where $x \in B_k$.

Obviously, x is a mild solution of (1) if and only if the operator equation $x = F_1x + F_2x$ has a solution $x \in B_k$. Therefore, the existence of a mild solution of (1) is equivalent to determining a positive constant k_0, such that $F_1 + F_2$ has a fixed point on B_{k_0}.

In fact, in view of (i) of Theorem 3.1, by choosing k_0 such that

$$\begin{aligned} k_0 &= M[\|\varphi\|_* + Lk_0 + \|g(\upsilon_{t_1},\dots,\upsilon_{t_n})\|_* + |A^{-\beta}|H_1(k_0+1)] + |A^{-\beta}|H_1(k_0+1) \\ &\quad + \frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}a^{(1+b)(1-q_1)} + \frac{\Gamma(1+\beta)C_{1-\beta}H_1(k_0+1)}{\beta\Gamma(1+q\beta)}a^{q\beta}, \end{aligned} \tag{18}$$

we can prove that $F_1 + F_2$ has a fixed point on B_{k_0}. Our proof will be divided into three steps.

Step I. $F_1x + F_2y \in B_{k_0}$ whenever $x, y \in B_{k_0}$.

For any fixed $x \in B_{k_0}$ and $0 \le t' < t'' \le a$, we get that

$$\begin{aligned} |(F_2x)(t'') - (F_2x)(t')| &= \left|\int_0^{t''}(t''-s)^{q-1}T_q(t''-s)f(s,x_s)\mathrm{d}s - \int_0^{t'}(t'-s)^{q-1}T_q(t'-s)f(s,x_s)\mathrm{d}s\right| \\ &\le \left|\int_{t'}^{t''}(t''-s)^{q-1}T_q(t''-s)f(s,x_s)\mathrm{d}s\right| \\ &\quad + \left|\int_0^{t'}(t''-s)^{q-1}T_q(t''-s)f(s,x_s)\mathrm{d}s - \int_0^{t'}(t'-s)^{q-1}T_q(t''-s)f(s,x_s)\mathrm{d}s\right| \\ &\quad + \left|\int_0^{t'}(t'-s)^{q-1}T_q(t''-s)f(s,x_s)\mathrm{d}s - \int_0^{t'}(t'-s)^{q-1}T_q(t'-s)f(s,x_s)\mathrm{d}s\right| \\ &= \left|\int_{t'}^{t''}(t''-s)^{q-1}T_q(t''-s)f(s,x_s)\mathrm{d}s\right| \\ &\quad + \left|\int_0^{t'}[(t''-s)^{q-1}-(t'-s)^{q-1}]T_q(t''-s)f(s,x_s)\mathrm{d}s\right| \\ &\quad + \left|\int_0^{t'}(t'-s)^{q-1}[T_q(t''-s)-T_q(t'-s)]f(s,x_s)\mathrm{d}s\right| \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = \left| \int_{t'}^{t''} (t''-s)^{q-1} T_q(t''-s) f(s, x_s) \mathrm{d}s \right|,$$

$$I_2 = \left| \int_0^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}] T_q(t''-s) f(s, x_s) \mathrm{d}s \right|,$$

$$I_3 = \left| \int_0^{t'} (t'-s)^{q-1} [T_q(t''-s) - T_q(t'-s)] f(s, x_s) \mathrm{d}s \right|.$$

Therefore, for every $x \in B_{k_0}$, $(F_1x)(t)$ is continuous in $t \in [-r, a]$. Using the similar argument and (12), we can conclude that for every $y \in B_{k_0}$, $(F_2y)(t)$ is also continuous in $t \in [-r, a]$.

For every pair $x, y \in B_{k_0}$ and $t \in [0, a]$, by using similar methods as we did in (14)–(17) and noting that (18), we have

$$\begin{aligned} |(F_1x)(t) + (F_2y)(t)| &\le |S_q(t)[\varphi(0) - (g(x_{t_1}, \ldots, x_{t_n}))(0) - h(0, x_0)]| + |h(t, x_t)| \\ &\quad + \left| \int_0^t (t-s)^{q-1} A T_q(t-s) h(s, x_s) \mathrm{d}s \right| + \left| \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_s) \mathrm{d}s \right| \\ &\le M[\|\varphi\|_* + Lk_0 + \|g(\upsilon_{t_1}, \ldots, \upsilon_{t_n})\|_* + |A^{-\beta}| H_1(k_0+1)] + |A^{-\beta}| H_1(k_0+1) \\ &\quad + \frac{qM_1 M}{\Gamma(1+q)(1+b)^{1-q_1}}\, a^{(1+b)(1-q_1)} + \frac{\Gamma(1+\beta) C_{1-\beta} H_1(k_0+1)}{\beta \Gamma(1+q\beta)}\, a^{q\beta} \\ &= k_0. \end{aligned} \tag{19}$$

Noting that $M \ge 1$, we have

$$|(F_1x)(\vartheta) + (F_2y)(\vartheta)| \le M[\|\varphi\|_* + Lk_0 + \|g(\upsilon_{t_1}, \ldots, \upsilon_{t_n})\|_*] \le k_0 \quad \vartheta \in [-r, 0].$$

Hence, $\|F_1x + F_2y\| \le k_0$ for every pair $x, y \in B_{k_0}$.

Step II. F_1 is a contraction on B_{k_0}.

For any $x, y \in B_{k_0}$ and $t \in [0, a]$, according to (11), (H_4), Lemmas 3.2 and 3.5, we have

$$\begin{aligned} |(F_1x)(t) - (F_1y)(t)| &\le |S_q(t)[(g(x_{t_1}, \ldots, x_{t_n}))(0) - (g(y_{t_1}, \ldots, y_{t_n}))(0)]| \\ &\quad + |S_q(t)[h(0, x_0) - h(0, y_0)]| + |h(t, x_t) - h(t, y_t)| \\ &\quad + \int_0^t (t-s)^{q-1} |A^{1-\beta} T_q(t-s)[A^\beta h(s, x_s) - A^\beta h(s, y_s)]| \mathrm{d}s \\ &\le ML\|x-y\| + (M+1)|A^{-\beta}| H \|x-y\| + \frac{\Gamma(1+\beta) C_{1-\beta} H}{\beta \Gamma(1+q\beta)}\, a^{q\beta} \|x-y\|, \\ &= \left(ML + (M+1)|A^{-\beta}| H + \frac{\Gamma(1+\beta) C_{1-\beta} H}{\beta \Gamma(1+q\beta)}\, a^{q\beta} \right) \|x-y\|. \end{aligned}$$

Noting that $M \ge 1$, we have

$$|(F_1x)(\vartheta) - (F_1y)(\vartheta)| \le ML\|x-y\|, \quad \vartheta \in [-r, 0],$$

which implies that $\|F_1x - F_1y\| \le \left(ML + (M+1)|A^{-\beta}| H + \frac{\Gamma(1+\beta) C_{1-\beta} H}{\beta \Gamma(1+q\beta)}\, a^{q\beta} \right) \|x-y\|$. According to (ii) of Theorem 3.1, we get that F_1 is a contraction.

Step III. F_2 is a completely continuous operator.

First, we will prove that F_2 is continuous on B_{k_0}. Let $\{x^n\} \subseteq B_{k_0}$ with $x^n \to x$ on B_{k_0}. Then by (H_2) and the fact that $x_t^n \to x_t$ for $t \in [0, a]$, we have

$$ff(s, x_s^n) \to f(s, x_s), \quad \text{a.e. } t \in [0, a] \quad \text{as } n \to \infty.$$

Noting that $|f(s, x_s^n) - f(s, x_s)| \le 2m(s)$, by the dominated convergence theorem, we have

$$\begin{aligned} |(F_2x^n)(t) - (F_2x)(t)| &= \left| \int_0^t (t-s)^{q-1} T_q(t-s)[f(s, x_s^n) - f(s, x_s)] \mathrm{d}s \right| \\ &\le \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} |f(s, x_s^n) - f(s, x_s)| \mathrm{d}s \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

which implies that F_2 is continuous.

Next, we will show that $\{F_2x,\ x \in B_{k_0}\}$ is relatively compact. It suffices to show that the family of functions $\{F_2x,\ x \in B_{k_0}\}$ is uniformly bounded and equicontinuous, and for any $t \in [0, a]$, $\{(F_2x)(t),\ x \in B_{k_0}\}$ is relatively compact in E.

For any $x \in B_{k_0}$, we have $\|F_2x\| \le k_0$, which means that $\{F_2x,\ x \in B_{k_0}\}$ is uniformly bounded. In the following, we will show that $\{F_2x,\ x \in B_{k_0}\}$ is a family of equicontinuous functions.

For any $x \in B_{k_0}$ and $0 \le t' < t'' \le a$, we get that $|(F_2x)(t'') - (F_2x)(t')| \le I_1 + I_2 + I_3$, where $I_1,\ I_2$ and I_3 are defined as in *Step* I. By using the analogous argument performed in (15) and (16), we can conclude that

$$
\begin{aligned}
I_1 &\le \frac{qM_1M(t''-t')^{(1+b)(1-q_1)}}{\Gamma(1+q)(1+b)^{1-q_1}},\\
I_2 &\le \frac{qM}{\Gamma(1+q)}\left(\int_0^{t'}\left((t'-s)^{q-1}-(t''-s)^{q-1}\right)^{\frac{1}{1-q_1}}\mathrm{d}s\right)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}[0,t']}\\
&\le \frac{qM_1M}{\Gamma(1+q)}\left(\int_0^{t'}\left((t'-s)^{b}-(t''-s)^{b}\right)\mathrm{d}s\right)^{1-q_1}\\
&= \frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}\left((t')^{1+b}-(t'')^{1+b}+(t''-t')^{1+b}\right)^{1-q_1}\\
&\le \frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}(t''-t')^{(1+b)(1-q_1)}.
\end{aligned}
$$

For $t' = 0,\ 0 < t'' \le a$, it is easy to see that $I_3 = 0$. For $t' > 0$ and $\varepsilon > 0$ small enough, we have

$$
\begin{aligned}
I_3 &\le \int_0^{t'-\varepsilon}(t'-s)^{q-1}|T_q(t''-s)-T_q(t'-s)|\|f(s,x_s)|\mathrm{d}s + \int_{t'-\varepsilon}^{t'}(t'-s)^{q-1}|T_q(t''-s)-T_q(t'-s)|\|f(s,x_s)|\mathrm{d}s\\
&\le \frac{M_1((t')^{1+b}-\varepsilon^{1+b})^{(1-q_1)}}{(1+b)^{1-q_1}}\sup_{s\in[0,t'-\varepsilon]}|T_q(t''-s)-T_q(t'-s)| + \frac{2qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}\,\varepsilon^{(1+b)(1-q_1)}.
\end{aligned}
$$

Since (H_1) and Lemma 3.4 imply that the continuity of $T_q(t)(t > 0)$ in t in the uniform operator topology, it is easy to see that I_3 tends to zero independently of $x \in B_{k_0}$ as $t''-t' \to 0,\ \varepsilon \to 0$. Thus, $|(F_2x)(t') - (F_2x)(t'')|$ tends to zero independently of $x \in B_{k_0}$ as $t''-t' \to 0$, which means that $\{F_2x,\ x \in B_{k_0}\}$ is equicontinuous.

It remains to prove that for any $t \in [-r, a]$, $V(t) = \{(F_2x)(t),\ x \in B_{k_0}\}$ is relatively compact in E.

Obviously, for $t \in [-r, 0]$, $V(t)$ is relatively compact in E. Let $0 < t \le a$ be fixed. For $\forall\,\varepsilon \in (0, t)$ and $\forall\,\delta > 0$, define an operator $F_{\varepsilon,\delta}$ on B_{k_0} by the formula

$$
\begin{aligned}
(F_{\varepsilon,\delta}x)(t) &= q\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s\\
&= q\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}\phi_q(\theta)[T(\varepsilon^q\delta)T((t-s)^q\theta-\varepsilon^q\delta)]f(s,x_s)\mathrm{d}\theta\mathrm{d}s,\\
&= T(\varepsilon^q\delta)q\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta-\varepsilon^q\delta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s,
\end{aligned}
$$

where $x \in B_{k_0}$. Then from the compactness of $T(\varepsilon^q\delta)(\varepsilon^q\delta > 0)$, we obtain that the set $V_{\varepsilon,\delta}(t) = \{(F_{\varepsilon,\delta}x)(t),\ x \in B_{k_0}\}$ is relatively compact in E for $\forall\,\varepsilon \in (0, t)$ and $\forall\,\delta > 0$. Moreover, for every $x \in B_{k_0}$, we have

$$
\begin{aligned}
|(F_2x)(t)-(F_{\varepsilon,\delta}x)(t)| &= q\left|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s\right.\\
&\quad + \int_0^t\int_\delta^\infty \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s\\
&\quad \left.- \int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s\right|\\
&\le q\left|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s\right|\\
&\quad + q\left|\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\phi_q(\theta)T((t-s)^q\theta)f(s,x_s)\mathrm{d}\theta\mathrm{d}s\right|\\
&\le qM\left(\int_0^t(t-s)^{\frac{q-1}{1-q_1}}\mathrm{d}s\right)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}[0,t]}\int_0^\delta \theta\phi_q(\theta)\mathrm{d}\theta
\end{aligned}
$$

$$+qM\left(\int_{t-\varepsilon}^{t}(t-s)^{\frac{q-1}{1-q_1}}\mathrm{d}s\right)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}[t-\varepsilon,t]}\int_0^\infty\theta\phi_q(\theta)\mathrm{d}\theta$$
$$\le\frac{qM_1Ma^{(1+b)(1-q_1)}}{(1+b)^{1-q_1}}\int_0^\delta\theta\phi_q(\theta)\mathrm{d}\theta+\frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}\varepsilon^{(1+b)(1-q_1)}.$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t),\ t>0$. Hence the set $V(t),\ t>0$ is also relatively compact in E.

Therefore, $\{F_2x,\ x\in B_{k_0}\}$ is relatively compact by Ascoli–Arzela Theorem. Thus, the continuity of F_2 and relative compactness of $\{F_2x,\ x\in B_{k_0}\}$ imply that F_2 is a completely continuous operator. Hence, Krasnoselskii's fixed point theorem shows that F_1+F_2 has a fixed point on B_{k_0}. Therefore, the nonlocal Cauchy problem (1) has a mild solution. The proof is complete. □

In the following, we give an existence result in the case where (H_4) is not satisfied. We need the following assumption.

$(\mathrm{H}_4)'$ g is completely continuous, and there exist positive constants $L_1, L_1{}'$ such that $\|g(x_{t_1},\ldots,x_{t_n})\|_*\le L_1\|x\|+L_1{}'$ for all $x\in C([-r,a],E)$.

Theorem 3.2. *If assumptions* (H_1)–(H_3), $(\mathrm{H}_4)'$ *and* (H_5) *are satisfied, then the nonlocal Cauchy problem* (1) *has a mild solution provided that*

(i) $ML_1+(M+1)|A^{-\beta}|H_1+\frac{\Gamma(1+\beta)C_{1-\beta}H_1}{\beta\Gamma(1+q\beta)}a^{q\beta}<1$,

(ii) $(M+1)|A^{-\beta}|H+\frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}a^{q\beta}<1$.

Proof. For any positive constant k and $x\in B_k$, according to (12), $(\mathrm{H}_4)'$ and Lemma 3.2, it follows that

$$|S_q(t)[\varphi(0)-(g(x_{t_1},\ldots,x_{t_n}))(0)-h(0,x_0)]|\le M[\|\varphi\|_*+L_1k+L_1{}'+|A^{-\beta}|H_1(k+1)].\tag{20}$$

Therefore, the function $S_q(t)[\varphi(0)-(g(x_{t_1},\ldots,x_{t_n}))(0)-h(0,x_0)]$ exists. According to (15)–(17), for $x\in B_k$, the functions $(t-s)^{q-1}T_q(t-s)f(s,x_s)$ and $(t-s)^{q-1}AT_q(t-s)h(s,x_s)$ are Bochner integrable with respect to $s\in[0,t]$ for all $t\in[0,a]$.

For each positive k, define two operators G_1 and G_2 on B_k as follows

$$\begin{cases}(G_1x)(t)=h(t,x_t)-S_q(t)h(0,x_0)+\displaystyle\int_0^t(t-s)^{q-1}AT_q(t-s)h(s,x_s)\mathrm{d}s & t\in[0,a],\\ (G_1x)(\vartheta)=0, \quad \vartheta\in[-r,0],\end{cases}$$

and

$$\begin{cases}(G_2x)(t)=S_q(t)[\varphi(0)-(g(x_{t_1},\ldots,x_{t_n}))(0)]+\displaystyle\int_0^t(t-s)^{q-1}T_q(t-s)f(s,x_s)\mathrm{d}s, & t\in[0,a],\\ (G_2x)(\vartheta)=\varphi(\vartheta)-(g(x_{t_1},\ldots,x_{t_n}))(\vartheta), \quad \vartheta\in[-r,0]\end{cases}$$

where $x\in B_k$. In view of (i) of Theorem 3.2, we can choose k_1 such that

$$\begin{aligned}k_1&=M[\|\varphi\|_*+L_1k_1+L_1'+|A^{-\beta}|H_1(k_1+1)]+|A^{-\beta}|H_1(k_1+1)\\&\quad+\frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}a^{(1+b)(1-q_1)}+\frac{\Gamma(1+\beta)C_{1-\beta}H_1(k_1+1)}{\beta\Gamma(1+q\beta)}a^{q\beta}.\end{aligned}\tag{21}$$

In the following, we will prove that F has a fixed point on B_{k_1}. Our proof will be divided into three steps.

Step I. $G_1x+G_2y\in B_{k_1}$ whenever $x,y\in B_{k_1}$.

Obviously, for every pair $x,y\in B_{k_1}$, $(G_1x)(t)$ and $(G_2y)(t)$ are continuous in $t\in[-r,a]$. For every pair $x,y\in B_{k_1}$ and $t\in[0,a]$, by using (20) and (21) and similar methods as we did in (15)–(17), we have

$$\begin{aligned}|(G_1x)(t)+(G_2y)(t)|&\le|S_q(t)[\varphi(0)-(g(x_{t_1},\ldots,x_{t_n}))(0)-h(0,x_0)]|+|h(t,x_t)|\\&\quad+\left|\int_0^t(t-s)^{q-1}AT_q(t-s)h(s,x_s)\mathrm{d}s\right|+\left|\int_0^t(t-s)^{q-1}T_q(t-s)f(s,x_s)\mathrm{d}s\right|\\&\le M[\|\varphi\|_*+L_1k_1+L_1'+|A^{-\beta}|H_1(k_1+1)]+|A^{-\beta}|H_1(k_1+1)\\&\quad+\frac{qM_1M}{\Gamma(1+q)(1+b)^{1-q_1}}a^{(1+b)(1-q_1)}+\frac{\Gamma(1+\beta)C_{1-\beta}H_1(k_1+1)}{\beta\Gamma(1+q\beta)}a^{q\beta}\\&=k_1.\end{aligned}\tag{22}$$

Noting that $M\ge1$, we have

$$|(G_1x)(\vartheta)+(G_2y)(\vartheta)|\le M[\|\varphi\|_*+L_1k_1+L_1']\le k_1\quad \vartheta\in[-r,0].$$

Hence, $\|G_1x+G_2y\|\le k_1$ for every pair $x,y\in B_{k_1}$.

Step II. G_1 is a contraction on B_{k_1}.

For any $x, y \in B_{k_1}$ and $t \in [0, a]$, according to (11), we have

$$\begin{aligned}|(G_1x)(t) - (G_1y)(t)| &\leq |S_q(t)[h(0, x_0) - h(0, y_0)]| + |h(t, x_t) - h(t, y_t)| \\ &\quad + \int_0^t (t-s)^{q-1}|A^{1-\beta}T_q(t-s)[A^\beta h(s, x_s) - A^\beta h(t, y_s)]|\mathrm{d}s \\ &\leq (M+1)|A^{-\beta}|H\|x-y\| + \frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}\, a^{q\beta}\|x-y\|, \\ &\leq \left((M+1)|A^{-\beta}|H + \frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}\, a^{q\beta}\right)\|x-y\|,\end{aligned}$$

which implies

$$\|G_1x - G_1y\| \leq \left((M+1)|A^{-\beta}|H + \frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}\, a^{q\beta}\right)\|x-y\|.$$

According to (ii) of Theorem 3.2, we get that G_1 is a contraction.

Step III. G_2 is a completely continuous operator.

First, we will show that $\{G_2x,\ x \in B_{k_1}\}$ is equicontinuous. For any $x \in B_{k_1}$ and $0 \leq t' < t'' \leq a$, we get that

$$|(G_2x)(t'') - (G_2x)(t')| \leq |[S_q(t'') - S_q(t')][\varphi(0) - (g(x_{t_1}, \dots, x_{t_n}))(0)]| + I_1 + I_2 + I_3,$$

where I_1, I_2 and I_3 are defined as in the proof of Theorem 3.1. According to the strongly continuity of $\{S_q(t)\}_{t\geq 0}$ and $(H_4)'$, we know that $|(Gx)(t') - (Gx)(t'')|$ tends to zero independently of $x \in B_{k_1}$ as $t'' - t' \to 0$, which means that $\{Gx,\ x \in B_{k_1}\}$ is equicontinuous.

It remains to prove that for $t \in [-r, a]$, the set $\{(G_2x)(t),\ x \in B_{k_1}\}$ is relatively compact in E. Obviously, for $t \in [-r, 0]$, the set $\{(G_2x)(t),\ x \in B_{k_1}\}$ is relatively compact in E by $(H_4)'$. According to the argument of Theorem 3.1, we only need prove that for any $t \in (0, a]$, the set $V'(t) = \{S_q(t)[\varphi(0) - (g(x_{t_1}, \dots, x_{t_n}))(0)],\ x \in B_{k_1}\}$ is relatively compact in E. In fact, the set $V'(t),\ t > 0$ is also relatively compact in E according to (H_1) and Lemma 3.4. Moreover, $\{G_2x,\ x \in B_{k_1}\}$ is uniformly bounded by (22). Therefore, $\{G_2x,\ x \in B_{k_1}\}$ is relatively compact by Ascoli–Arzela Theorem.

Using a similar argument as that we did in the proof of Theorem 3.1, we know that G_2 is continuous on B_{k_1} by (H_2) and $(H_4)'$. Thus, G_2 is a completely continuous operator. Hence, Krasnoselskii's fixed point theorem shows that $G_1 + G_2$ has a fixed point on B_{k_1}, which means that the nonlocal Cauchy problem (1) has a mild solution. The proof is complete. □

The following existence and uniqueness result for the nonlocal Cauchy problem (1) is based on the Banach contraction principle. We will need the following assumptions.

(H_6) $f(t, x_t)$ is strongly measurable for any $x \in C([-r, a], B_k)$ and almost all $t \in [0, a]$,

(H_7) there exists a constant $q_2 \in [0, q)$ and $\rho \in L^{\frac{1}{q_2}}([0, a], R^+)$ such that for any $x, y \in C([-r, a], B_k)$, we have $|f(t, x_t) - f(t, y_t)| \leq \rho(t)\|x - y\|$, $t \in [0, a]$, where k is a positive constant.

Theorem 3.3. *Assume that* (H_3)–(H_7) *are satisfied. If* (i) *of Theorem 3.1 holds, then the nonlocal Cauchy problem (1) has a unique mild solution provided that*

$$ML + (M+1)|A^{-\beta}|H + \frac{\Gamma(1+\beta)C_{1-\beta}Ha^{q\beta}}{\beta\Gamma(1+q\beta)} + \frac{qM_2Ma^{(1+b')(1-q_2)}}{\Gamma(1+q)(1+b')^{1-q_2}} < 1, \tag{23}$$

where $b' = \frac{q-1}{1-q_2} \in (-1, 0),\ M_2 = \|\rho\|_{L^{\frac{1}{q_2}}[0,a]}$.

Proof. It is easy to see that $S_q(t)[\varphi(0) - (g(x_{t_1}, \dots, x_{t_n}))(0) - h(0, x_0)]$ exists, $(t-s)^{q-1}AT_q(t-s)h(s, x_s)$ and $(t-s)^{q-1}T_q(t-s)f(s, x_s)$ are Bochner integrable with respect to $s \in [0, t]$ for all $t \in [0, a]$. For $x \in B_k$, define the operator F on B_k by

$$\begin{cases}(Fx)(t) = S_q(t)[\varphi(0) - (g(x_{t_1}, \dots, x_{t_n}))(0) - h(0, x_0)] + h(t, x_t) \displaystyle\int_0^t (t-s)^{q-1}AT_q(t-s)h(s, x_s)\mathrm{d}s \\ \qquad + \displaystyle\int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_s)\mathrm{d}s, \quad t \in [0, a], \\ (Fx)(\vartheta) = -(g(x_{t_1}, \dots, x_{t_n}))(\vartheta) + \varphi(\vartheta), \quad \vartheta \in [-r, 0].\end{cases}$$

Obviously, it is sufficient to prove that F has a unique fixed point on B_{k_0}, where k_0 is defined as in (18).

According to (19), we know that F is an operator from B_{k_0} into itself. For any $x, y \in B_{k_0}$ and $t \in [0, a]$, according to (H_4), (H_5) and (H_7), we have

$$\begin{aligned}
|(Fx)(t)-(Fy)(t)| &\le |S_q(t)[(g(x_{t_1},\dots,x_{t_n}))(0)-(g(y_{t_1},\dots,y_{t_n}))(0)]| \\
&\quad + |S_q(t)[h(0,x_0)-h(0,y_0)]| + |h(t,x_t)-h(t,y_t)| \\
&\quad + \int_0^t (t-s)^{q-1}|A^{1-\beta}T_q(t-s)[A^\beta h(s,x_s)-A^\beta h(s,y_s)]|\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{q-1}|T_q(t-s)[f(s,x_s)-f(s,y_s)]|\mathrm{d}s \\
&\le ML\|x-y\| + (M+1)|A^{-\beta}|H\|x-y\| + \frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}\, a^{q\beta}\|x-y\| \\
&\quad + \frac{qM}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\rho(s)\|x-y\|\mathrm{d}s \\
&\le ML\|x-y\| + (M+1)|A^{-\beta}|H\|x-y\| + \frac{\Gamma(1+\beta)C_{1-\beta}H}{\beta\Gamma(1+q\beta)}\, a^{q\beta}\|x-y\| \\
&\quad + \frac{qM}{\Gamma(1+q)}\left(\int_0^t (t-s)^{\frac{q-1}{1-q_2}}\mathrm{d}s\right)^{1-q_2}\|\rho\|_{L^{\frac{1}{q_2}}[0,t]}\|x-y\| \\
&\le \left(ML + (M+1)|A^{-\beta}|H + \frac{\Gamma(1+\beta)C_{1-\beta}Ha^{q\beta}}{\beta\Gamma(1+q\beta)} + \frac{qM_2Ma^{(1+b')(1-q_2)}}{\Gamma(1+q)(1+b')^{1-q_2}}\right)\|x-y\|,
\end{aligned}$$

which means that F is a contraction according to (23). By applying the Banach contraction principle, we know that F has a unique fixed point on B_{k_0}. The proof is complete. □

Theorem 3.4. *Assume that assumptions* (H_1), (H_3), $(H_4)'$ *and* (H_5)–(H_7) *are satisfied. If* (i)*of* Theorem 3.2 *holds, then the nonlocal Cauchy problem* (1) *has a mild solution provided that*

$$(M+1)|A^{-\beta}|H + \frac{\Gamma(1+\beta)C_{1-\beta}Ha^{q\beta}}{\beta\Gamma(1+q\beta)} + \frac{qM_2Ma^{(1+b')(1-q_2)}}{\Gamma(1+q)(1+b')^{1-q_2}} < 1.$$

Proof. In fact, for each positive k, define two operators U_1 and U_2 on B_k as follows

$$\begin{cases}
(U_1x)(t) = -S_q(t)h(0,x_0) + h(t,x_t) + \displaystyle\int_0^t (t-s)^{q-1}AT_q(t-s)h(s,x_s)\mathrm{d}s \\
\qquad + \displaystyle\int_0^t (t-s)^{q-1}T_q(t-s)f(s,x_s)\mathrm{d}s, \quad t\in[0,a], \\
(U_1x)(\vartheta) = 0, \quad \vartheta\in[-r,0],
\end{cases}$$

and

$$\begin{cases}
(U_2x)(t) = S_q(t)[\varphi(0)-(g(x_{t_1},\dots,x_{t_n}))(0)], \quad t\in[0,a], \\
(U_2x)(\vartheta) = \varphi(\vartheta)-(g(x_{t_1},\dots,x_{t_n}))(\vartheta), \quad \vartheta\in[-r,0].
\end{cases}$$

According to the arguments above, we can easily get that $U_1x+U_2y\in B_{k_1}$ whenever $x,y\in B_{k_1}$, where k_1 is defined as in (21). Furthermore, we can obtain that U_1 is a contraction on B_{k_1} according to (H_3) and (H_5)–(H_7) and U_2 is a completely continuous operator according to (H_1) and $(H_4)'$. Hence, Krasnoselskii's fixed point theorem shows that U_1+U_2 has a fixed point on B_{k_1}, which means that the nonlocal Cauchy problem (1) has a mild solution and this completes the proof. □

4. An example

Let $E=L^2([0,\pi],R)$. Consider the following fractional partial differential equations.

$$\begin{cases}
\partial_t^q\left(u(t,z)-\displaystyle\int_0^\pi U(z,y)u_t(\vartheta,y)\mathrm{d}y\right) = \partial_z^2u(t,z)+\partial_zG(t,u_t(\vartheta,z)) \quad t\in(0,a], \\
\qquad u(t,0)=u(t,\pi)=0, \quad t\in[0,a], \\
u(\vartheta,z)+\displaystyle\sum_{i=0}^n\int_0^\pi k(z,y)u_{t_i}(\vartheta,y)\mathrm{d}y = (\varphi(\vartheta))(z), \quad \vartheta\in[-r,0],
\end{cases} \tag{24}$$

where ∂_t^q is a Caputo fractional partial derivative of order $0<q<1$, $a>0$, $z\in[0,\pi]$, G is a given function, n is a positive integer, $0<t_0<t_1<\cdots<t_n<a$, $\varphi\in C([-r,0],E)$, that is $\varphi(\vartheta)\in E=L^2([0,\pi],R)$, $k(z,y)\in L^2([0,\pi]\times[0,\pi],R)$ and $u_t(\vartheta,z)=u(t+\vartheta,z)$, $t\in[0,a]$, $\vartheta\in[-r,0]$.

We define an operator A by $Av=-v''$ with the domain

$$D(A)=\{v(\cdot)\in E: v, v' \text{ absolutly continuous},\ v''\in E,\ v(0)=v(\pi)=0\}.$$

Then $-A$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ which is compact, analytic and self-adjoint. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2, n \in N$, with corresponding normalized eigenvectors $u_n(z) = (2/\pi)^{1/2}\sin(nz)$. We also use the following properties.

(i′) For each $v \in E$, $T(t)v = \sum_{n=1}^{\infty} e^{-n^2 t}\langle v, u_n\rangle u_n$. In particular, $T(\cdot)$ is a uniformly stable semigroup and $\|T(t)\|_{L^2[0,\pi]} \leq e^{-t}$.

(ii′) for each $v \in E$, $A^{-\frac{1}{2}}v = \sum_{n=1}^{\infty} 1/n\langle v, u_n\rangle u_n$. In particular, $\|A^{-\frac{1}{2}}\|_{L^2[0,\pi]} = 1$,

(iii′) the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}v = \sum_{n=1}^{\infty} n\langle v, u_n\rangle u_n$$

on the space $D(A^{\frac{1}{2}}) = \{v(\cdot) \in E, \sum_{n=1}^{\infty} n\langle v, u_n\rangle u_n \in E\}$.

Clearly (2), (3) and (H_1) are satisfied.

The system (24) can be reformulated as the following nonlocal Cauchy problem in E

$$\begin{cases} {}^cD^q(x_t - h(t, x_t)) + Ax(t) = f(t, x_t) & t \in (0, a], \\ x_0(\vartheta) + (g(x_{t_1}, \dots, x_{t_n}))(\vartheta) = \varphi(\vartheta) & \vartheta \in [-r, 0], \end{cases}$$

where $x_t = u_t(\vartheta, \cdot)$, that is $(x(t+\vartheta))(z) = u(t+\vartheta, z)$, $t \in [0, a]$, $z \in [0, \pi]$, $\vartheta \in [-r, 0]$. The function $h : [0, a] \times \mathcal{C} \to E$ is given by

$$(h(t, x_t))(z) = \int_0^{\pi} U(z, y)u_t(\vartheta, y)\mathrm{d}y.$$

Let $(U_h v)(z) = \int_0^{\pi} U(z, y)v(y)\mathrm{d}y$, for $v \in E = L^2([0, \pi], R)$, $z \in [0, \pi]$.

The function $f : [0, a] \times \mathcal{C} \to E$ is given by

$$(f(t, x_t))(z) = \partial_z G(t, u_t(\vartheta, z)),$$

and the function $g : \mathcal{C}^n \to \mathcal{C}$ is given by

$$(g(x_{t_1}, \dots, x_{t_n}))(\vartheta) = \sum_{i=0}^{n} K_g x_{t_i}(\vartheta),$$

where $(K_g v)(z) = \int_0^{\pi} k(z, y)v(y)\mathrm{d}y$, for $v \in E = L^2([0, \pi], R)$, $z \in [0, \pi]$.

We can take $q = 1/2$ and $f(t, x_t) = \frac{1}{t^{1/3}}\sin x_t$, then (H_2), (H_3), (H_6) and (H_7) are satisfied. Furthermore, assume that $L = L_1 = (n+1)[\int_0^{\pi}\int_0^{\pi} k^2(z, y)\mathrm{d}y\mathrm{d}z]^{1/2}$. Then (H_4) and $(H_4)'$ are satisfied (noting that $K_g : E \to E$ is completely continuous). In fact, for $v_1, v_2 \in E$, we have

$$\begin{aligned} \|K_g v_1 - K_g v_2\|_{L^2[0,\pi]} &= \left(\int_0^{\pi}\left(\int_0^{\pi} k(z, y)(v_1(y) - v_2(y))\mathrm{d}y\right)^2 \mathrm{d}z\right)^{1/2} \\ &\leq \left(\int_0^{\pi}\left(\int_0^{\pi} k^2(z, y)\mathrm{d}y\int_0^{\pi}[v_1(y) - v_2(y)]^2\mathrm{d}y\right)\mathrm{d}z\right)^{1/2} \\ &= \left(\int_0^{\pi}\int_0^{\pi} k^2(z, y)\mathrm{d}y\mathrm{d}z\right)^{1/2} \|v_1 - v_2\|_{L^2[0,\pi].} \end{aligned}$$

Moreover, we assume that the following conditions hold.

(a) The function $U(z, y)$, $z, y \in [0, \pi]$ is measurable and

$$\int_0^{\pi}\int_0^{\pi} U^2(z, y)\mathrm{d}y\mathrm{d}z < \infty,$$

(b) the function $\partial_z U(z, y)$ is measurable, $U(0, y) = U(\pi, y) = 0$, and let

$$\overline{H} = \left(\int_0^{\pi}\int_0^{\pi} (\partial_z U(z, y))^2 \,\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{2}} < \infty.$$

From (a) it is clear that U_h is a bounded linear operator on E. Furthermore, $U_h(v) \in D(A^{\frac{1}{2}})$, and $\|A^{\frac{1}{2}}U_h\|_{L^2[0,\pi]} < \infty$. In fact, from the definition of U_h and (b) it follows that

$$\begin{aligned} \langle U_h(v), u_n\rangle &= \int_0^{\pi} u_n(z)\left(\int_0^{\pi} U(z, y)v(y)\mathrm{d}y\right)\mathrm{d}z \\ &= \frac{1}{n}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \langle \overline{U}(v), \cos(nz)\rangle, \end{aligned}$$

where $\overline{U}$ is defined by

$$(\overline{U}(v))(z) = \int_0^\pi \partial_z U(z, y)v(y)\mathrm{d}y.$$

From (b) we know that $\overline{U} : E \to E$ is a bounded linear operator with $\|\overline{U}\|_{L^2[0,\pi]} \le \overline{H}$. Hence we can write $\|A^{\frac{1}{2}} U_h(v)\|_{L^2[0,\pi]} = \|\overline{U}(v)\|_{L^2[0,\pi]}$, which implies that (12) holds. Obviously, (11) holds according to (b).

Hence, According to Theorem 3.1/ 3.2, system (24) has a mild solution provided that (i) and (ii) of Theorem 3.1/ 3.2 hold. From Theorem 3.3, system (24) admits a unique mild solution provided that (i) of Theorem 3.1 and (23) hold. From Theorem 3.4, system (24) has a mild solution provided that the inequalities in Theorem 3.4 hold.

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