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Profile minimization on products of graphs☆

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Abstract

The profile minimization problem arose from the study of sparse matrix technique. In terms of graphs, the problem is to determine the profile of a graph G which is defined as

$$P(G) = \min_f \sum_{v \in V(G)} \max_{x \in N[v]} (f(v) - f(x)),$$

where f runs over all bijections from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ and $N[v] = \{v\} \cup \{x \in V(G) : xv \in E(G)\}$. The main result of this paper is to determine the profiles of $K_m \times K_n$, $K_{s,t} \times K_n$ and $P_m \times K_n$.

Keywords: Profile; Product; Complete graph; Complete bipartite graph; Path

1. Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless and without multiple edges. For a graph G, we use $V(G)$ to denote the set of vertices of G and $E(G)$ the set of edges. The profile minimization problem arose from the study of sparse matrix technique. It can be defined in terms of graphs as follows.

A *proper numbering* of a graph G of n vertices is a 1–1 mapping $f : V(G) \to \{1, 2, \ldots, n\}$. Given a proper numbering f, the *profile width* of a vertex v in G is

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x)),$$

where $N[v] = \{v\} \cup \{x \in V(G) : xv \in E(G)\}$. The *profile* of a proper numbering f of G is

$$P_f(G) = \sum_{v \in V(G)} w_f(v),$$

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and the *profile* of G is

$$P(G) = \min\{P_f(G) : f \text{ is a proper numbering of } G\}.$$

A *profile numbering* of G is a proper numbering f such that $P_f(G) = P(G)$.

The profile minimization problem is equivalent to the interval graph completion problem described as below. Recall that an *interval graph* is a graph whose vertices correspond to closed intervals in the real line, and two vertices are adjacent if and only if their corresponding intervals intersect. It is well-known that a graph G is an interval graph if and only if there exists an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ such that

$$i < j < k \text{ and } v_i v_k \in E(G) \text{ imply } v_j v_k \in E(G).$$

We call this ordering an *interval ordering* of G. This property can be re-stated as: a graph G of n vertices is an interval graph if and only if there is a proper numbering f such that

$$f(x) < f(y) < f(z) \text{ and } xz \in E(G) \text{ imply } yz \in E(G). \tag{1}$$

We call this property the *interval property*, which will be used frequently in this paper. This property leads to the *perfect elimination property* which is also useful in this paper:

$$f(x) < f(y) \text{ with } xy \in E(G) \text{ and } f(x) < f(z) \text{ with } xz \in E(G) \text{ imply } yz \in E(G). \tag{2}$$

The perfect elimination property in turn implies the *chordality property* which is also useful in this paper:

Every cycle of length greater than three has at least one chord. (3)

Having the interval property (1) in mind, it is then easy to see that for any proper numbering f of G, the graph G_f defined by the following is an interval super-graph of G with $|E(G_f)| = P_f(G)$:

$$V(G_f) = V(G) \quad \text{and} \quad E(G_f) = \{yz : f(x) \leqslant f(y) < f(z), xz \in E(G)\}.$$

In other words, we have:

Proposition 1 (*Lin and Yuan [10]*). *The profile minimization problem is the same as the interval graph completion problem. Namely*,

$$P(G) = \min\{|E(H)| : H \text{ is an interval super-graph of } G\}.$$

The profile minimization problem has been extensively studied in the literature [2–16], for a good survey see [9]. From an algorithmic point of view, the problem is known to be NP-complete (see [1]). While many approximation algorithms for profiles of various graphs have been developed, [5,6] gave a polynomial-time algorithm for finding profiles of trees. Among the non-algorithmic results for profiles, we are most interested in those graphs which are obtained from graph operations. The classes of graphs in this line include Cartesian product of certain graphs [11,13], sum of two graphs [10], composition of certain graphs [7], and corona of certain graphs [7].

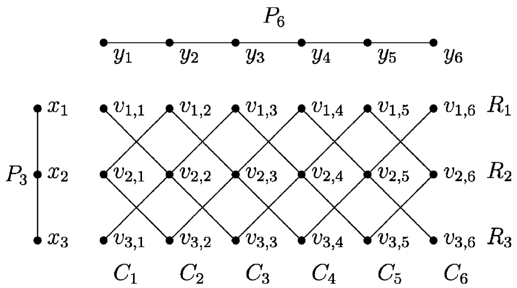

Fig. 1. The graph $P_3 \times P_6$.

The purpose of this paper is to study the profiles of product of graphs. The *product* (or *tensor product*) of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$ such that (x, y) is adjacent to (x', y') in $G \times H$ if $xx' \in E(G)$ and $yy' \in E(H)$. Notice that $G \times H$ has $|V(G)||V(H)|$ vertices and $2|E(G)||E(H)|$ edges.

For convenience, suppose $V(G) = \{x_i : 1 \leqslant i \leqslant |V(G)|\}$ and $V(H) = \{y_j : 1 \leqslant j \leqslant |V(H)|\}$, we may write (x_i, y_j) as $v_{i,j}$ and let $R_i = \{v_{i,j} : 1 \leqslant j \leqslant |V(H)|\}$ and $C_j = \{v_{i,j} : 1 \leqslant i \leqslant |V(G)|\}$ represent the ith row and the jth column of $V(G) \times V(H)$, respectively. See Fig. 1 for the example $P_3 \times P_6$.

The main result of this paper is to determine the profiles of $K_m \times K_n$, $K_{s,t} \times K_n$ and $P_m \times K_n$.

2. Profile of $K_m \times K_n$

This section establishes the profile of $K_m \times K_n$.

Theorem 2. *If $m = 1$ or $n \geqslant \max\{m, 4\}$, then* $P(K_m \times K_n) = \frac{1}{2}(m-1)(mn^2 + n^2 - n - 4)$.

Proof. As the case of $m = 1$ is obvious, we may assume that $m \geqslant 2$ and $n \geqslant \max\{m, 4\}$.

First, consider a proper numbering g of $K_m \times K_n$ satisfying

$$g(v_{i,j}) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \leqslant j \leqslant n-1, \\ mn & \text{for } i = 1 \text{ and } j = n, \\ i + n - 2 & \text{for } 2 \leqslant i \leqslant m \text{ and } j = n, \end{cases}$$

while the other vertices are assigned numbers arbitrarily, see Fig. 2 for g of $K_5 \times K_9$ in which the edges are not drawn for simplicity.

The profile width of vertex $v_{i,j}$ is

$$w_g(v_{i,j}) = \begin{cases} 0 & \text{for } i = 1 \text{ and } 1 \leqslant j \leqslant n-1, \\ mn - n - m + 1 & \text{for } i = 1 \text{ and } j = n, \\ g(v_{i,j}) - 2 & \text{for } 2 \leqslant i \leqslant m \text{ and } j = 1, \\ g(v_{i,j}) - 1 & \text{for } 2 \leqslant i \leqslant m \text{ and } 2 \leqslant j \leqslant n. \end{cases}$$

Therefore,

$$\begin{aligned} P(K_m \times K_n) &\leqslant P_g(K_m \times K_n) \\ &= (mn - n - m + 1) + \sum_{k=n}^{mn-1} (k-1) - (m-1) \\ &= \tfrac{1}{2}(m-1)(mn^2 + n^2 - n - 4). \end{aligned}$$

Next, we shall prove that $P(K_m \times K_n) \geqslant \frac{1}{2}(m-1)(mn^2 + n^2 - n - 4)$. Choose a profile numbering f of $K_m \times K_n$. Notice that $P(K_m \times K_n) = |E((K_m \times K_n)_f)|$. Without loss of generality, we may assume that $f(v_{1,1}) = 1$. For positive integers a and b, let $e_{a,b} = 2\binom{a}{2}\binom{b}{2} + (a-1)\binom{b}{2} + (b-2)\binom{a}{2} + 2\binom{a-1}{2}$. We consider the following three cases.

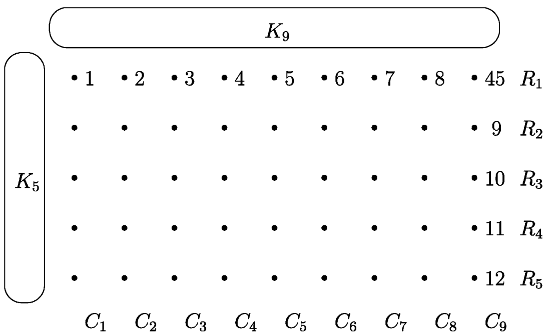

Fig. 2. A proper numbering g of $K_5 \times K_9$.

Case 1: $f^{-1}(2) \in R_1$, say $f(v_{1,j}) = j$ for $1 \leqslant j \leqslant r$ but $f(v_{s,t}) = r+1$ with $s \neq 1$ for some $r \geqslant 2$.

We shall count the number of edges in $(K_m \times K_n)_f$. Notice that besides the edges in $K_m \times K_n$, extra edges are due to the following cliques in $(K_m \times K_n)_f$ which are independent sets in $K_m \times K_n$.

Each row R_i with $2 \leqslant i \leqslant m$ is a clique in $(K_m \times K_n)_f$, since for $v_{i,p}, v_{i,q} \in R_i$ with $f(v_{i,p}) < f(v_{i,q})$, we can choose $k \in \{1,2\} - \{q\}$, such that $f(v_{1,k}) = k < f(v_{i,p}) < f(v_{i,q})$ and $v_{1,k}v_{i,q} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$, which imply $v_{i,p}v_{i,q} \in E((K_m \times K_n)_f)$. Notice that we use the interval property (1) in this implication. As the property will be used frequently, we shall not mention it every time.

Each column C_j with $2 \leqslant j \leqslant r$ is a clique in $(K_m \times K_n)_f$, since for $v_{p,j}, v_{q,j} \in C_j$ with $f(v_{p,j}) < f(v_{q,j})$, we have $q \geqslant 2$, and so $f(v_{1,1}) = 1 < f(v_{p,j}) < f(v_{q,j})$ and $v_{1,1}v_{q,j} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$, which imply $v_{p,j}v_{q,j} \in E((K_m \times K_n)_f)$.

For the case $r+1 \leqslant n$, any column C_j with $j \geqslant r+1$ but $j \neq t$ is a clique in $(K_m \times K_n)_f$, since for $v_{p,j}, v_{q,j} \in C_j$ with $f(v_{p,j}) < f(v_{q,j})$, we can choose $x = v_{1,1}$ (when $q \neq 1$) or $v_{s,t}$ (when $q = 1$), such that $f(x) < f(v_{p,j}) < f(v_{q,j})$ and $xv_{q,j} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$, which imply $v_{p,j}v_{q,j} \in E((K_m \times K_n)_f)$.

Similarly, $C_j - \{v_{1,j}\}$ is cliques in $(K_m \times K_n)_f$ for $1 \leqslant j \leqslant n$. In particular, this is true for $j = 1, t$.

Therefore, totally the graph $(K_m \times K_n)_f$ has at least $e_{m,n} = 2\binom{m}{2}\binom{n}{2} + (m-1)\binom{n}{2} + (n-2)\binom{m}{2} + 2\binom{m-1}{2} = \frac{1}{2}(m-1)(mn^2+n^2-n-4)$ edges, which gives that $P(K_m \times K_n) \geqslant \frac{1}{2}(m-1)(mn^2+n^2-n-4)$.

Case 2: $f^{-1}(2) \in C_1$.

Since $n \geqslant m$ and $n+m \geqslant 5$, we have $e_{n,m} - e_{m,n} = \binom{m}{2} - \binom{n}{2} + 2\binom{n-1}{2} - 2\binom{m-1}{2} = \frac{1}{2}(n+m-5)(n-m) \geqslant 0$. By an argument similar as Case 1, $P(K_m \times K_n) \geqslant e_{n,m} \geqslant e_{m,n} = \frac{1}{2}(m-1)(mn^2+n^2-n-4)$.

Case 3: $f^{-1}(2) \notin R_1 \cup C_1$, say $f(v_{2,2}) = 2$.

By an argument similar as Case 1, $R_1 - \{v_{1,1}, v_{1,2}\}$, $R_2 - \{v_{2,1}\}$, R_i for $3 \leqslant i \leqslant m$, $C_1 - \{v_{1,1}, v_{2,1}\}$, $C_2 - \{v_{1,2}\}$, C_j for $3 \leqslant j \leqslant n$ are all cliques in $(K_m \times K_n)_f$. Let $f^{-1}(3) = v_{s,t}$. Then, either $v_{s,t} \notin R_1 \cup C_2$ or $v_{s,t} \notin R_2 \cup C_1$. We may assume $v_{s,t} \notin R_1 \cup C_2$. Suppose $3 \leqslant q \leqslant n$. For the case $f(v_{1,2}) < f(v_{1,q})$, we have $f(v_{2,2}) = 2 < f(v_{1,2}) < f(v_{1,q})$ and $v_{2,2}v_{1,q} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$ implying $v_{1,2}v_{1,q} \in E((K_m \times K_n)_f)$. For the case $f(v_{1,2}) > f(v_{1,q})$, we have $f(v_{s,t}) = 3 < f(v_{1,q}) < f(v_{1,2})$ and $v_{s,t}v_{1,2} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$implying $v_{1,q}v_{1,2} \in E((K_m \times K_n)_f)$. So, in any case, $v_{1,2}v_{1,q} \in E((K_m \times K_n)_f)$. Similarly, $v_{1,2}v_{p,2} \in E((K_m \times K_n)_f)$ for $3 \leqslant p \leqslant m$. There are totally $n+m-4$ such edges. So $(K_m \times K_n)_f$ has at least $2\binom{m}{2}\binom{n}{2} + \binom{n-2}{2} + \binom{n-1}{2} + (m-2)\binom{n}{2} + \binom{m-2}{2} + \binom{m-1}{2} + (n-2)\binom{m}{2} + (n+m-4)$ edges. As $n \geqslant 4$, this number is greater than $e_{m,n}$ by $(n-1)(n-4)/2 \geqslant 0$ edges. Again, we have $P(K_m \times K_n) \geqslant \frac{1}{2}(m-1)(mn^2+n^2-n-4)$. $\square$

The other cases remain are: $P(K_2 \times K_2) = 2$, $P(K_2 \times K_3) = 9$ and $P(K_3 \times K_3) = 28$.

3. Profile of $K_{s,t} \times K_n$

This section determines the profile of $K_{s,t} \times K_n$.

The notations we use in this section are the same as above except now we let $m = s+t$ and $V(K_{s,t}) = S \cup T$, where $S = \{x_1, x_2, \ldots, x_s\}$ and $T = \{x_{s+1}, x_{s+2}, \ldots, x_{s+t}\}$. We also let $S_j = \{v_{i,j} : x_i \in S\}$ and $T_j = \{v_{i,j} : x_i \in T\}$ for $1 \leqslant j \leqslant n$. Notice that $C_j = S_j \cup T_j$.

Theorem 3. *If* $r = \min\{s,t\}$ *and* $n \geqslant 4$, *then* $P(K_{s,t} \times K_n) = \binom{nr}{2} + (n^2-2)st$.

Proof. To prove $P(K_{s,t} \times K_n) \leqslant \binom{nr}{2} + (n^2-2)st$, without loss of generality we may assume that $r = t$. Consider the proper numbering g of $K_{s,t} \times K_n$ defined by

$$g(v_{i,j}) = \begin{cases} i + (j-1)s & \text{for } 1 \leqslant i \leqslant s \text{ and } 1 \leqslant j \leqslant n-1, \\ i + (n-1)s + t & \text{for } 1 \leqslant i \leqslant s \text{ and } j = n, \\ i + jt + (n-1)s & \text{for } s+1 \leqslant i \leqslant s+t \text{ and } 1 \leqslant j \leqslant n-1, \\ i + (n-2)s & \text{for } s+1 \leqslant i \leqslant s+t \text{ and } j = n. \end{cases}$$

See Fig. 3 for g of $K_{4,3} \times K_9$ in which the edges are not drawn for simplicity.

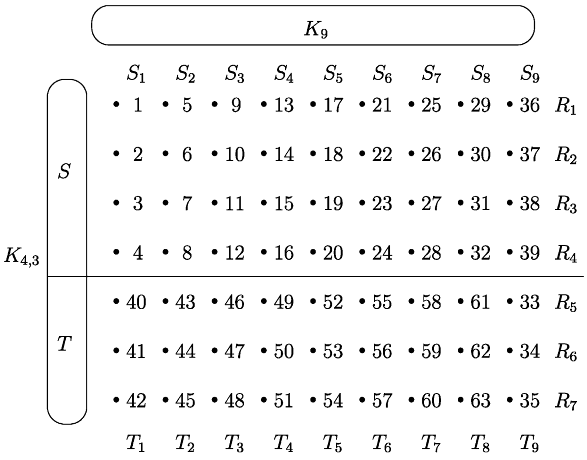

Fig. 3. A proper numbering g of $K_{4,3} \times K_9$.

Notice that two vertices are adjacent in $K_{s,t} \times K_n$ if and only if one is in S_j and the other in $T_{j'}$ for some $j \neq j'$. As no vertex in S_i is adjacent to a vertex with smaller numbering in $K_{s,t} \times K_n$, $S \times V(K_n)$ is an independent set in $(K_{s,t} \times K_n)_g$.

For any two vertices $v_{i,j}$ and $v_{i',j'}$ in $T \times K_n$ with $g(v_{i,j}) < g(v_{i',j'})$, we may choose k from $\{1, 2\}$ such that $k \neq j'$. So, $g(v_{1,k}) < g(v_{i,j}) < g(v_{i',j'})$ and $v_{1,k}v_{i',j'} \in E(K_{s,t} \times K_n) \subseteq E((K_{s,t} \times K_n)_g)$ imply that $v_{i,j}v_{i',j'} \in E((K_{s,t} \times K_n)_g)$. This proves that $T \times V(K_n)$ is a clique in $(K_{s,t} \times K_n)_g$, which gives $\binom{nt}{2}$ edges.

For any $v_{i,j} \in S_j$ and $v_{i',j} \in T_j$ with $2 \leqslant j \leqslant n-1$, we have $g(v_{1,1}) < g(v_{i,j}) < g(v_{i',j})$ and $v_{1,1}v_{i',j} \in E(K_{s,t} \times K_n) \subseteq E((K_{s,t} \times K_n)_g)$ implying that $v_{i,j}v_{i',j} \in E((K_{s,t} \times K_n)_g)$. It is also the case that no vertex in S_j is adjacent to a vertex in T_j in $(K_{s,t} \times K_n)_g$ for $j = 1$ or n. So, vertices in S_j are adjacent to vertices in $T_{j'}$ in $(K_{s,t} \times K_n)_g$ for all j and j' except $j = j' \in \{1, n\}$. These give $(n^2 - 2)st$ edges.

Therefore, $P(K_{s,t} \times K_n) \leqslant |E((K_{s,t} \times K_n)_g)| = \binom{nt}{2} + (n^2 - 2)st = \binom{nr}{2} + (n^2 - 2)st$.

Next, we shall prove that $P(K_{s,t} \times K_n) \geqslant \binom{nr}{2} + (n^2 - 2)st$. Choose a profile numbering f of $K_{s,t} \times K_n$. Without loss of generality, assume that $f(v_{1,1}) = 1$. Let $f(v_{a,b}) = \min\{f(v_{i,j}) : v_{i,j} \in T_2 \cup \cdots \cup T_n\}$.

For any vertices $v_{i,j} \in S_j$ and $v_{i',j'} \in T_{j'}$, by the definition, $v_{i,j}v_{i',j'} \in E(K_{s,t} \times K_n) \subseteq E((K_{s,t} \times K_n)_f)$ if $j \neq j'$. Suppose $j = j' \notin \{1, b\}$. If $f(v_{i,j}) < f(v_{i',j'})$, then $f(v_{1,1}) < f(v_{i,j}) < f(v_{i',j'})$ and $v_{1,1}v_{i'j'} \in E((K_{s,t} \times K_n)_f)$ imply that $v_{i,j}v_{i'j'} \in E((K_{s,t} \times K_n)_f)$. If $f(v_{i,j}) > f(v_{i',j'})$, then $f(v_{a,b}) < f(v_{i',j'}) < f(v_{i,j})$ and $v_{a,b}v_{i,j} \in E((K_{s,t} \times K_n)_f)$ imply that $v_{i,j}v_{i'j'} \in E((K_{s,t} \times K_n)_f)$. So, vertices in S_j are adjacent to vertices in $T_{j'}$ for all j and j' except $j = j' \in \{1, b\}$. These give $(n^2 - 2)st$ edges.

Consider any two vertices $v_{i,j}$ and $v_{i',j'}$ in $T_1 \cup T_2 \cup \cdots \cup T_n$ such that $f(v_{i,j}) < f(v_{i',j'})$. For $j' \geqslant 2$, we have $f(v_{1,1}) < f(v_{i,j}) < f(v_{i',j'})$ and $v_{1,1}v_{i',j'} \in E((K_{s,t} \times K_n)_f)$ implying $v_{i,j}v_{i',j'} \in E((K_{s,t} \times K_n)_f)$. So, $T_2 \cup T_3 \cup \cdots \cup T_n$ is a clique in $(K_{s,t} \times K_n)_f$. This gives $\binom{(n-1)t}{2}$ edges. If $T_1 \cup T_2 \cup \cdots \cup T_n$ is a clique, then these give $\binom{nt}{2} \geqslant \binom{nr}{2}$ edges. Therefore, $P(K_{s,t} \times K_n) \geqslant \binom{nr}{2} + (n^2 - 2)st$. Now, we may assume that there are two non-adjacent vertices $v_{p,q}$ and $v_{p',q'}$ in $T_1 \cup T_2 \cup \cdots \cup T_n$ with $f(v_{p,q}) < f(v_{p',q'})$ and $q' = 1$.

For any two vertices $v_{i,j}$ and $v_{i',j'}$ in $S_2 \cup S_3 \cup \cdots \cup S_n$ such that $f(v_{i,j}) < f(v_{i',j'})$. If $f(v_{p,q}) > f(v_{i,j})$, then $f(v_{i,j}) < f(v_{p,q}) < f(v_{p',q'})$ and $v_{i,j}v_{p',q'} \in E((K_{s,t} \times K_n)_f)$ imply $v_{p,q}v_{p',q'} \in E((K_{s,t} \times K_n)_f)$, a contradiction. Therefore, it is always the case that $f(v_{p,q}) < f(v_{i,j}) < f(v_{i',j'})$. Except for the case when $q = j' = b$, we have $v_{p,q}v_{i',j'} \in E((K_{s,t} \times K_n)_f)$, which together with the above inequalities gives that $v_{i,j}v_{i',j'} \in E((K_{s,t} \times K_n)_f)$.

Now, if $q \neq b$, we have that $S_2 \cup S_3 \cup \cdots \cup S_n$ is a clique. This gives $\binom{(n-1)s}{2}$ edges. And so $P(K_{s,t} \times K_n) \geqslant \binom{(n-1)s}{2} + \binom{(n-1)t}{2} + (n^2 - 2)st \geqslant 2\binom{(n-1)r}{2} + (n^2 - 2)st \geqslant \binom{nr}{2} + (n^2 - 2)st$ as $n \geqslant 4$. Hence we may assume that if $v_{p,q}$ and $v_{p',q'}$ are non-adjacent in $T_1 \cup T_2 \cup \cdots \cup T_n$ with $f(v_{p,q}) < f(v_{p',q'})$, then $q = b$ and $q' = 1$. In this case, $S_2 \cup S_3 \cup \cdots \cup S_{b-1} \cup S_{b+1} \cup S_{b+2} \cup \cdots \cup S_n$ is a clique and $T_1 \cup T_2 \cup \cdots \cup T_n$ is a clique except that vertices in T_1

are not necessarily adjacent to vertices inT_b. This gives $P(K_{s,t}\times K_n)\geqslant \binom{(n-2)s}{2}+\binom{nt}{2}-t^2+(n^2-2)st$. Notice that $\binom{nt}{2}-t^2=((n^2-2)t^2-nt)/2\geqslant((n^2-2)r^2-nr)/2$ as $t\geqslant r$. Thus, $P(K_{s,t}\times K_n)\geqslant \binom{(n-2)r}{2}+((n^2-2)r^2-nr)/2+(n^2-2)st\geqslant\binom{nr}{2}+(n^2-2)st$. □

4. Profile of $P_m \times K_n$

Finally, we study the profile of $P_m\times K_n$.

The results in the previous sections cover the case for $P_1\times K_n=K_1\times K_n$, $P_2\times K_n=K_2\times K_n=K_{1,1}\times K_n$ and $P_3\times K_n=K_{1,2}\times K_n$. In the following, we consider only for $m\geqslant 4$.

Theorem 4. *If $m,n\geqslant 4$, then $P(P_m\times K_n)=(m-2)\binom{n}{2}+(m-1)(n^2-1)$.*

Proof. For $P(P_m\times K_n)\leqslant(m-2)\binom{n}{2}+(m-1)(n^2-1)$, consider the proper numbering g of $P_m\times K_n$ defined by

$$g(v_{i,j})=\begin{cases}(i-1)n+j & \text{for } 1\leqslant i\leqslant m-2 \text{ and } 1\leqslant j\leqslant n,\\ (m-1)n+j & \text{for } i=m-1 \text{ and } 1\leqslant j\leqslant n-1,\\ (m-1)n & \text{for } i=m-1 \text{ and } j=n,\\ (m-2)n+j & \text{for } i=m \text{ and } 1\leqslant j\leqslant n-1,\\ mn & \text{for } i=m \text{ and } j=n,\end{cases}$$

see Fig. 4 for g of $P_5\times K_9$ in which the edges are not drawn for simplicity.

The profile width of vertex $v_{i,j}$ is

$$w_g(v_{i,j})=\begin{cases}0 & \text{for } i=1 \text{ and} 1\leqslant j\leqslant n,\\ n-1 & \text{for } 2\leqslant i\leqslant m-2 \text{ and } j=1,\\ n-1+j & \text{for } 2\leqslant i\leqslant m-2 \text{ and } 2\leqslant j\leqslant n,\\ 2n-1 & \text{for } i=m-1 \text{ and } j=1,\\ 2n-1+j & \text{for } i=m-1 \text{ and } 2\leqslant j\leqslant n-1,\\ 2n-1 & \text{for } i=m-1 \text{ and } j=n,\\ 0 & \text{for } i=m \text{ and } 1\leqslant j\leqslant n-1,\\ n-1 & \text{for } i=m \text{ and } j=n.\end{cases}$$

Therefore,

$$\sum_{j=1}^{n} w_g(v_{i,j})=\begin{cases}0 & \text{for } i=1,\\ \binom{n}{2}+(n^2-1) & \text{for } 2\leqslant i\leqslant m-2,\\ \binom{n}{2}+(2n^2-n-1) & \text{for } i=m-1,\\ n-1 & \text{for } i=m,\end{cases}$$

and so $P(K_m\times K_n)\leqslant P_g(K_m\times K_n)=\sum_{i=1}^{m}\sum_{j=1}^{n}w_g(v_{i,j})=(m-2)\binom{n}{2}+(m-1)(n^2-1)$.

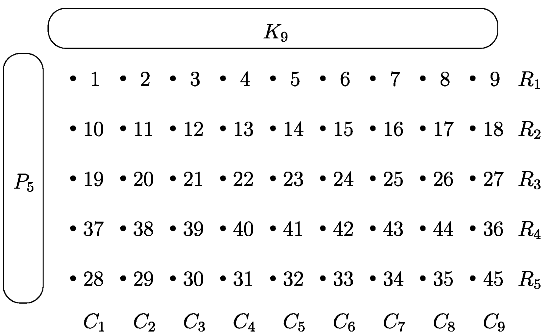

Fig. 4. A proper numbering g of $P_5\times K_9$.

To prove that $P(K_m \times K_n) \geqslant (m-2)\binom{n}{2} + (m-1)(n^2-1)$, choose a profile numbering f of $P_m \times K_n$. We use the following notation:

Let $a_i = \min_{v_{i,j} \in R_i} f(v_{i,j})$ and $f(v_{i,b_i}) = a_i$ for $1 \leqslant i \leqslant m$.

Let $A = \{i : 2 \leqslant i \leqslant m-1$ and R_i is not a clique in $(P_m \times K_n)_f\}$ and $p = |A|$.

Let $B = \{i : 2 \leqslant i \leqslant m-1$ and $a_i < \min\{a_{i-1}, a_{i+1}\}\}$ and $q = |B|$.

Let $\Lambda_{i,i'} = \{v_{i,j}v_{i',j'} \in E((P_m \times K_n)_f) : 1 \leqslant j, j' \leqslant n\}$ and $\lambda_{i,i'} = |\Lambda_{i,i'}|$ for $1 \leqslant i, i' \leqslant m$.

Let $\Lambda^{=}_{i,i'} = \{v_{i,j}v_{i',j'} \in E((P_m \times K_n)_f) : 1 \leqslant j = j' \leqslant n\}$ and $\lambda^{=}_{i,i'} = |\Lambda^{=}_{i,i'}|$ for $1 \leqslant i, i' \leqslant m$.

Let $\Lambda^{\leqslant}_{i,i'} = \{v_{i,j}v_{i',j'} \in E((P_m \times K_n)_f) : 1 \leqslant j \leqslant j' \leqslant n\}$ and $\lambda^{\leqslant}_{i,i'} = |\Lambda^{\leqslant}_{i,i'}|$ for $1 \leqslant i, i' \leqslant m$.

Claim 1. *Suppose $|i-i'|=1$. Then $\lambda^{=}_{i,i'} \geqslant n-2$ and so $\lambda_{i,i'} \geqslant n^2-2$. Furthermore, if $b_i = b_{i'}$, or $f(v_{i,b_{i'}}) < f(v_{i',b_{i'}})$, or R_i is a clique in $(P_m \times K_n)_f$ with $a_i < a_{i'}$, then $\lambda^{=}_{i,i'} \geqslant n-1$ and so $\lambda_{i,i'} \geqslant n^2-1$.*

Proof of Claim 1. Consider any $j \notin \{b_i, b_{i'}\}$. If $f(v_{i,j}) < f(v_{i',j})$, then $f(v_{i,b_i}) < f(v_{i,j}) < f(v_{i',j})$ and $v_{i,b_i}v_{i',j} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ imply $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$. If $f(v_{i,j}) > f(v_{i',j})$, then $f(v_{i',b_{i'}}) < f(v_{i',j}) < f(v_{i,j})$ and $v_{i',b_{i'}}v_{i,j} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ imply $v_{i',j}v_{i,j} \in E((P_m \times K_n)_f)$. In any case, $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$ for $j \notin \{b_i, b_{i'}\}$, which give $\lambda^{=}_{i,i'} \geqslant n-2$. There are already other $n(n-1)$ edges between R_i and $R_{i'}$ in $E(P_m \times K_n)$, so we have $\lambda_{i,i'} \geqslant n^2-2$.

For the case $b_i = b_{i'}$, there are at least $n-1$ edges $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$ for $j \notin \{b_i, b_{i'}\}$. So, $\lambda^{=}_{i,i'} \geqslant n-1$ and $\lambda_{i,i'} \geqslant n^2-1$.

Now suppose $b_i \neq b_{i'}$. For the case $f(v_{i,b_{i'}}) < f(v_{i',b_{i'}})$, besides the $n-2$ edges $v_{i,j}v_{i',j}$ for $j \notin \{b_i, b_{i'}\}$, we also have the edge $v_{i,b_{i'}}v_{i',b_{i'}}$, since $f(v_{i,b_i}) < f(v_{i,b_{i'}}) < f(v_{i',b_{i'}})$ and $v_{i,b_i}v_{i',b_{i'}} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ implying $v_{i,b_{i'}}v_{i',b_{i'}} \in E((P_m \times K_n)_f)$. For the case when $f(v_{i,b_{i'}}) > f(v_{i',b_{i'}})$ and R_i is a clique with $a_i < a_{i'}$, again $f(v_{i,b_i}) = a_i < a_{i'} = f(v_{i',b_{i'}}) < f(v_{i,b_{i'}})$ and $v_{i,b_i}v_{i,b_{i'}} \in E((P_m \times K_n)_f)$ imply $v_{i',b_{i'}}v_{i,b_{i'}} \in E((P_m \times K_n)_f)$. In any case, $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$ for $j \neq b_i$, which gives $\lambda^{=}_{i,i'} \geqslant n-1$ and $\lambda_{i,i'} \geqslant n^2-1$. □

Claim 2. *If $i \in A$, then $\lambda^{\leqslant}_{i-1,i+1} \geqslant \binom{n-1}{2} \geqslant 3$.*

Proof of Claim 2. As R_i is not a clique in $(P_m \times K_n)_f$, we may choose $c \neq d$ such that $v_{i,c}v_{i,d} \notin E((P_m \times K_n)_f)$. Consider any $j, j' \notin \{c, d\}$ with $1 \leqslant j \leqslant j' \leqslant n$. In the 4-cycle $(v_{i,c}, v_{i-1,j}, v_{i,d}, v_{i+1,j'}, v_{i,c})$, we have $v_{i,c}v_{i,d} \notin E((P_m \times K_n)_f)$ implying $v_{i-1,j}v_{i+1,j'} \in E((P_m \times K_n)_f)$ by the chordality property (3). This gives that $\lambda^{\leqslant}_{i-1,i+1} \geqslant (1+2+\cdots+(n-2)) = \binom{n-1}{2} \geqslant 3$. □

Claim 3. *If $i \in B$, then $\lambda^{\leqslant}_{i-1,i+1} \geqslant \binom{n}{2} \geqslant 6$.*

Proof of Claim 3. For any $j, j' \notin \{b_i\}$ with $1 \leqslant j \leqslant j' \leqslant n$, since $f(v_{i,b_i}) = a_i < a_{i-1} \leqslant f(v_{i-1,j})$ with $v_{i,b_i}v_{i-1,j} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ and $f(v_{i,b_i}) = a_i < a_{i+1} \leqslant f(v_{i+1,j'})$ with $v_{i,b_i}v_{i+1,j'} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$, by perfect elimination property (2), $v_{i-1,j}v_{i+1,j'} \in E((P_m \times K_n)_f)$. These give $\lambda^{\leqslant}_{i-1,i+1} \geqslant 1+2+\cdots+(n-1) = \binom{n}{2} \geqslant 6$. □

Having these three claims in mind, we are ready to prove the theorem. As $n \geqslant 4$, there is a bijection from $\{\{j,k\} : 1 \leqslant j < k \leqslant n\}$ to itself such that $\{j,k\}$ is disjoint from its image $\{j',k'\}$. This can be done by setting $\{j',k'\} = \{(j+\delta) \bmod n, (k+\delta) \bmod n\}$, where $\delta = 2$ when j and k are consecutive under modula n, and $\delta = 1$ otherwise. We may assume that $j' > k'$ for our convenience. Consider the following $(m-2)\binom{n}{2}$ disjoint sets:

$$S_{i,j,k} = \{v_{i,j}v_{i,k}, v_{i-1,j'}v_{i+1,k'}\},$$

where $2 \leqslant i \leqslant m-2$ and $1 \leqslant j < k \leqslant n$. In the 4-cycle $(v_{i,j}, v_{i-1,j'}, v_{i,k}, v_{i+1,k'}, v_{i,j})$ (see Fig. 5), at least one of the edge in $S_{i,j,k}$ must exist. These give totally at least $(m-2)\binom{n}{2}$ edges.

Among the $m-2$ rows $R_2, R_3, \ldots, R_{m-1}$, there are p rows that are not cliques in $(P_m \times K_n)_f$ and the other $m-2-p$ rows are cliques. Among the $m-2-p$ clique rows, let there be p' consecutive pairs, that is, cliques R_i and $R_{i'}$ with

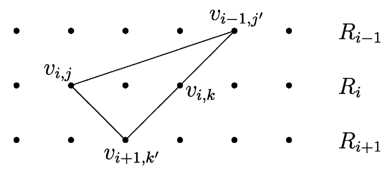

Fig. 5. The 4-cycle $(v_{i,j}, v_{i-1,j'}, v_{i,k}, v_{i+1,k'}, v_{i,j})$.

$|i-i'|=1$. By Claim 1, $\lambda_{i,i'} \geqslant n^2-1$ for these p' pairs and $\lambda_{i,i'} \geqslant n^2-2$ for the remaining $m-1-p'$ pairs of i and i' with $|i-i'|=1$. These give totally at least $p'(n^2-1)+(m-1-p')(n^2-2)=(m-1)(n^2-1)+(p'+1-m)$ edges.

By Claim 3, there are at least $6q$ extra edges from the sets $\Lambda^{\leqslant}_{i-1,i+1}$ for $i \in B$. By Claim 2, there are at least $3(p-q)$ extra edges from the sets $\Lambda^{\leqslant}_{i-1,i+1}$ for $i \in A \backslash B$. These give at least $3p+3q$ extra edges. So, we have

$$P(P_m \times K_n) \geqslant (m-2)\binom{n}{2}+(m-1)(n^2-1)+(p'+1-m+3p+3q).$$

In particular, $P(P_m \times K_n) \geqslant (m-2)\binom{n}{2}+(m-1)(n^2-1)$ when $p'+1-m+3p+3q \geqslant 0$. So, now assume that $p'+1-m+3p+3q \leqslant -1$ or $p' \leqslant m-3p-3q-2$.

Notice that there are p non-clique rows R_i with $2 \leqslant i \leqslant m-1$. These rows separate the other rows into $p+1$ runs. Each run with α clique rows in $R_2, R_3, \ldots, R_{m-1}$ has $\max\{0, \alpha-1\} \geqslant \alpha-1$ consecutive pairs of cliques. Therefore, $p' \geqslant m-2-p-(p+1)=m-2p-3$ with equality holds if and only if $\alpha \geqslant 1$ for each run of clique rows. Or equivalently, any two rows in $A \cup \{R_1, R_m\}$ are not consecutive, which implies that $3 \leqslant i \leqslant m-2$ for $i \in A$.

Now, $m-2p-3 \leqslant p' \leqslant m-3p-3q-2$ imply that $p+3q \leqslant 1$. This is possible only when $p \leqslant 1$ and $q=0$. Suppose $p=1$, say $A=\{R_i\}$. Then, the above inequalities are in fact equalities, i.e., $m-2p-3=p'$ and so $3 \leqslant i \leqslant m-2$. Therefore, R_{i-1} and R_{i+1} are clique rows. As $q=0$, we have $i \notin B$ and so either $a_{i-1}<a_i$ or $a_{i+1}<a_i$. By Claim 1, either $\lambda^{=}_{i-1,i} \geqslant n-1$ or $\lambda^{=}_{i,i+1} \geqslant n-1$. So in the above calculation, we in fact have $p'+1$, rather than p', consecutive pairs of i and i' with $\lambda_{i,i'} \geqslant n^2-1$. Thus,

$$P(P_m \times K_n) \geqslant (m-2)\binom{n}{2}+(m-1)(n^2-1)+(p'+2-m+3p+3q),$$

where $p'+2-m+3p+3q \geqslant (m-2p-3)+2-m+3p+3q=p+3q-1=0$ and so again $P(P_m \times K_n) \geqslant (m-2)\binom{n}{2}+(m-1)(n^2-1)$.

Now we may suppose that $p=q=0$. In other words, $R_2, R_3, \ldots, R_{m-1}$ are cliques and

$$a_1<a_2<\cdots<a_{r-1}<a_r \quad \text{and} \quad a_r>a_{r+1}>a_{r+2}>\cdots>a_m \tag{4}$$

for some r. By Claim 1, we have

$$\lambda_{1,2} \geqslant n^2-2, \quad \lambda_{i,i+1} \geqslant n^2-1 \text{ for } 2 \leqslant i \leqslant m-2, \quad \lambda_{m-1,m} \geqslant n^2-2.$$

These together with the $m-2$ clique rows gives at least $(m-2)\binom{n}{2}+(m-1)(n^2-1)-2$ edges. In the following, two extra edges, one with an end vertex in R_1 and the other with an end vertex in R_m, are to be found to make $P(P_m \times K_n) \geqslant (m-2)\binom{n}{2}+(m-1)(n^2-1)$. Assume, by symmetric, there is no such extra edge with a vertex in R_1 which we call an R_1-edge, we shall either get a contradiction or find two other extra edges.

First, we may assume that $b_1 \neq b_2$ and $a_1<a_2$ and $f(v_{1,b_2})>f(v_{2,b_2})$, for otherwise Claim 1 gives that $\lambda_{1,2} \geqslant n^2-1$ rather than only $\lambda_{1,2} \geqslant n^2-2$ which give an extra R_1-edge, a contradiction. Notice that the two non-edges between R_1 and R_2 are $v_{1,b_1}v_{2,b_1}$ and $v_{1,b_2}v_{2,b_2}$.

We claim that in fact $a_1=1$. Suppose to the contrary that $a_1>1$. By (4), we have $a_m=1$. This together with $a_m<a_1<a_2 \leqslant a_r$ implies that there is some i such that $a_r \geqslant a_{i-1}>a_1>a_i \geqslant a_m=1$. Then, for each $j \neq b_i$, we have $f(v_{i,b_i})<f(v_{1,b_1})<f(v_{i-1,j})$ and $v_{i,b_i}v_{i-1,j} \in E((P_m \times K_n)_f)$ implying $v_{1,b_1}v_{i-1,j} \in E((P_m \times K_n)_f)$, which gives $n-1$ extra R_1-edges, a contradiction. Thus, $a_1=1$.

As $a_1=1$ and $f(v_{1,b_2})>a_2$, without loss of generality, we may assume that $f(v_{1,j})=j$ for $1 \leqslant j \leqslant \ell-1$ but $f^{-1}(\ell)=v_{i^*,j^*} \notin R_1$, where $\ell \leqslant n$. Notice that we assume $b_1=1$ now. By the inequalities in (4), we have $\ell=a_m$

or $\ell = a_2$. For the case $\ell = a_m$, for any $j \neq 1$, we have $f(v_{1,1}) = 1 < \ell = a_m = f(v_{m,b_m}) < f(v_{2,j})$ and $v_{1,1}v_{2,j} \in E((P_m \times K_n)_f)$, implying $v_{m,b_m}v_{2,j'} \in E((P_m \times K_n)_f)$, which are $n-1 \geqslant 2$ extra edges as desired. For the case $\ell = a_2$, we may assume that $b_2 = n$. If $\ell < n$, then for any $j < n$, we have $f(v_{2,n}) < f(v_{1,\ell})$ with $v_{2,n}v_{1,\ell} \in E((P_m \times K_n)_f)$ and $f(v_{2,n}) < f(v_{3,j})$ with $v_{2,n}v_{3,j} \in E((P_m \times K_n)_f)$, implying $v_{1,\ell}v_{3,j} \in E((P_m \times K_n)_f)$by the perfect elimination property (2). This gives $n-1 \geqslant 2$ extra edges as desired. So, we may assume that $\ell = n$.

Next, $f(v_{1,n}) > f(v_{3,1})$, for otherwise, $f(v_{1,n}) < f(v_{3,1})$ gives that $f(v_{2,n}) < f(v_{1,n}) < f(v_{3,1})$, this together with $v_{2,n}v_{3,1} \in E((P_m \times K_n)_f)$ implying $v_{1,n}v_{3,1} \in E((P_m \times K_n)_f)$, which is an extra R_1-edge, a contradiction. Similarly, for each j with $2 \leqslant j \leqslant n-1$ we have $f(v_{2,j}) > f(v_{3,1})$, for otherwise, $f(v_{2,j}) < f(v_{3,1})$ gives that $f(v_{2,j}) < f(v_{3,1}) < f(v_{1,n})$, this together with $v_{2,j}v_{1,n} \in E((P_m \times K_n)_f)$ implying $v_{3,1}v_{1,n} \in E((P_m \times K_n)_f)$, which is an extra R_1-edge, a contradiction. Also, $f(v_{4,2}) > f(v_{3,1})$, for otherwise, $f(v_{4,2}) < f(v_{3,1})$ gives that for each j with $2 \leqslant j \leqslant n-1$, we have $f(v_{1,1}) < f(v_{4,2}) < f(v_{3,1}) < f(v_{2,j})$, this together with $v_{1,1}v_{2,j} \in E((P_m \times K_n)_f)$ implying $v_{4,2}v_{2,j} \in E((P_m \times K_n)_f)$, which are $n-2 \geqslant 2$ extra edges as desired. Now, for each j with $2 \leqslant j \leqslant n-1$, we have $f(v_{3,1}) < f(v_{2,j})$ with $v_{3,1}v_{2,j} \in E((P_m \times K_n)_f)$, and $f(v_{3,1}) < f(v_{4,2})$ with $v_{3,1}v_{4,2} \in E((P_m \times K_n)_f)$, implying $v_{2,j}v_{4,2} \in E((P_m \times K_n)_f)$, which are $n-2 \geqslant 2$ extra edges as desired. $\square$

5. Conclusion

In this paper, we determine the profiles of $K_m \times K_n$, $K_{s,t} \times K_n$ and $P_n \times K_n$. It is desirable to find the profile of $G \times H$ for general graphs G and H, or at least for a general G with $H = K_n$.

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