CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 295 (2004) 127–133
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations

Gian-Luigi Forti

Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, I-20133, Milano, Italy

Received 4 November 2003
Available online 8 May 2004
Submitted by S.R. Grace

Abstract

In this short paper the core of the direct method for proving stability of functional equations is described in a clear way and in a quite general form.
© 2004 Elsevier Inc. All rights reserved.

Keywords: Hyers–Ulam stability; Functional equations in several variables

1. Introduction

In the last years a great number of papers dealing with the Hyers–Ulam stability of functional equations have been published. Almost all of them treat functional equations in several variables and in order to prove stability they perform certain manipulations. The core of these manipulations (sometimes called *direct method*) is always the same and essentially goes back to a result of mine published in 1980 [3] (see also [6,8]). This is not always recognized (or the authors are not aware of this result) and so hundreds of pages have been written repeating essentially the same procedure.

The manipulations come down to the following: there is (in an appropriate framework) a functional equation

$$E_1(F) = E_2(F)$$

E-mail address: gianluigi.forti@mat.unimi.it.

0022-247X/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2004.03.011

in the unknown function F and appear several variables (note that F is a one-place function); moreover we have a function f satisfying the related inequality

$$d\big(E_1(f), E_2(f)\big) \leqslant \Delta,$$

where Δ is a certain function depending (or not) on the variables involved (d is a distance).

After a certain number of manipulations in the inequality, only one variable remains and we get something of this form

$$d\big(H\big\{f\big[G(x)\big]\big\}, f(x)\big) \leqslant \delta(x).$$

It is exactly at this moment that a standard procedure can be applied to get a solution F of the functional equation (in one variable!)

$$H\big\{F\big[G(x)\big]\big\} = F(x)$$

which is *near* the function f. The aim of this short paper is simply to enucleate in a clear way and in a quite general form the procedure of construction of F, in order to have a standard tool ready to be used.

To conclude the stability result it is then necessary to show that the function obtained is indeed a solution of the original equation $E_1(F) = E_2(F)$ and this part strongly depends on the form of the functional equation involved, on the set S and on the space X. Various results and rich bibliographies about stability can be found in [2,4,7].

2. Main result

Let (X, d) be a complete metric space, S a set, $G: S \to S$ and $H\colon X \to X$ be two given functions. From now on we assume that $f: S \to X$ is a function satisfying the following inequality:

$$d\big(H\big\{f\big[G(x)\big]\big\}, f(x)\big) \leqslant \delta(x) \tag{1}$$

for all $x \in S$ and for some function $\delta: S \to \mathbb{R}^+$.

Lemma 1. *Assume that the function H satisfies the following inequality*:

$$d\big(H(u), H(v)\big) \leqslant \phi\big(d(u, v)\big), \quad u, v \in X, \tag{2}$$

for a certain non-decreasing function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$. Then, for each integer n, we have

$$d\big(H^{n+1}\big\{f\big[G^{n+1}(x)\big]\big\}, H^n\big\{f\big[G^n(x)\big]\big\}\big) \leqslant \phi^n\big(\delta\big[G^n(x)\big]\big), \quad x \in S, \tag{3}$$

where H^i, G^i and ϕ^i denote the i-th iterate of H, G and ϕ, respectively.

Proof. Setting in (1) $G(x)$ instead of x we get

$$d\big(H\big\{f\big[G^2(x)\big]\big\}, f\big[G(x)\big]\big) \leqslant \delta\big[G(x)\big].$$

Then by (2) we obtain

$$\begin{aligned} d\big(H^2\big\{f\big[G^2(x)\big]\big\}, H\big\{f\big[G(x)\big]\big\}\big) &\leqslant \phi\big(d\big(H\big\{f\big[G^2(x)\big]\big\}, f\big[G(x)\big]\big)\big) \\ &\leqslant \phi\big(\delta\big[G(x)\big]\big), \end{aligned}$$

since ϕ is non-decreasing. The lemma follows by induction. □

Now, we take the sequence of functions

$$Q_n(x) := H^n\{f[G^n(x)]\}, \quad x \in S,$$

and consider the problem of its convergence. Since the metric space X is complete, this is equivalent to find conditions assuring that $\{Q_n(x)\}$ is a Cauchy sequence for every x in S.

Lemma 2. *In the hypotheses of Lemma* 1, *if the series*

$$\sum_{i=0}^{\infty} \phi^i(\delta[G^i(x)])$$

is convergent for every $x \in S$ then $\{Q_n(x)\}$ is a Cauchy sequence. Defined

$$F(x) = \lim_{n\to+\infty} Q_n(x),$$

we have

$$d(F(x), f(x)) \leqslant \sum_{i=0}^{\infty} \phi^i(\delta[G^i(x)]). \tag{4}$$

Proof. Let $m > n$; then

$$d(Q_n(x), Q_m(x)) \leqslant \sum_{i=n}^{m-1} d(Q_{i+1}(x), Q_i(x)) \leqslant \sum_{i=n}^{m-1} \phi^i(\delta[G^i(x)]),$$

thus the first part of the lemma follows immediately.

Using (3) we get

$$\begin{aligned} d(Q_n(x), f(x)) &= d(H^n\{f[G^n(x)]\}, f(x)) \\ &\leqslant \sum_{i=1}^{n} d(H^i\{f[G^i(x)]\}, H^{i-1}\{f[G^{i-1}(x)]\}) \\ &\leqslant \sum_{i=1}^{n} \phi^{i-1}(\delta[G^{i-1}(x)]). \end{aligned}$$

Taking the limit as n goes to infinity we obtain (4). □

Lemma 3. *Assume the hypotheses of Lemmas* 1 *and* 2. *If the function H is continuous, then the function F is a solution of the functional equation*

$$H\{F[G(x)]\} = F(x), \quad x \in S. \tag{5}$$

Moreover, if ϕ is subadditive, then F is the only function satisfying Eq. (5) *and inequality* (4).

Proof. By the continuity of H we have the following chain of equalities:

$$H\{F[G(x)]\} = H\Big\{\lim_{n\to+\infty} Q_n[G(x)]\Big\} = \lim_{n\to+\infty} H\{Q_n[G(x)]\}$$
$$= \lim_{n\to+\infty} H^{n+1}\{f[G^{n+1}(x)]\} = F(x).$$

Suppose that a function $\hat{F}$ satisfies (4) and (5) and ϕ is subadditive. Thus

$$\begin{aligned} d\big(\hat{F}(x), Q_n(x)\big) &= d\big(H^n\{\hat{F}[G^n(x)]\}, H^n\{f[G^n(x)]\}\big) \\ &\leqslant \phi^n\big(d\big(\hat{F}[G^n(x)], f[G^n(x)]\big)\big) \leqslant \phi^n\left(\sum_{i=0}^{\infty}\phi^i\big(\delta[G^{n+i}(x)]\big)\right) \\ &\leqslant \sum_{i=0}^{\infty}\phi^{n+i}\big(\delta[G^{n+i}(x)]\big). \end{aligned}$$

Taking the limit as n goes to infinity, since the last term goes to zero we obtain

$$\lim_{n\to+\infty} d\big(\hat{F}(x), Q_n(x)\big) = d\big(\hat{F}(x), F(x)\big) = 0. \qquad \square$$

We may now summarize the previous results in the following theorem.

Theorem 1. *Assume that $f : S \to X$ is a function satisfying the inequality*

$$d\big(H\{f[G(x)]\}, f(x)\big) \leqslant \delta(x).$$

If the function $H : X \to X$ is continuous and satisfies the inequality

$$d\big(H(u), H(v)\big) \leqslant \phi\big(d(u, v)\big), \quad u, v \in X,$$

for a certain non-decreasing subadditive function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and the series

$$\sum_{i=0}^{\infty}\phi^i\big(\delta[G^i(x)]\big)$$

is convergent for every $x \in S$, then there exists a unique function $F : S \to X$ solution of the functional equation

$$H\{F[G(x)]\} = F(x), \quad x \in S,$$

and satisfying the following inequality:

$$d\big(F(x), f(x)\big) \leqslant \sum_{i=0}^{\infty}\phi^i\big(\delta[G^i(x)]\big).$$

The function F is given by

$$F(x) = \lim_{n\to+\infty} H^n\{f[G^n(x)]\}.$$

In the case the functions G and H are invertible, we immediately obtain the following result.

Theorem 2. *Assume that $f : S \to X$ is a function satisfying the inequality*

$$d\bigl(H\bigl\{f\bigl[G(x)\bigr]\bigr\}, f(x)\bigr) \leqslant \delta(x)$$

and suppose that the functions G and H are invertible. If the function $H^{-1} : X \to X$ is continuous and satisfies the inequality

$$d\bigl(H^{-1}(u), H^{-1}(v)\bigr) \leqslant \psi\bigl(d(u, v)\bigr), \quad u, v \in X,$$

for a certain non-decreasing subadditive function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and the series

$$\sum_{i=1}^{\infty} \psi^i\bigl(\delta\bigl[G^{-i}(x)\bigr]\bigr)$$

is convergent for every $x \in S$, then there exists a unique function $F : S \to X$ solution of the functional equation

$$H\bigl\{F\bigl[G(x)\bigr]\bigr\} = F(x), \quad x \in S,$$

and satisfying the following inequality:

$$d\bigl(F(x), f(x)\bigr) \leqslant \sum_{i=1}^{\infty} \psi^i\bigl(\delta\bigl[G^{-i}(x)\bigr]\bigr).$$

The function F is given by

$$F(x) = \lim_{n\to+\infty} H^{-n}\bigl\{f\bigl[G^{-n}(x)\bigr]\bigr\}.$$

Appendix A

Following a valuable suggestion of the referee (I thank him/her for the contribution), in this section I would like to outline a procedure analogous to the one presented above, but in the spirit of paper [5] of R. Ger. For the sake of simplicity this presentation is not carried out in a more general setting as in the previous section.

Again, we consider the functional equation

$$H\bigl\{F\bigl[G(x)\bigr]\bigr\} = F(x) \tag{A.1}$$

and we assume that F is a real function and $H : \mathbb{R} \to \mathbb{R}$ (thus, $X = \mathbb{R}$). If F is a solution of (A.1), on the set where it is different from zero, Eq. (A.1) is equivalent to

$$\frac{H\{F[G(x)]\}}{F(x)} = 1.$$

Hence, for a given function $f : S \to \mathbb{R} \setminus \{0\}$ we can "measure" how far we are from solving (A.1) by the quantity

$$\left|\frac{H\{f[G(x)]\}}{f(x)} - 1\right|.$$

We have the following stability theorem.

Theorem A.1. *Assume that* $f : S \to \mathbb{R} \setminus \{0\}$ *is a function satisfying the inequality*

$$\left| \frac{H\{f[G(x)]\}}{f(x)} - 1 \right| \leqslant \delta(x) < 1. \tag{A.2}$$

Suppose that the following hypotheses are satisfied:

(i) *the function* $H : \mathbb{R} \to \mathbb{R}$ *is continuous and there are two non-decreasing functions* $\phi_1, \phi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi_1(1) = \phi_2(1) = 1$, *such that*

$$\phi_1\left(\frac{u}{v}\right) \leqslant \frac{H(u)}{H(v)} \leqslant \phi_2\left(\frac{u}{v}\right) \tag{A.3}$$

for $u \in \mathbb{R}$, $v \in \mathbb{R} \setminus \{0\}$ *and* $u/v > 0$;

(ii) *the series*

$$\sum_{i=0}^{\infty} \max\bigl\{-\log\bigl[\phi_1^i\bigl(1 - \delta\bigl[G^i(x)\bigr]\bigr)\bigr], \log\bigl[\phi_2^i\bigl(1 + \delta\bigl[G^i(x)\bigr]\bigr)\bigr]\bigr\}$$

is convergent for every $x \in S$ *and let us denote by* $\Lambda(x)$ *its sum.*

Then there exists a function $F : S \to \mathbb{R}$ *solution of* (A.1) *such that*

$$\exp\bigl(-\Lambda(x)\bigr) \leqslant \frac{F(x)}{f(x)} \leqslant \exp \Lambda(x).$$

Proof. Since $\delta(x) < 1$, $H\{f[G(x)]\}$ and $f(x)$ have the same sign and we may assume without loss of generality that are both positive. We can write (A.2) as

$$1 - \delta(x) \leqslant \frac{H\{f[G(x)]\}}{f(x)} \leqslant 1 + \delta(x).$$

From the above and relation (A.3), we can easily get by induction the following relation:

$$\phi_1^n\bigl(1 - \delta\bigl[G^n(x)\bigr]\bigr) \leqslant \frac{H^{n+1}\{f[G^{n+1}(x)]\}}{H^n\{f[G^n(x)]\}} \leqslant \phi_2^n\bigl(1 + \delta\bigl[G^n(x)\bigr]\bigr). \tag{A.4}$$

Consider the sequence $Q_n(x) := H^n\{f[G^n(x)]\}$ and let $P_n(x) = \log Q_n(x)$. If $m > n$, from (A.4) we obtain that

$$\bigl|P_m(x) - P_n(x)\bigr| \leqslant \sum_{i=n}^{m-1} \max\bigl\{-\log\bigl[\phi_1^i\bigl(1 - \delta\bigl[G^i(x)\bigr]\bigr)\bigr], \log\bigl[\phi_2^i\bigl(1 + \delta\bigl[G^i(x)\bigr]\bigr)\bigr]\bigr\}.$$

Since we have assumed that the series

$$\sum_{i=0}^{\infty} \max\bigl\{-\log\bigl[\phi_1^i\bigl(1 - \delta\bigl[G^i(x)\bigr]\bigr)\bigr], \log\bigl[\phi_2^i\bigl(1 + \delta\bigl[G^i(x)\bigr]\bigr)\bigr]\bigr\}$$

is convergent, we conclude that $\{P_n(x)\}$ is a Cauchy sequence for every x and so it converges to a function $g(x)$. But $Q_n(x) = \exp P_n(x)$, hence

$$\lim_{n\to\infty} Q_n(x) = \exp g(x) =: F(x).$$

From the continuity of H we can conclude that

$$H\{F[G(x)]\} = H\Big\{\lim_{n\to\infty} Q_n[G(x)]\Big\} = \lim_{n\to\infty} H\{Q_n[G(x)]\}$$
$$= \lim_{n\to\infty} H^{n+1}\{f[G^{n+1}(x)]\} = F(x),$$

thus F is a solution of the functional equation (A.1).

From

$$\frac{Q_n(x)}{f(x)} = \prod_{i=1}^{n} \frac{H^i\{f[G^i(x)]\}}{H^{i-1}\{f[G^{i-1}(x)]\}}$$

we obtain the inequality

$$\exp(-\Lambda(x)) \leqslant \frac{F(x)}{f(x)} \leqslant \exp \Lambda(x). \qquad \square$$

Note that for proving the uniqueness of the obtained solution it is necessary to add some conditions to the functions ϕ_1 and ϕ_2.

As a conclusion, I would like to note that we can consider more general functional equations as

$$H\{F[G(x)], x\} = F(x)$$

and produce analogous stability theorems (see, for instance, [1,9]).

References

[1] C. Borelli, On Hyers–Ulam stability for a class of functional equations, Aequationes Math. 54 (1997) 74–86.

[2] C. Borelli, G.L. Forti, On a general Hyers–Ulam stability result, Internat. J. Math. Math. Sci. 18 (1995) 229–236.

[3] G.L. Forti, An existence and stability theorem for a class of functional equations, Stochastica 4 (1980) 23–30.

[4] G.L. Forti, Hyers–Ulam stability of functional equations in several variables, Aequationes Math. 50 (1995) 143–190.

[5] R. Ger, Superstability is not natural, Rocznik Nauk.-Dydakt. Prace Mat. 13 (1993) 109–123.

[6] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27 (1941) 222–224.

[7] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, in: Progress in Nonlinear Differential Equations and Their Applications, vol. 34, Birkhäuser, 1998.

[8] Th.M. Rassias, On the stability of linear mappings in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.

[9] T. Trif, On the stability of a general gamma-type functional equation, Publ. Math. Debrecen 60 (2002) 47–61.